Print your first and last name legibly above the line:

Solutions

Calculus III

Professor Piotr Hajłasz

Second Exam

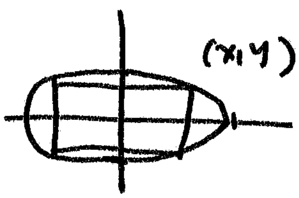
March 30, 2015.

Problem	Possible points	Score
1	20	
2	10	
3	10	
4	20	
5	10	
6	20	
7	10	
8	10	
Total	110	

You need 100 to have a perfect score. 10 additional points is a bonus.

Problem 1. (20p.) Using the method of Lagrange multipliers, find the area of the largest rectangle with pairs of sides parallel to the coordinate axes that can be inscribed in the ellipse $x^2 + 4y^2 = 1$. Also give the coordinates of the corner of the rectangle in the first quadrant.

If (x, y) is the vertex located in the first quadrant, the sides have lengths $2x$ and $2y$ and the area equals $f(x,y) = (2x)\cdot(2y) = 4xy$



Note that $x, y > 0$ and $x^2+4y^2=1$. Thus we need to find:

$$\begin{cases} \text{maximum of } f(x,y) = 4xy \\ \text{subject to } g(x,y) = x^2+4y^2=1 \end{cases}$$

We have
$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases}$$

i.e.
$$\begin{cases} 4y = \lambda \cdot 2x \\ 4x = \lambda \cdot 8y \\ x^2+4y^2=1 \end{cases}$$

$\lambda = \frac{4y}{2x} = \frac{2y}{x}$, $4x = \frac{2y}{x}\cdot 8y$, $4x^2 = 16y^2$,

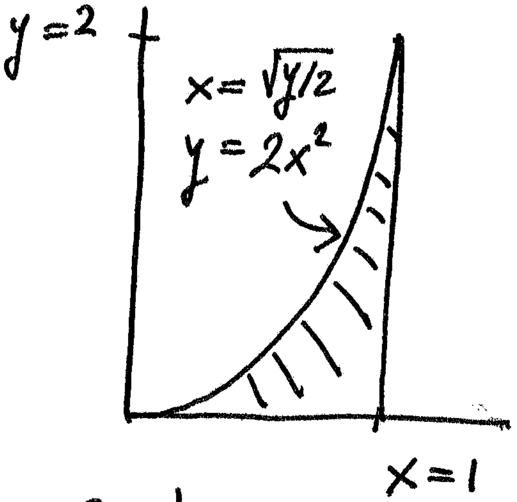
$2x = 4y$, $x = 2y$, $(2y)^2+4y^2=1$, $8y^2=1$,

$y = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$, $x = 2\cdot\frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$

$f(x,y) = 4xy = 4\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2\sqrt{2}} = \boxed{1}$

Vertex $(x,y) = \boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)}$

Problem 2. (10p.) Evaluate the integral $I = \int_0^2 \int_{\sqrt{y/2}}^1 y \exp(x^5)\, dx\, dy.$



$$\int_0^2 \int_{\sqrt{y/2}}^1 y \exp(x^5)\, dx\, dy = \int_0^1 \int_0^{2x^2} y \exp(x^5)\, dy\, dx$$

$$= \int_0^1 \frac{y^2}{2} \exp(x^5) \Big|_{y=0}^{y=2x^2} dx = \int_0^1 \frac{(2x^2)^2}{2} \exp(x^5)\, dx$$

$$= \int_0^1 2x^4 \exp(x^5)\, dx = \frac{2}{5} \exp(x^5) \Big|_0^1 =$$

$$= \boxed{\frac{2}{5}(e-1)}$$

Exercise 3. (10p.) Evaluate the integral

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\, dy\, dx$$

by converting to polar coordinates. (In order to compute the integral it might be useful to note that $(-(\cos\theta)^5/5)' = (\cos\theta)^4 \sin\theta$.)

We have to find out over what domain we integrate

$0 \le x \le 2, \quad 0 \le y \le \sqrt{2x - x^2}$

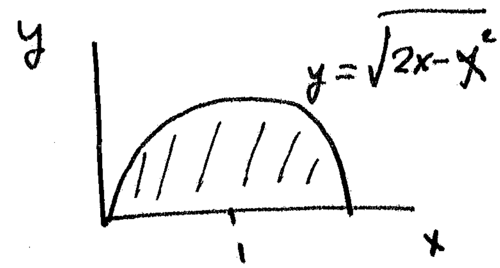
$y = \sqrt{2x - x^2}$

$y^2 = 2x - x^2$

(*) $x^2 + y^2 = 2x$

$x^2 - 2x + 1 + y^2 = 1$

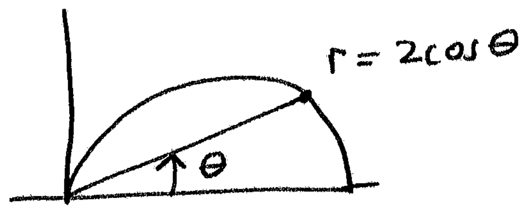
$(x-1)^2 + y^2 = 1$ — a circle



In polar coordinates (*) is

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$



$D = \{(r,\theta) \mid 0 \le \theta \le \pi/2, \quad 0 \le r \le 2\cos\theta\}$

Hence

$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \underbrace{r\sin\theta}_{y} \sqrt{\underbrace{r^2}_{x^2+y^2}}\, \underbrace{r\,dr\,d\theta}_{dA} =$$

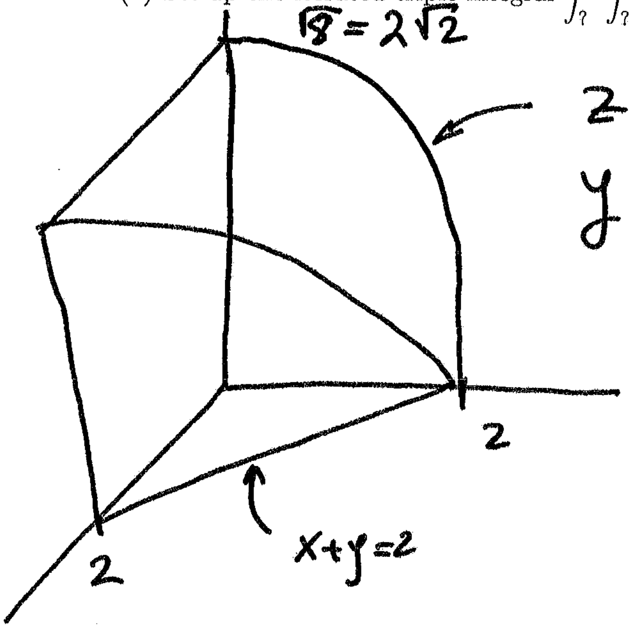
$$\int_0^{\pi/2}\int_0^{2\cos\theta} \sin\theta\, r^3\, dr\, d\theta = \int_0^{\pi/2} \sin\theta \left.\frac{r^4}{4}\right|_{r=0}^{r=2\cos\theta} d\theta =$$

$$\int_0^{\pi/2} \sin\theta \frac{(2\cos\theta)^4}{4}\, d\theta = 4\int_0^{\pi/2} \sin\theta(\cos\theta)^4\, d\theta =$$

$$4\left[-\frac{(\cos\theta)^5}{5}\right]_0^{\pi/2} = \boxed{\frac{4}{5}}$$

Problem 4. (10p=5p+5p). Let E be the region in the first octant bounded by the surfaces $2y^2 + z^2 = 8$ and $x + y = 2$, and let $f(x, y, z)$ be a function whose domain contains E. Denote $I = \iiint_E f(x, y, z), dV$.

(a) Set up the iterated triple integral $\int_?^? \int_?^? \int_?^? f(x, y, z)\, dz\, dx\, dy$.



$$z = \sqrt{8-2y^2}$$

$$y = \sqrt{\frac{8-z^2}{2}}$$

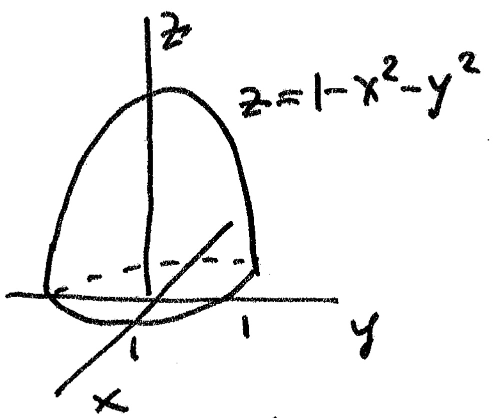
$$\int_0^2 \int_0^{2-y} \int_0^{\sqrt{8-2y^2}} f(x, y, z)\, dz\, dx\, dy$$

(b) Set up the iterated triple integral $\int_?^? \int_?^? \int_?^? f(x, y, z)\, dx\, dy\, dz$

$$\int_0^{\sqrt{8}} \int_0^{\sqrt{\frac{8-z^2}{2}}} \int_0^{2-y} f(x, y, z)\, dx\, dy\, dz$$

Problem 5 (10p.) Find the volume of the solid bounded by the plane $z = 0$ and the surface $z = 1 - x^2 - y^2$.

The surface intersects with the plane $z = 0$ along the circle $0 = 1 - x^2 - y^2$, i.e. $x^2 + y^2 = 1$. Thus we have to find the volume under the graph of $z = 1 - x^2 - y^2$ over the disc $D = \{(x,y) \mid x^2 + y^2 \le 1\}$. In other words we need to evaluate the integral



$$I = \iint_D 1 - x^2 - y^2 \, dx\, dy$$

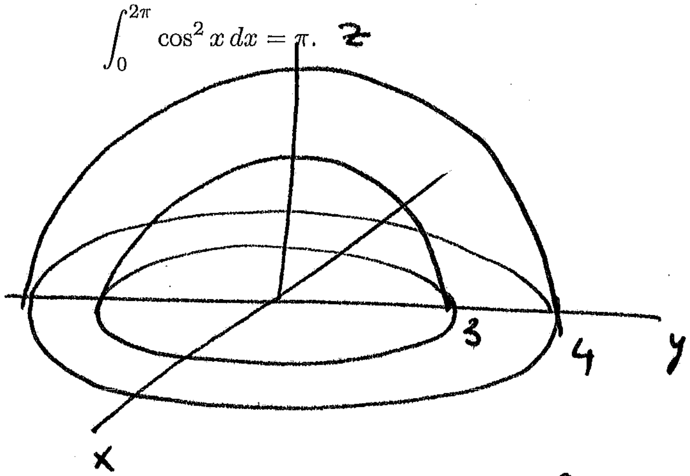
Using polar coordinates we get

$$I = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 r - r^3 dr$$

$$= 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Bigg|_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \boxed{\frac{\pi}{2}}.$$

Problem 6 (20p.) Evaluate the integral $\iiint_E x^2\,dV$ where E is bounded by the xy-plane and the hemispheres $z = \sqrt{9 - x^2 - y^2}$, $z = \sqrt{16 - x^2 - y^2}$.

Hint: You can use without any explanations the following integrals: $\int_0^{\pi/2} \sin^3 x\,dx = 2/3$, $\int_0^{2\pi} \cos^2 x\,dx = \pi$.



E – shell between the hemispheres of radii 3 and 4.

In spherical coordinates

$$E = \{ (\rho, \theta, \phi) \mid 3 \le \rho \le 4,\ 0 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi \}$$

$$\iiint_E x^2\,dV = \int_0^{\pi/2} \int_0^{2\pi} \int_3^4 \underbrace{(\rho \sin\phi \cos\theta)^2}_{x^2} \underbrace{\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi}_{dV}$$

$$= \int_0^{\pi/2} \sin^3\phi\, d\phi \int_0^{2\pi} \cos^2\theta\, d\theta \int_3^4 \rho^4\, d\rho =$$

$$\frac{2}{3} \cdot \pi \cdot \left. \frac{\rho^5}{5} \right|_3^4 = \frac{2}{3} \cdot \pi \cdot \frac{1024 - 243}{5} = \boxed{\frac{1562\pi}{15}}$$

Problem 7 (10p=5p+5p.) Let $\mathbf{F} = \langle yz(2x+y), xz(x+2y), xy(x+y)\rangle$.
(a) Find a function $f(x,y,z)$ such that $\nabla f = \mathbf{F}$.

$$\begin{cases} f_x = yz(2x+y) \\ f_y = xz(x+2y) \\ f_z = xy(x+y) \end{cases} \quad \text{i.e.} \quad \begin{cases} f_x = 2xyz + y^2 z \\ f_y = x^2 z + 2xyz \\ f_z = x^2 y + xy^2 \end{cases}$$

$$f = \int 2xyz + y^2 z \, dx = x^2 yz + xy^2 z + C(y,z)$$

$$f_y = \cancel{x^2 z} + \cancel{2xyz} + C_y(y,z) = \cancel{x^2 z} + \cancel{2xyz}$$

2nd equation

$$C_y(y,z) = 0$$

$$C(y,z) = d(z)$$

$$f = x^2 yz + xy^2 z + d(z)$$

$$f_z = \cancel{x^2 y} + \cancel{xy^2} + d'(z) = \cancel{x^2 y} + \cancel{xy^2}$$

3rd equation

$$d'(z) = 0$$

$$d(z) = \text{const.}$$

We can take for example $d = 0$ so

$$\boxed{f(x,y,z) = x^2 yz + xy^2 z}$$

satisfies $\nabla f = \vec{F}$.

(b) (Continuation from the previous page.) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C has parametrization $\mathbf{r} = \langle 1 + t + \sin(\pi t), 1 + 2(1 + \sin(\pi t))t^2, 1 + 3t^2 \rangle$, $0 \leq t \leq 1$.

$$f(x, y, z) = x^2 y z + x y^2 z, \quad \nabla f = \vec{F}$$

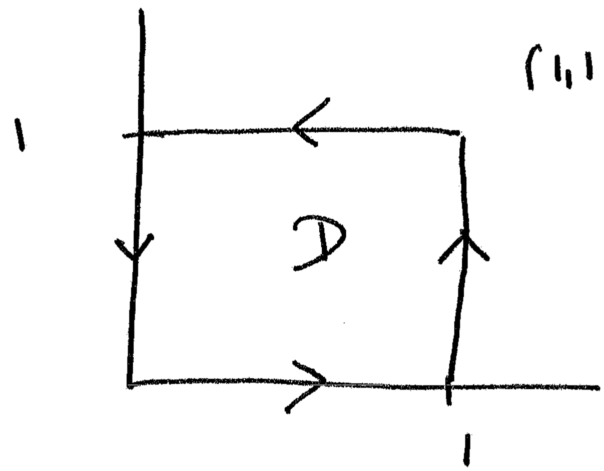
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = \heartsuit$$

$$\vec{r}(1) = \langle 2, 3, 4 \rangle, \quad \vec{r}(0) = \langle 1, 1, 1 \rangle$$

$$\heartsuit = f(2,3,4) - f(1,1,1) =$$

$$(2^2 \cdot 3 \cdot 4 + 2 \cdot 3^2 \cdot 4) - (1+1) = \boxed{118}$$

Problem 8 (10p.) Use Green's theorem to find the integral $\int_C (x^2 - y)\,dx + (x + y^2)\,dy$, where C is the perimeter of the unit square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$, with positive orientation.[1]



$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA =$$

$$\iint_D \frac{\partial(x+y^2)}{\partial x} - \frac{\partial(x^2-y)}{\partial y}\,dA = \iint_D 1-(-1)\,dA$$

$$= 2\iint_D dA = 2\cdot \text{Area}(D) = \boxed{2}$$

[1]No Green's theorem, no credit :(