Calculus 3 Quiz 8 (30 min) NAME:

Problem 1. *Show that if the vector field* $\mathbf{F} = \langle P, Q\rangle$ *is conservative, then necessarily* $P_y = Q_x$.

$\vec{F} = \nabla f, \quad P = f_x, \quad Q = f_y$

$P_y = f_{xy} = f_{yx} = Q_x$

$P_y = Q_x$

Problem 2. *Find a function* f *such that* $\mathbf{F} = \nabla f$ *and use it to evaluate the integral* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F} = (1+xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$ *and* C *is given by* $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le \pi/2$.

$$\begin{cases} f_x = (1+xy)e^{xy} \\ f_y = x^2 e^{xy} \end{cases}$$

From the second equation

$f = xe^{xy} + g(x)$

so

$f_x = e^{xy} + xye^{xy} + g'(x) = (1+xy)e^{xy}$

$g'(x) = 0, \quad g = K$

and we can take $K = 0$ so

$f = xe^{xy}$

$\vec{r}(\frac{\pi}{2}) = \langle 0,2\rangle, \quad \vec{r}(0) = \langle 1,0\rangle$

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(\vec{r}(\frac{\pi}{2})) - f(\vec{r}(0))$

$= f(0,2) - f(1,0) = 0 - 1 = \boxed{-1}$

Problem 3. *Evaluate* $\int_C xyz\,ds$, *where* C *is given the parametrization* $x = 2\sin t$, $y = t$, $z = -2\cos t$, $0 \le t \le \pi$. *(You need to use* $2\sin t\cos t = \sin(2t)$ *and integration by parts).*

$$\int_C xyz\,ds = \int_0^{\pi} x(t)\,y(t)\,z(t)\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$$

$$= \int_0^{\pi} 2\sin t \cdot t \cdot (-2\cos t)\sqrt{(2\cos t)^2+1^2+(2\sin t)^2}\,dt$$

$$= \int_0^{\pi} -4t\sin t\cos t\sqrt{4(\cos^2 t+\sin^2 t)+1}\,dt$$

$$= -2\sqrt{5}\int_0^{\pi} t\cdot 2\sin t\cos t\,dt$$

$$= -2\sqrt{5}\int_0^{\pi} t\sin(2t)\,dt$$

$$= -2\sqrt{5}\int_0^{\pi} t\left(-\frac{\cos(2t)}{2}\right)'dt$$

$$= -2\sqrt{5}\left[-\frac{t\cos(2t)}{2}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos(2t)}{2}\,dt\right]$$

$$= -2\sqrt{5}\left[-\frac{t\cos(2t)}{2}\Big|_0^{\pi} + \frac{\sin(2t)}{4}\Big|_0^{\pi}\right]$$

$$= -2\sqrt{5}\left[-\frac{\pi}{2} - 0 + 0 - 0\right] = \sqrt{5}\,\pi$$

Problem 4. *Find the work done by the force field* $\mathbf{F} = \langle x^2, ye^x \rangle$ *on a particle that moves along the parabola* $x = y^2 + 1$ *from* $(1, 0)$ *to* $(2, 1)$.

The parabola has a parametrization

$$\vec{r}(t) = \langle t^2+1, t \rangle, \qquad 0 \le t \le 1$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle (t^2+1)^2, t e^{t^2+1} \rangle \cdot \langle 2t, 1 \rangle \, dt$$

$$= \int_0^1 2t(t^2+1)^2 + t e^{t^2+1} \, dt$$

$$= \frac{1}{3}(t^2+1)^3 + \frac{1}{2} e^{t^2+1} \Big|_0^1$$

$$= \left(\frac{8}{3} + \frac{1}{2} e^2\right) - \left(\frac{1}{3} + \frac{1}{2} e\right)$$

$$= \frac{7}{3} + \frac{e^2}{2} - \frac{e}{2}$$