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Solutions

Calculus III

Professor Piotr Hajłasz

Second Exam

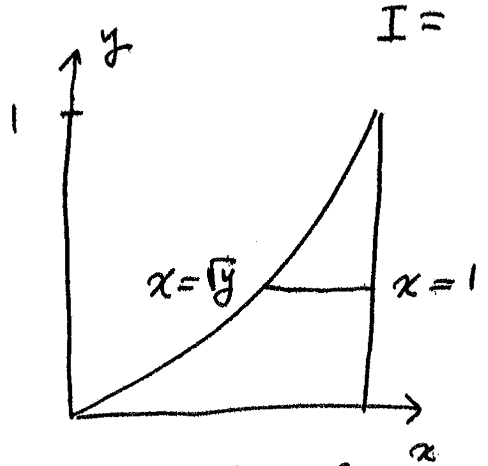
November 13, 2015.

Problem	Possible points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

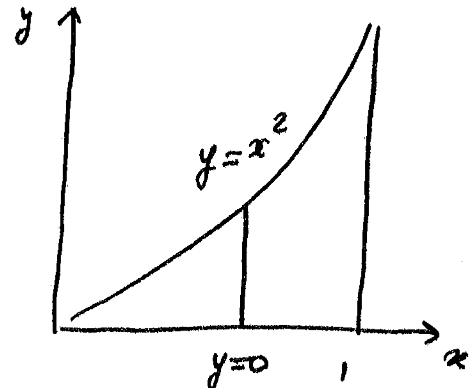
Problem 1.

(a) Evaluate the integral

$$I = \int_0^1 \int_{\sqrt{y}}^1 \frac{y}{x}\, dx\, dy.$$



or



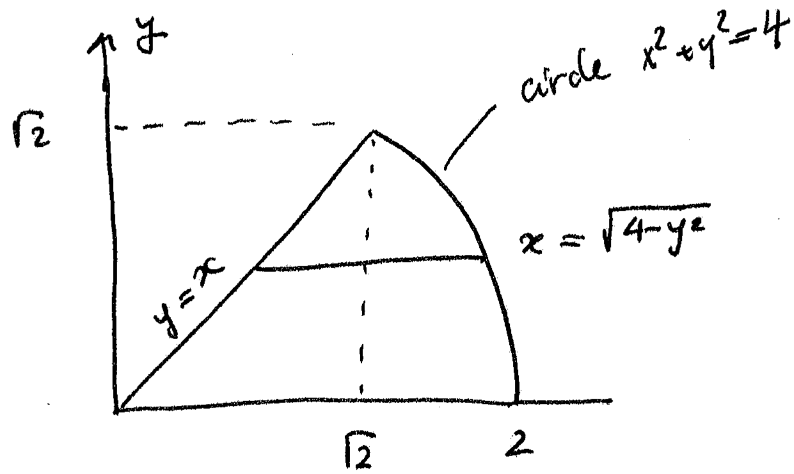
$$I = \int_0^1 \int_0^{x^2} \frac{y}{x}\, dy\, dx = \int_0^1 \frac{y^2}{2x}\Big|_{y=0}^{y=x^2} dx$$

$$= \frac{1}{2}\int_0^1 \frac{x^4}{x}\, dx = \frac{1}{2}\int_0^1 x^3\, dx = \frac{x^4}{8}\Big|_0^1 = \boxed{\frac{1}{8}}$$

(b) Evaluate the integral

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\, dx\, dy$$

by converting it to polar coordinates.



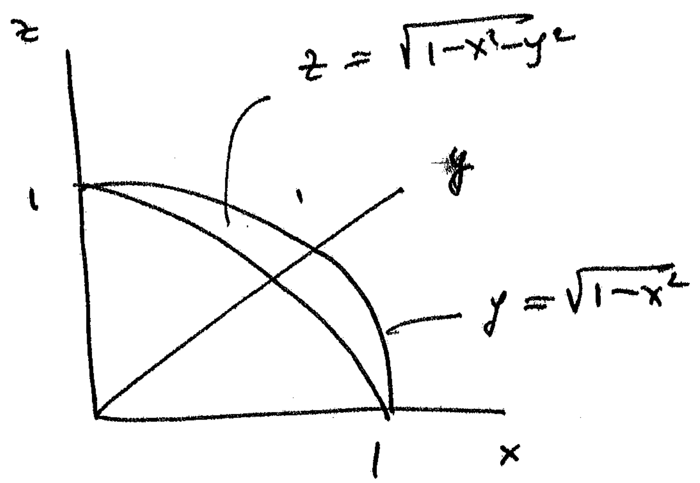
$$D = \left\{ (r, \theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{4} \right\}$$

$$I = \int_0^{\pi/4} \int_0^{2} \frac{1}{\sqrt{1+r^2}}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/4} d\theta \int_0^{2} \frac{r}{\sqrt{1+r^2}}\, dr = \frac{\pi}{4} \cdot \sqrt{1+r^2}\Big|_0^2 = \boxed{\frac{\pi}{4}(\sqrt{5}-1)}$$

Problem 2. Evaluate the integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx.$$



We are integrating over the part of the unit ball that is contained in the first octant.
In spherical coordinates

$$D = \{ (\rho, \theta, \phi) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \rho \le 1 \}$$

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta =$$

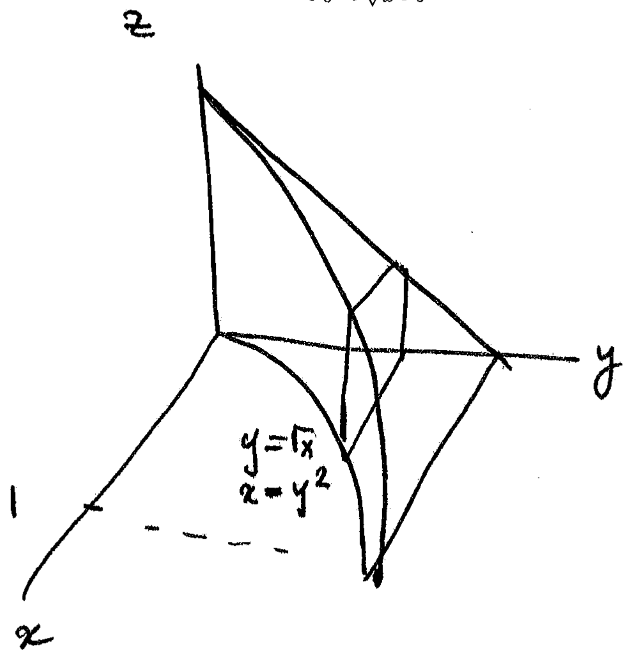
$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi\, d\phi \int_0^1 e^{\rho^3} \rho^2\, d\rho =$$

$$= \frac{\pi}{2} \left[-\cos\phi \right]_0^{\pi/2} \left. \frac{e^{\rho^3}}{3} \right|_0^1 =$$

$$= \frac{\pi}{2} \cdot 1 \cdot \left(\frac{e}{3} - \frac{1}{3} \right) = \boxed{\frac{\pi(e-1)}{6}}$$

Problem 3. Rewrite the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx \quad \text{as} \quad \int_?^? \int_?^? \int_?^? f(x,y,z)\,dx\,dz\,dy.$$



$$0 \le y \le 1$$

$$0 \le x \le y^2$$

$$0 \le z \le 1-y$$

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

Problem 4. Find a function f such that $\mathbf{F} = \nabla f$ and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + (x \cos y + \cos z)\,\mathbf{j} - y \sin z\,\mathbf{k}$ and C is parametrized by $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}$, $0 \le t \le \pi/2$.

$$\begin{cases} f_x = \sin y \\ f_y = x \cos y + \cos z \\ f_z = -y \sin z \end{cases}$$

$$f = x \sin y + g(y, z)$$

$$f_y = \cancel{x \cos y} + g_y(y, z) = \cancel{x \cos y} + \cos z$$

$$g_y(y, z) = \cos z$$

$$g(y, z) = y \cos z + h(z)$$

$$f = x \sin y + y \cos z + h(z)$$

$$f_z = \cancel{-y \sin z} + h'(z) = \cancel{-y \sin z}$$

$$h'(z) = 0$$

$$h(z) = 0 \quad (\text{or } h = k \text{ in general})$$

$$f(x, y, z) = x \sin y + y \cos z$$

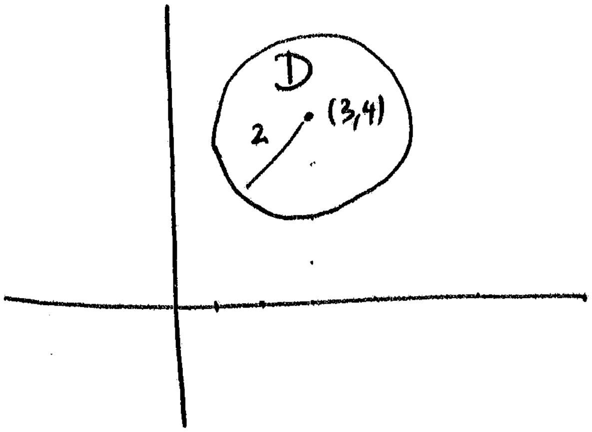
$$\vec{r}(0) = \langle 0, 0, 0 \rangle, \quad \vec{r}\left(\tfrac{\pi}{2}\right) = \left\langle 1, \tfrac{\pi}{2}, \pi \right\rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f\left(\vec{r}\left(\tfrac{\pi}{2}\right)\right) - f(\vec{r}(0))$$

$$= f\left(1, \tfrac{\pi}{2}, \pi\right) - f(0, 0, 0) = 1 \cdot \sin \tfrac{\pi}{2} + \tfrac{\pi}{2} \cos \pi - 0$$

$$= \boxed{1 - \frac{\pi}{2}}$$

Problem 5. Use Green's theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle y - \cos y, x\sin y\rangle$ and C is the circle $(x-3)^2 + (y+4)^2 = 4$ oriented **clockwise**.



clockwise = negative orientation so we have "−" in front of the integral

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= -\iint_D \frac{\partial}{\partial x}(x\sin y) - \frac{\partial}{\partial y}(y - \cos y)\, dA$$

$$= -\iint_D (\cancel{\sin y} - 1 - \cancel{\sin y})\, dA = \iint_D dA =$$

$$= \pi\cdot 2^2 = \boxed{4\pi}$$