Geometric approach to Sobolev spaces and badly degenerated elliptic equations

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Introduction. In this note we will be concerned with the geometric properties of the functions from the Sobolev space

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \,|\, D^\alpha u \in L^p(\Omega), |\alpha| \leq m\}.$$

Here $\Omega \subset \mathbb{R}^n$ is an open set. In the most part of the paper we will be concerned with the case of the first order derivatives $m = 1$.

We will see that it is possible to find an equivalent characterization of the space $W^{1,p}(\Omega)$ which does not make use of the notion of derivative. This characterization leads to a definition of the Sobolev space, with the first order derivatives, on an arbitrary metric space equipped with a Borel measure. There are three particular interesting cases: Sobolev spaces on a fractal type sets, on graphs, and the Sobolev spaces with respect to the Carnot–Carathéodory metric.

The geometric point of view, which is presented here, seems to be important in the study of traces of Sobolev functions on fractal type sets, boundary behaviour of solutions to elliptic equations, the regularity theory for solutions to strongly degenerated elliptic equations, in the nonlinear potential theory, diffusions on fractal type sets, analysis on graphs and probably in the finite element method.

In the last section we will see how to extend the "pointwise inequalities" — the basic tool for the development of Sobolev spaces on metric spaces — to the case of higher order derivatives.

It was professor Bogdan Bojarski who pointed my attention on the importance of "pointwise inequalities".

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Most of the results are given here without proofs. Some of them announces forthcoming papers [41], [44], [45], [46], [11]. For the simplicity of presentation, very often, we deal with the most simple cases.

The paper is divided into four sections which concern slightly different aspects of the same geometric approach. For reader's convenience we tried to make these sections as independent as possible. This paper is addressed also to the people who are familiar with the basic concepts of the theory of Sobolev spaces, but who does not work in the field.

The structure of the paper is the following. In Section 1 we are concerned with the "pointwise inequalities" for Sobolev functions with the first order derivatives. We show some elementary applications. The main result stated this section is Theorem 3, which provides a characterization of the Sobolev space, without using the notion of derivative. This result is a starting point for the Section 2, where we develop the theory of Sobolev spaces on metric spaces, initiated in [41]. In Section 2 we remark relations to diffusions on fractals, to the analysis on infinite graphs, and to the theory of infinite resistive networks. Then we explain, following the recent results of the author and Koskela [44], [45], how the metric approach to Sobolev spaces applies to nonlinear potential theory and to strongly degenerated elliptic equations. The last part of Section 2 contains a survey of the theory of strongly degenerated elliptic equations related to Hörmander's vector fields. In Section 3, following the forthcoming paper of the author and Martio, [46], we remark the applications of the Sobolev spaces on metric spaces to the theory of traces on fractal type sets. Finally in Section 4 we show, following the papers of the author and Bojarski [9] and the author, Bojarski and Strzelecki [11], how to extend the pointwise inequalities to the higher order derivatives. We also show some applications of these higher order inequalities.

We will very often consider bounded domains with the sufficiently regular boundary, without specifying what we mean by sufficient regularity. Boundary which is locally a graph of a Lipschitz function is sufficiently regular in *all* our considerations, however in most of the cases the results apply to much more irregular domains. Since we do not want to deal with the problems of the regularity of the boundary, we leave the assumptions indefinite.

We will frequently write $u_K = \fint_K u\, d\mu = \mu(K)^{-1}\int_K u\, d\mu$ to denote the average of u over K. By $u \approx v$ we mean there exists two positive constants c_1, c_2 such that $c_1 u \leq v \leq c_2 u$. If $A \subset \mathbb{R}^n$ is a measurable subset, then $|A|$ denotes its Lebesgue measure. Symbol B is reserved for a ball, and B_R, $B(R)$ for a ball with the radius R. By kB we will denote a ball concentric with B and with the radius k times that of B. Finally by C we denote a general constant which can change from line to line.

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1. Pointwise inequalities.

The above mentioned characterization of the Sobolev space (Theorem 3 below) is based on the following elementary inequality. If $u \in W^{1,p}_{\rm loc}(\mathbb{R}^n)$ and $0 \leq \lambda < 1$ then the inequality

$$|u(x) - u(y)| \leq C|x-y|^{1-\lambda}\left(M^\lambda_{|x-y|}|\nabla u|(x) + M^\lambda_{|x-y|}|\nabla u|(y)\right) \qquad (1)$$

holds for almost all $x \neq y$. Here $M^\lambda_R g(x) = \sup_{r<R} r^\lambda \fint_{B(x,r)} |g(z)|\, dz$ denotes the fractional maximal function. We put the assumption $x \neq y$ in (1) only because the expression $M^\lambda_0 g$

is not defined. Proof of (1) is elementary and is given in [42]. This inequality generalizes the inequality given in [8] (the case $\lambda = 0$). Various versions of (1), with $\lambda = 0$, appear also in [40], [50], [61].

Obviously (1) generalizes also to $u \in W^{1,p}(\Omega)$, where Ω is a bounded domain with the sufficiently regular boundary. In particular (1) implies

$$|u(x) - u(y)| \le C|x-y|^{1-\lambda}\left(M^{\lambda}_{\operatorname{diam}\Omega}|\nabla(Eu)|(x) + M^{\lambda}_{\operatorname{diam}\Omega}|\nabla(Eu)|(y)\right), \tag{2}$$

almost everywhere in Ω, where $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ is a bounded extension operator (i.e., $Eu|_{\Omega} = u$), and $\operatorname{diam}\Omega$ denotes the diameter od Ω. Sufficient regularity of $\partial\Omega$ guarantees existence of E.

In the last section we will show following [9], [11] how to generalize (1) to the higher order derivatives.

Before we state the above mentioned characterization, we show some direct applications of (2).

First we note that one can regard (2) as a refined form of Morrey's lemma. Namely, as a direct consequence of (2), we obtain the following corollary.

Corollary 1 (Morrey's lemma [78] [79], [80]). *Let $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ and $1 \le p < \infty$. Suppose that for some constants μ, M, $0 < \mu \le 1$, $M > 0$,*

$$\int_{B(x,R)} |\nabla u|^p \le M^p R^{n-p+p\mu},$$

holds whenever $B(x,R) \subset \Omega$. Then $u \in C^{0,\mu}_{\mathrm{loc}}(\Omega)$, and in each ball $B(x,R)$ such that $B(x,3R) \subset \Omega$ the estimate for the oscillation

$$\operatorname*{osc}_{B(x,r)} u = \sup_{y,z\in B(x,r)} |u(y) - u(z)| \le CMR^{\mu}$$

holds with a constant C which depends on n, p and μ only.

Indeed, the hypothesis of the corollary implies that the suitable fractional maximal function with $\lambda = 1 - \mu$ is finite.

Sobolev function $u \in W^{1,p}(\Omega)$ is defined except the set of measure zero. On the other hand it is possible to restrict u to a $(n-1)$-dimensional subspace (trace theorem). It suggests that one can prescribe values of u, in a reasonable way, outside the set of Hausdorff measure H^{n-1} zero, or even more precisely. This is well known fact. It seems that the first comprehensive treatment of this problem was made by Fuglede [36]. There are also many other approaches, different from that of Fuglede. However most of these methods are rather technical and involve the notion of capacity or extremal length. Current sources to the topic are e.g., [104], [28], [49].

We will show one tricky and elementary argument which allows us to avoid the use of capacity in many important cases. We will follow [40], [9], [42].

If $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, then we define values of $\tilde{u}$ *everywhere* by the formula

$$\tilde{u}(x) = \limsup_{r\to 0} ⨍_{B(x,r)} u(z)\,dz. \tag{3}$$

By the Lebesgue differentiation theorem (see [93, Chapter 1]) integral averages converge to $u(x)$, as $r \to 0$, for almost all x, and hence $\tilde{u}$ is a Borel measurable representative of u (u is an equivalence class of functions which differ on a set of measure zero). Function $\tilde{u}$ is a natural representative of u, however one should be very careful since, for example, it may happen that for some x, $\tilde{u}(x) \neq -(\widetilde{-u})(x)$. In what follows we will very often identify $\tilde{u}$ with u and omit tilde sign, however we will always point it out.

Now one can prove [40], [9], [42], that such representative of u satisfies the inequalities (1) and (2) for *all* $x \neq y$ (maybe with the slightly worse constant C). It may happen that the left hand side of (1) or (2) is of the indefinite form like $|\infty - \infty|$, then we adopt the convention $|\infty - \infty| = \infty$. In such a case the inequalities are still valid since $\tilde{u}(z) = \pm\infty$ implies $M_R^\lambda |\nabla u|(z) = \infty$ for any $R > 0$. Now we show an application.

Assume for a moment that $u \in W^{1,p}(\mathbb{R}^n)$ has compact support with diameter less than 1/2. Than it easily follows from (1) that

$$|u(x) - u(y)| \leq C|x-y|^{1-\lambda} \left(M_1^\lambda |\nabla u|(x) + M_1^\lambda |\nabla u|(y) \right), \tag{4}$$

for *all* $x, y \in \mathbb{R}^n$, $x \neq y$. Let $E_{\lambda,t} = \{x \in \mathbb{R}^n \,|\, M_1^\lambda |\nabla u|(x) \leq t\}$. Obviously $u|_{E_{\lambda,t}}$ is $C^{0,1-\lambda}$ Hölder continuous with the constant $2tC$. Note, we used that (4) holds *everywhere.*

A Lipschitz function defined on an arbitrary subset of an arbitrary metric space can be extended to the entire space with the same Lipschitz constant (see [91, Theorem 5.1], [31, 2.10.4]). Hence such result holds also for the extensions of $C^{0,\mu}$–Hölder continuous functions (indeed, $C^{0,\mu}$ function is Lipschitz with respect to a new metric $d'(x,y) = d(x,y)^\mu$). This implies that there exists a $C^{0,1-\lambda}$ function $u_{\lambda,t}$ defined on the entire $\mathbb{R}^n$ such that $u_{\lambda,t}|_{E_{\lambda,t}} = u|_{E_{\lambda,t}}$.

For a set $A \subset \mathbb{R}^n$ we define $H_\infty^d(A) = \inf \sum r_i^d$, where infimum is taken over all coverings of A by balls B_i with radius r_i. The number $H_\infty^d(A)$ is sometimes called Hausdorff d–content. By H^d we will denote the standard Hausdorff d-dimensional measure. Contrary to the properties of the measure H^d, content H_∞^d is finite on *all* bounded sets, however $H^d(A) = 0$ if and only if $H_\infty^d(A) = 0$.

Standard application of Vitali type covering lemma (see [49, Lemma 2.29], see also [93, Chapter 1]) leads to the following weak type estimate.

Lemma 1 $H_\infty^{n-\lambda p}(\mathbb{R}^n \setminus E_{\lambda,t}) \leq Ct^{-p} \int_{\mathbb{R}^n} |\nabla u(z)|^p \, dz.$

If $\lambda \to 1$ and $t \to \infty$, then $\mathbb{R}^n \setminus E_{\lambda,t}$ is a decreasing sequence of open sets and it follows from the lemma that the intersection of these sets has Hausdorff dimension less than or equal to $n - p$.

Using the partition of unity this argument can be applied to $u \in W_{\rm loc}^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an arbitrary open set. This leads to the following theorem.

Theorem 1 *If $u \in W_{\rm loc}^{1,p}(\Omega)$ is defined everywhere by the formula (3), then there exists a sequence of compact sets $X_1 \subset X_2 \subset \ldots \subset X_k \subset \ldots \subset \Omega$ and a sequence of Hölder continuous functions u_k on Ω such that $u_k|_{X_k} = u|_{X_k}$ and $\Omega \setminus \bigcup_k X_k$ has the Hausdorff dimension less than or equal to $n - p$.*

This theorem shows that it is reasonable to talk about values of $u \in W^{1,p}(\Omega)$ except the set of dimension $n - p$. Hence if $p > 1$, we can define a trace of the Sobolev function on a

$(n-1)$-dimensional submanifold of Ω just as a restriction. If we want to define the trace on the boundary of Ω (provided it is sufficiently regular), we first extend the Sobolev function to $W^{1,p}(\mathbb{R}^n)$ and then we proceed as above. In the case $p = 1$ one also can define a trace, but the above approach does not cover that case. We will come back to the problem of traces in Section 3, where we will be concerned with the traces on fractal type subsets.

More sophisticated version of Theorem 1 was proven by Malý [71] (see [74], [72] for applications). For the case of higher order derivatives see [11] and Section 4.

In general, the Hölder exponent of the functions u_k in Theorem 1 has to go to 0 as $k \to \infty$. If we make the assumption that all the functions u_k have a fixed Hölder exponent $1 - \lambda$, then by the same argument as above we get $H^{n-\lambda p}(\Omega \setminus \bigcup_k X_k) = 0$, which is more than to say that the Hausdorff dimension is less than or equal to $n - \lambda p$. In particular we get the classical imbedding into Hölder continuous functions: if $p > n$ and $\lambda = n/p$, then $n - \lambda p = 0$ and hence $u \in C^{0,1-n/p}_{\mathrm{loc}}(\Omega)$ (cf. Corollary 1).

On the other hand if $\lambda = 0$ and $1 \le p < \infty$, we get that the Sobolev function coincide with a Lipschitz function outside a set of an arbitrary small measure. Lipschitz functions belong also to the Sobolev spaces $W^{1,p}$ for all p. Careful study of the $W^{1,p}$ norm of these Lipschitz functions shows that they approximate our Sobolev function in the Sobolev norm. Hence we can approximate given Sobolev function in the norm by a Lipschitz functions which coincide with our Sobolev function outside the set of an arbitrary small measure. It appears that the same phenomen holds in the general setting of Sobolev spaces on metric spaces (these spaces will be introduced in the next section). Theorem 5 covers both, Euclidean and metric case.

It seems that in the case of classical Sobolev spaces the property 1. of Theorem 5 (with μ being the Lebesgue measure) was known (even in the slightly stronger form) since the paper by Whitney [100]. By an analogy with the Lusin's theorem, this property is called a Lusin type property of a Sobolev function. In the classical setting, Calderón and Zygmund [18, Theorem 13] extended part 1. of Theorem 5 to the case of $W^{m,p}$ functions, with $h \in C^m(\Omega)$. Liu [66] extended the result of Calderón and Zygmund and proved that both 1. and 2. hold for suitable $h \in C^m$, with the approximation in 2. taken with respect to the $W^{m,p}$ norm. We will come back to this type results for higher order derivatives in Section 4 and we will discuss then some further generalizations.

Theorem 5 shows that Sobolev functions inherit many properties of Lipschitz functions. Hence it is not surprising that this result (in the setting of classical Sobolev spaces) found many applications in P.D.E. and Calculus of Variations [1], [20], [65], [63], [38], [70], [50], [43], [101], [61].

Also one can use the above Lusin type property to extend change of variables formula to the case when the change is made by a Sobolev mapping $u : \Omega_1 \to \Omega_2$ (i.e., components of $u = (u_1, \ldots, u_n)$ belong to the Sobolev space $u_i \in W^{1,p}(\Omega_1)$) [40], [96], however change of variables formula of that type seems to be known for a long time. Also as shown in the author's paper [43], Federer's *co–area* formula [30], [31, Theorem 3.2.12], [17] (the generalization of the change of variables formula) extends to the case of Sobolev mappings. This extension is useful in the Calculus of Variations, (see [43]).

Now as an application of Theorem 5 (still in the setting of classical Sobolev spaces) we show how to extend Brouwer fixed point theorem to the case of discontinuous Sobolev mappings. It is known that Brouwer theorem is equivalent with the nonexistence of continuous retraction from the ball onto its boundary. We will state our theorem in that form.

Theorem 2 *For* $u \in W^{1,n}(B^n, B^n)$, $u|_{\partial B^n} = \mathrm{id}$, *there is* $|B^n \setminus u(B^n)| = 0$.

Of course $u \in W^{1,n}(B^n, B^n)$ means $u = (u_1, \ldots, u_n)$, $u_i \in W^{1,n}(B^n)$ for $i = 1, 2, \ldots, n$ and $|u| \leq 1$. The restriction of u to the boundary is understood as a trace of a Sobolev function. More sophisticated results, related to the degree theory for traces of Sobolev mappings were proven (using different methods) by Bethuel [2]. Results of this type are important in the Ginzburg–Landau theory [3], and in the nonlinear elasticity [96], [82].

Proof. By Theorem 5, there exists a sequence $u_k \in \mathrm{Lip}\,(B^n, B^n)$, such that $u_k \to u$ in $W^{1,n}$ and $|\{u_k \neq u\}| \to 0$ as $k \to \infty$. It is not difficult to see that we can assume in addition that $u_k|_{\partial B^n} = \mathrm{id}$ (restriction in the classical sense). According to Brouwer's theorem $u_k(B^n) = B^n$, and hence there exists $E_k \subset B^n$ with $u_k(E_k) = B^n \setminus u(B^n)$. Obviously $E_k \subset \{u_k \neq u\}$, and hence $|E_k| \to 0$, as $k \to \infty$. Now we have

$$|B^n \setminus u(B^n)| = |u_k(E_k)| \leq \int_{E_k} |\det Du_k| \to 0,$$

since $\det Du_k \to \det Du$ in L^1 and $|E_k| \to 0$. We used here the change of variables formula for Lipschitz mappings, which implies the inequality $|u_k(E_k)| \leq \int_{E_k} |\det Du_k|$ (cf. [31, Theorem 3.2.3], [17], see also [12, Theorem 8.3], [40]).

As a last topic of this section we will give a characterization of the Sobolev space, which does not make use of the notion of derivative.

If $1 < p \leq \infty$, then the Theorem of Hardy, Littlewood and Wiener (see [93, Chapter 1], [97], [104, Theorem 2.8.2]) states that the maximal operator $Mu(x) = \sup_{r>0} ⨍_{B(x,r)} |u(z)|\, dz$ is bounded in $L^p(\mathbb{R}^n)$ i.e., $||Mu||_p \leq C||u||_p$. Obviously $||u||_p \leq ||Mu||_p$. This in connection with (2) for $\lambda = 0$, implies that to every $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently regular boundary, and $1 < p \leq \infty$, there exists $g \in L^p(\Omega)$, $g \geq 0$, such that

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)), \tag{5}$$

almost everywhere in Ω. It appears that this inequality characterizes $W^{1,p}(\Omega)$. Namely we have the following result.

Theorem 3 (Hajłasz [41]). *Let* Ω *be a bounded domain with the sufficiently regular boundary and* $1 < p \leq \infty$. *Then* $u \in W^{1,p}(\Omega)$ *if and only if there exists* $g \in L^p(\Omega)$, $g \geq 0$ *such that (5) holds. Moreover*

$$||\nabla u||_{L^p(\Omega)} \approx \inf_g ||g||_{L^p(\Omega)},$$

where the infimum is taken over all functions g which satisfy (5).

In the case $p = \infty$ we recover a classical result which states that $W^{1,\infty}(\Omega) = \mathrm{Lip}\,(\Omega)$. Hence it is natural to call the above theorem a Lipschitz type characterization of Sobolev functions.

We have already explained the implication $\Rightarrow$. Now we will give a new proof of the converse implication $\Leftarrow$. The following proposition and the example are the outcomes of my discussions with Jan Malý.

Proposition 1 *If* Ω *is as above, and* u *satisfies (5), with* $g \in L^p(\Omega)$, $g \geq 0$, *where* $1 \leq p \leq \infty$, *then* $u \in W^{1,p}(\Omega)$ *and* $|\nabla u(x)| \leq 4\sqrt{n} g(x)$ *a.e.*

Of course it suffices to consider the case $p = 1$, but before we will do it we answer a natural question which arise now. It is natural to ask whether for $u \in W^{1,1}(\Omega)$ there exists $g \in L^1(\Omega)$ such that (5) holds. The answer is negative, as the following example shows.
EXAMPLE. Let $\Omega = (-1/2, 1/2)$, and $u(x) = -x/(|x| \log |x|)$. Hence $u \in W^{1,1}(\Omega)$, since $u'(x) = |x|^{-1}(\log |x|)^{-2} \in L^1(-1/2, 1/2)$. Suppose that there exists $g \in L^1(-1/2, 1/2)$, such that (5) holds. For $0 < x < 1/2$, we have $|u(x) - u(-x)| \leq 2x(g(x) + g(-x))$ and hence

$$\frac{-2}{\log x} \leq 2x(g(x) + g(-x)),$$

thus

$$\int_{-1/2}^{1/2} g(x)\, dx = \int_0^{1/2} (g(x) + g(-x))\, dx \geq \int_0^{1/2} \frac{-dx}{x \log x} = \infty.$$

This contradicts the summability of g.

Proposition 1, together with the example, show that the set of all functions u satisfying (5) with certain nonnegative $g \in L^1$, forms a strict subspace of $W^{1,1}(\Omega)$ (one can expect that such functions locally belong to the Hardy space (cf. [23], [89]), however as was proven by Jan Malý, [73], this is not true).

Proof of Proposition 1. In the proof we need so called ACL characterization of the Sobolev space. We say that $u \in \mathrm{ACL}\,(\Omega)$ if the function u is Borel measurable and absolutely continuous on almost all lines parallel to coordinate axes. Since absolutely continuous functions are differentiable almost everywhere, $u \in \mathrm{ACL}\,(\Omega)$ has partial derivatives a.e., and hence the gradient ∇u is defined a.e. Now we say that $u \in \mathrm{ACL}^p\,(\Omega)$ if $u \in L^p(\Omega) \cap \mathrm{ACL}\,(\Omega)$ and $|\nabla u| \in L^p$. The following characterization of the Sobolev space is due to Nikodym.

Theorem 4 (Nikodym [84], [64, Theorems 5.6.2–3], [76, Section 1.1.3]). $W^{1,p}(\Omega) = \mathrm{ACL}^p\,(\Omega)$.

Since maybe it is not evident how to understand this theorem, we shall comment it now. This theorem states that each $\mathrm{ACL}^p\,(\Omega)$ function belongs to $W^{1,p}(\Omega)$ and the gradient ∇u, which is defined a.e. for $u \in \mathrm{ACL}^p\,(\Omega)$ is just the distributional gradient. On the other hand, each element $u \in W^{1,p}(\Omega)$ (which is an equivalence class of functions equal except the set of measure zero) admit a Borel representative, which belongs to the space $\mathrm{ACL}^p\,(\Omega)$.

According to the ACL characterization it suffices to consider the case $\Omega = (0, 1) \subset \mathbb{R}$ and prove that u is absolute continuous with $|u'| \leq 4g$. It would follow if we proved $|u(a) - u(b)| \leq 4 \int_a^b g(x)\, dx$ for almost all $a, b \in (0, 1)$, $a < b$. Fix such a and b with the additional condition $g(a), g(b) < \infty$. Divide $[a, b]$ into n segments I_i, $i = 1, \ldots, n$, each of the length $(b - a)/n$. To every i there exists $x_i \in I_i$ with $g(x_i) \leq ⨍_{I_i} g$. Let $x_0 = a$ and $x_{n+1} = b$. We have

$$|u(a) - u(b)| \;\; \leq \;\; \sum_{i=0}^{n} |u(x_i) - u(x_{i+1})| \leq \frac{2(b-a)}{n} \sum_{i=0}^{n} (g(x_i) + g(x_{i+1}))$$

$$\leq \frac{4(b-a)}{n}\sum_{i=1}^{n} \fint_{I_i} g + \frac{2(b-a)}{n}(g(a)+g(b)) \longrightarrow 4\int_a^b g,$$

which completes the proof.

2. Sobolev spaces on metric spaces. The above characterization (Theorem 3) hints the way how one should define the Sobolev space on an arbitrary metric space.

DEFINITION (Hajłasz [41]). Let (X,d,μ) be a metric space (X,d), μ a Borel measure, finite on bounded sets, and $1 \leq p \leq \infty$. We define the Sobolev space on the triple (X,d,μ) as follows

$$\begin{aligned} W^{1,p}(X,d,\mu) \quad = \quad & \{u \in L^p(X,\mu) \mid \exists E \subset X,\ \mu(E)=0, \text{ and } \exists g \in L^p(X,\mu),\ g \geq 0, \\ & \text{such that } |u(x)-u(y)| \leq d(x,y)(g(x)+g(y)), \\ & \text{for all } x,\, y \in X \setminus E\}. \end{aligned}$$

Moreover we set $||u||_{W^{1,p}} = ||u||_{L^p} + ||u||_{L^{1,p}}$, where $||u||_{L^{1,p}} = \inf_g ||g||_{L^p}$. If $Y \subset X$ is a measurable subset of a positive measure then by $||u||_{W^{1,p}(Y)}$ we denote the norm of $u|_Y \in W^{1,p}(Y,d,\mu)$. For our convenience we will often call the above definition a metric definition of the Sobolev space.

If $X = \Omega = \mathbb{R}^n$ or $X = \Omega \subset \mathbb{R}^n$ is a bounded domain with the sufficiently regular boundary, d is the Euclidean metric, μ is the Lebesgue measure, and $1 < p \leq \infty$, then the space $W^{1,p}(X,d,\mu)$ is equivalent with the classical Sobolev space $W^{1,p}(\Omega)$. Moreover $||u||_{L^{1,p}}$ is equivalent with $||\nabla u||_p$. In the case $p=1$ the metric and the classical definitions *are not* equivalent (see the example after Proposition 1). However for some reasons it is natural to consider metric definition in the case $p=1$ as well. We will see later how to define a Sobolev type space, in a reasonable way, for $0 < p < 1$.

It is not difficult to prove (see [41]) that $W^{1,p}(X,d,\mu)$ is a Banach space. Moreover if $Y \subset X$ is a subset of positive measure and finite diameter, then using pointwise inequality from the definition and Hölder's inequality, we get a version of the Poincaré inequality

$$||u-u_Y||_{L^p(Y)} \leq 2(\operatorname{diam} Y)||u||_{L^{1,p}(Y)}. \tag{6}$$

The following theorem is the already mentioned Lusin type approximation.

Theorem 5 (Hajłasz [41]). *If $u \in W^{1,p}(X,d,\mu)$, where $1 \leq p \leq \infty$, then to every $\varepsilon > 0$ there exists a Lipschitz function h such that*

1. $\mu(\{x \,|\, u(x) \neq h(x)\}) < \varepsilon$,
2. $\|u-h\|_{W^{1,p}} < \varepsilon$.

It is natural to ask whether the Sobolev imbedding theorems extend to the metric case. The problem is that in the imbedding theorems the dimension of the space plays a crucial role of critical exponent. This shows that we have to add a condition which will be a counterpart of the dimension in the metric setting. The condition is the following.

DEFINITION. Let (X,d,μ) be as above. We say that the measure μ restricted to a measurable subset $Y \subset X$ is *s-regular* $(s>0)$ if there exists a constant $b>0$ such that for all $x \in Y$ and all $r \leq \operatorname{diam} Y$

$$\mu(B(x,r)\cap Y) \geq br^s.$$

We do not require $\operatorname{diam} Y < \infty$. Of course $\mathbb{R}^n$ and any bounded domain in $\mathbb{R}^n$ with sufficiently regular boundary are n–regular with respect to the Lebesgue measure. Also many natural fractal sets are s–regular. For example the standard ternary Cantor set is $\log_3 2$-regular with respect to the natural Hausdorff measure. Many important domains and their boundaries, which are useful in the theory of Sobolev spaces, are regular with respect to the Hausdorff measure (see [59] for details). Consult also [75] for related results. Later we will see that measures from a large class of the so called doubling measures are s–regular for a suitable s, which depend on a measure.

On certain fractal type subsets of $\mathbb{R}^n$, Jonsson and Wallin, [59], defined Hardy and Besov type spaces. Their resluts are somehow related to that of ours, but involve different methods and work in the Euclidean setting only. Quite recently Han and Sawyer [48], and in a more general form Han [47], defined Besov and Triebel–Lizorkin spaces on quasi-metric spaces equipped with a doubling measure. One should also mention a recent work of Semmes [90], which seems to be strongly related to the approach presented in this section.

Now we state the imbedding theorem.

Theorem 6 (Hajłasz [41]). *Let $u \in W^{1,p}(X, d, \mu)$, where $1 \leq p \leq \infty$. Assume that μ restricted to a measurable subset $Y \subset X$ with $\operatorname{diam} Y < \infty$ is s–regular.*

1. *If $p < s$, then $u \in L^{p^*}(Y, \mu)$, where $p^* = \frac{sp}{s-p}$ and*

$$\|u - u_Y\|_{L^{p^*}(Y)} \leq C\|u\|_{L^{1,p}(Y)}.$$

2. *If $p = s$, then there exist constants C_1 and C_2 such that*

$$⨍_Y \exp\left(C_1 \frac{\mu(Y)^{1/s}}{\operatorname{diam} Y} \frac{|u - u_Y|}{\|u\|_{L^{1,s}(Y)}}\right) d\mu \;\leq\; C_2.$$

3. *If $p > s$, then u is bounded on Y and*

$$|u(x) - u(y)| \leq C\mu(Y)^{\frac{1}{s}-\frac{1}{p}}\|u\|_{L^{1,p}(Y)} \ a.e.$$

Here the constants C, C_1, C_2 depend on p, s and b only.

Remarks. 1) generalizes the classical Sobolev imbedding theorem. In the Euclidean case 2) is just the inequality of John and Nirenberg (see [57], [97]) applied to $W^{1,n}(Q^n) \subset \mathrm{BMO}$. Also in the Euclidean case inequality 3) leads to Hölder continuity of u. Later we will see that we also have Hölder continuity in the metric setting.

Assume that the measure μ is s-regular on the entire space X, and let $B_R \subset X$ be a ball. If $x \in B_R$, and $r \leq R$, then by the hypothesis $\mu(B(x, r)) \geq br^s$, however in general it is not true that $\mu(B(x, r) \cap B_R) \geq Cr^s$ with a constant C which does not depend on x and r. That means in general a s-regular measure on X does not has to be s-regular when restricted to a certain ball B. In such a case the assumptions of the above theorem are not satisfied for $Y = B$, and hence one can doubt if the theorem extends to this case. Fortunately one can prove then a slightly weaker version of the above result. Detailed statement and the proof will be given in the forthcoming paper of the author and Koskela [45]. Now we give only a rough statement.

If the measure μ is s-regular on X and $u \in W^{1,p}(X,d,\mu)$, $1 \leq p < s$, then for any ball $B \subset X$ the following weak version of the Sobolev inequality

$$||u - u_B||_{L^{p^*}(B)} \leq C||u||_{L^{1,p}(2B)}$$

holds. If $p = s$ or $p > s$, then also "weak" counterparts of the other two cases of Theorem 6 hold. In particular if $p > s$, then this "weak" inequality leads to Hölder continuity of u, just as in the Euclidean case.

Sobolev type inequality with the norm on the right hand side in a bigger ball than that on the left hand side will be called weak Sobolev inequality. If the ball on both sides is the same, we will simply say Sobolev inequality or strong Sobolev inequality.

The above metric approach to Sobolev spaces applies when one deals with the analysis on "nonsmooth" spaces like fractals or graphs. There are many papers which deal with diffusions on fractal sets, see [62] and references therein. Given the "Brownian motion" on a fractal set one obtains the "Laplace" operator — the infinitesimal generator of "Brownian motion". It would be interesting to explain in details how the Sobolev spaces on fractal sets apply in this setting. This is also strongly related to the analysis on graphs (roughly speaking Brownian motion on a fractal set can be defined as a certain limit of random walks on graphs which approximate given fractal).

Given a graph and a function u defined on its vertices, one can define Dirichlet norm of u (a counterpart of $||\nabla u||_2$), replacing differentiation by a differences and the integration by summation. Also the Laplace opeartor can be defined on a graph via the mean value property of a harmonic function, see [98], [92] and references therein. This approach found, in particular, applications to infinite electrical networks (Soardi [92] and references therein). Namely it is easy to show that, when the total power available in the infinite resistive network, is finite, the solutions to Kirchoff's equations (which define electric current) are unique if and only if the only harmonic function with finite Dirichlet norm is constant equal to zero. It also applies to random walks on graphs, and to the theory of finite generated groups (such a group can be represented as a graph), see [98].

In such setting it is important to have the imbedding theorem on a graph. There are plenty of papers dealing with this problem. For Varopoulos' contribution see [98, Chapter 6] and references therein, see also papers of Coulhon [25] and Coulhon and Saloff-Coste [26] and the paper of Saloff-Coste [88]. In fact the imbedding theorems are usually reduced to the verification of a certain discrete version of the isoperimetric inequality. Since the graph is a metric space (metric is a length of a shortest path), equipped with a counting measure, Theorem 6 provides another approach. Details will be given in the forthcoming paper. The s-regularity condition is also natural in this setting, in fact condition of this type is required in the above mentioned papers dealing with the imbedding theorems on graphs.

Now let's come back to the metric theory of Sobolev spaces. The *classical* Poincaré and Sobolev inequalities do not hold for $0 < p < 1$. An elementary counter example, in the setting of the spaces on an interval, in provided in [15]. However if one carefully study the proof of Theorem 6 and its "weak" generalization, it is clear that the assumption $p \geq 1$ is used only in the one step. It is possible to avoid this step, thus obtaining the imbedding (in the metric setting), with the slightly modified statement, even for $0 < p < 1$. This approach to Sobolev inequalities for $p > 0$ is presented in details in the forthcoming paper of the author and Koskela [45]. Motivation for consideration of the Sobolev inequalities in the range $0 < p < 1$ comes from the paper by Buckley and Koskela [15] where the authors prove Sobolev inequality for u for $0 < p < 1$ provided u is a solution to a suitable elliptic equation (or more generally: provided $|\nabla u|$ satisfies weak reverse Hölder inequality). The result of Buckley and Koskela was extended to the setting of more general equations related to vector fields satisfying so called Hörmander's condition (which is defined below), by Buckley, Koskela and Lu [16]. We will come back to the case $0 < p < 1$ later.

A very big and a very important class of measures satisfying s-regularity condition is a class of measures satisfying so called doubling condition. There are several modifications of the notion of a doubling condition, depending on a given context, so we will not give a formal definition, but we explain this notion in the statement of the following lemma.

Lemma 2 *Let μ be a Borel measure on a metric space X, finite on bounded sets. Assume that μ is doubling on a bounded subset $Y \subset X$, in the following sense: there is a constant $C_d \geq 1$, such that*

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)),$$

whenever $x \in Y$, and $r \leq \operatorname{diam} Y$. Then

$$\mu(B(x, r)) \geq (2 \operatorname{diam} Y)^{-s} \mu(Y) r^s,$$

for $s = \log_2 C_d$, $x \in Y$ and $r \leq \operatorname{diam} Y$.

The constant C_d is called doubling constant. This lemma shows that properties of μ strongly rely on the exact value of the doubling constant. If μ is the Lebesgue measure on $\mathbb{R}^n$, then $C_d = 2^n$, and hence $s = n$ is just dimension.

Metric space (or more generally quasi-metric, whatever it means) equipped with a doubling measure is called a space of homogeneous type. This notion was introduced by Coifman and Weiss [24].

It is known that doubling measures inherit many properties of the Lebesgue measure, and it is possible to extend many results from the classical Harmonic Analysis, including for example singular integrals (Coifman Weiss [24]), into the general setting of spaces of homogeneous type (see also [22]). The class of doubling measures is very large, since as was proven by Volberg and Konyagin [102], [103], every compact subset of $\mathbb{R}^n$ supports a doubling measure. Moreover it is possible to construct an example of a doubling measure, in the whole $\mathbb{R}^n$, singular with respect to the Lebesgue measure, and an absolutely continuous doubling measure which vanishes on a set of positive Lebesgue measure, see [94, pp. 40–41].

Later we will give one more example of a doubling measure, on the so called Carnot–Carathéodory space. This example is very important in the regularity theory of strongly degenerated elliptic equations. For further examples of doubling measures see [22], [94].

The above lemma is well known. Proof is very easy, and we sketch it now. Iterating k times the doubling inequality, we get $\mu(2^k B) \leq C_d^k \mu(B)$. Now for any B we can find the smallest integer k, with $Y \subset 2^k B$, and hence we can estimate $\mu(Y)$ in terms of $\mu(B)$, but this is just the statement of the lemma. This iteration argument resembles that used in the proof of the fact that Harnack's inequality implies Hölder continuity (cf. [81] and later in this section).

It was proven in Section 1 that u is in the Sobolev space if and only if certain pointwise inequality is satisfied (Theorem 3). However one can prove a stronger result. Roughly speaking it was proven by the author and Koskela [44], [45] that if μ is a doubling measure on X, $u \in L^1_{\rm loc}(X,\mu)$, $g \in L^p_{\rm loc}(X,\mu)$, $g \geq 0$, and the following weak version of the Poincaré inequality

$$\fint_B |u-u_B|\, d\mu \leq r \left(\fint_{2B} g^p\ d\mu \right)^{1/p}, \tag{7}$$

holds, whenever B is a ball with radius r, then $u \in W^{1,q}(X,d,\mu)$ for all $q < p$. Thus applying imbedding theorem we obtain that there is $p^* > p$, such that the weak Sobolev inequality

$$\left(\fint_B |u-u_B|^{p^*}\, d\mu \right)^{1/p^*} \leq Cr \left(\fint_{2B} g^p\, d\mu \right)^{1/p} \tag{8}$$

holds for all balls B. Moreover if we add certain conditions concerning the geometry of the balls (it suffices, for example, to assume that every two points in the ball can be connected by the shortest path), then we obtain also the strong Sobolev imbedding

$$\left(\fint_B |u-u_B|^{p^*}\, d\mu \right)^{1/p^*} \leq Cr \left(\fint_{B} g^p\, d\mu \right)^{1/p}. \tag{9}$$

This result extends also to the case $0 < p < 1$.

For our convenience the inequality of the type (9), with $p^* = p$ will be called Poincaré inequality, and when $p^* > p$, Sobolev–Poincaré inequality.

Note that in the classical setting, when $u \in W^{1,p}(2B)$, $g = |\nabla u|$, and μ is the Lebesgue measure, (7) follows from the classical Poincaré inequality.

One could expect that if he assumed the strong version of (7), replacing $2B$ by B, he would obtain the strong Sobolev–Poincaré inequality (9), for general doubling measure, without making any requirements concerning the geometry of the balls. However it seems to be not true. According to our proof, even strong version of (7) leads only to the weak inequality (8), unless one makes some additional assumptions.

The paper of the author and Koskela [44] contains also another proof of the above result, which states that *under certain assumptions concerning the geometry of balls, family of weak Poincaré inequalities (7) implies the family of strong Sobolev–Poincaré inequalities (9), for all* $0 < p < \infty$. This proof is direct, surprisingly elementary, and it involves different idea from that presented above. In particular it avoids the use of the theory of Sobolev spaces on metric spaces.

The just stated result generalizes a part of related results of Saloff-Coste [87], Biroli and Mosco [4], [5], [6] and Maheux and Saloff-Coste [68], (see later in this section).

Moreover, roughly speaking, if $|\nabla u|$ satisfies weak reverse Hölder inequality, then one can prove inequality of the type (7) (with $g = |\nabla u|$), for any $p > 0$, and hence our result applies to the previously mentioned result of Buckley and Koskela [15]. This application is presented in details in the paper by the author and Koskela [45].

Also the "chain technique" developed in [44], [45], in order to deduce strong inequalities from its weak form, simplifies, in many cases, an earlier techniques based on the so called

Boman chain condition, and developed in the papers of Boman [13], Iwaniec and Nolder [54], Bojarski [7], Chua [27], Jerison [55], Franchi Gutiérrez and Wheeden [34] and many others.

The above theorem directly applies to the regularity theory of degenerated elliptic equations. Consider the following degenerate equation in $\Omega \subset \mathbb{R}^n$

$$\operatorname{div} A(x, \nabla u) = 0, \tag{10}$$

where $A(x,\xi) \cdot \xi \geq C_1 \omega(x)|\xi|^p$, $|A(x,\xi)| \leq C_2 \omega(x)|\xi|^{p-1}$, and $\omega > 0$, $\omega \in L^1_{\mathrm{loc}}(\Omega)$. The prototype equation is the weighted p-Laplacian

$$\operatorname{div}(\omega(x)|\nabla u|^{p-2}\nabla u) = 0.$$

Weak solutions of (10) are defined in the weighted Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \omega)$ (where the L^p norms of u and $|\nabla u|$ are taken with respect to the measure $\omega(x)dx$). It was observed by Fabes, Kenig and Serapioni, [29], that in order to extend Moser's technique, [81], to the degenerated linear equation of the form (10), $p = 2$, it suffices to put conditions upon ω which are collected in the definition below. Moser's iteration technique implies that, for suitable ω, there exists a constant $C = C(n, p, C_2/C_1, \omega)$, such that whenever $2B \subset \Omega$, and u is a weak solutions of (10), nonnegative in $2B$, then the following scale invariant (s.i.) Harnack's inequality

$$\sup_B u \leq C \inf_B u, \tag{11}$$

holds (since the constant C does not depend on the radius of B we call this inequality scale invariant). Then iterating s.i. Harnack's inequality one easily obtains Hölder continuity of the solution. As we already mentioned this last step resembles the iteration used in the proof of Lemma 2.

Definition (Heinonen, Kilpeläinen and Martio [49]). We say that $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\omega > 0$ a.e. is p-admissible, $1 < p < \infty$, if the measure defined by $d\mu = \omega(x)\,dx$ satisfies the following four conditions:

1. (Doubling condition) $\mu(2B) \leq C_d \mu(B)$ for all balls $B \subset \mathbb{R}^n$.

2. (Uniqueness condition) If Ω is an open subset of $\mathbb{R}^n$ and $\varphi_i \in C^\infty(\Omega)$ is a sequence such that $\int_\Omega |\varphi_i|^p\, d\mu \to 0$ and $\int_\Omega |\nabla\varphi_i - v|^p\, d\mu \to 0$, where $v \in L^p(\mu)$, then $v \equiv 0$.

3. (Sobolev inequality) There exists an exponent $p^* > p$, such that for all balls $B \subset \mathbb{R}^n$ and all $\varphi \in C_0^\infty(B)$

$$\left(\fint_B |\varphi|^{p^*}\, d\mu \right)^{1/p^*} \leq C_2 r \left(\fint_B |\nabla\varphi|^p\, d\mu \right)^{1/p}.$$

4. (Poincaré inequality) If $B \subset \mathbb{R}^n$ is a ball and $\varphi \in C^\infty(B)$, then

$$\fint_B |\varphi - \varphi_B|^p\, d\mu \leq C_3 r^p \fint_B |\nabla\varphi|^p\, d\mu.$$

During the 1980's Granlund, Heinonen, Kilpeläinen, Lindqvist, Malý, Martio and the others (see [49] and references therein) shown that not only Moser's technique extends to the general case of (10), when ω is p-admissible weight, $1 < p < \infty$, but also one obtains a rich potential theory.

It appears that above theorem which asserts that (7) implies (9) leads to the following elementary characterization of p-admissible weights.

Theorem 7 (Hajłasz and Koskela [44]) *Let $\omega > 0$ be a locally integrable function. Then the weight ω is p-admissible if and only if the measure μ associated with ω is doubling, (i.e., $\mu(2B) \leq C_d\mu(B)$ for all balls $B \subset \mathbb{R}^n$) and*

$$\fint_B |u - u_B|\, d\mu \leq C_2 r \left(\fint_{2B} |\nabla u|^p \, d\mu \right)^{1/p},$$

whenever B is a ball with radius r and $u \in C^\infty(2B)$.

In the above application we were concerned with the Euclidean metric, so we didn't use full power of our "metric" result.

Now we show that equations degenerated in a different way than that above lead to Sobolev spaces with respect to a non Euclidean metric. First consider an example. Elliptic operator $Lu = \partial^2 u/\partial x_1^2 + x_1^2 \partial^2 u/\partial x_2^2$ degenerates along the line $x_1 = 0$. It is of a divergence form $\operatorname{div} A(x, \nabla u)$, with $A(x, \xi) = [\xi_1, x_1^2 \xi_2]$. If we try to apply the above described approach to degenerated equations via weighted Sobolev spaces we fail, since we have bad estimates (assume $|x| < 1$)

$$A(x,\xi)\cdot\xi = \xi_1^2 + x_1^2\xi_2^2 \geq x_1^2|\xi|^2, \qquad |A(x,\xi)| \leq |\xi|.$$

Problem is that in the above approach to degenerated elliptic equations of the type (10), conditions describing how strongly the equation is degenerated, were isotropic with respect to all variables, while the operator L is degenerated in an "anisotropic" way. Hence we have to find a different approach. The idea is to consider a certain non Euclidean metric (so called Carnot–Carathéodory metric) which reflects the anisotropic degeneracy of the operator. Then we will see that our metric approach to Sobolev spaces applies to Carnot–Carathéodory metric, and hence applies to the regularity theory of related elliptic equations. First let us define a certain, general class of degenerated elliptic equations.
DEFINITION (see [98], [83], [55]). Let $\Omega \subset \mathbb{R}^n$ be an open, connected set, and let the vector fields X_1, X_2,…,X_k be defined in a neighborhood of $\overline{\Omega}$, real valued, and C^∞-smooth. We say that these vector fields satisfy Hörmander's condition, provided there is an integer p such that the family of commutators of $X_1, X_2, \ldots, X_k$ up to length p i.e., the family of vector fields $X_1,\ldots,X_k$, $[X_{i_1}, X_{i_2}]$,…, $[X_{i_1}, [X_{i_2}, [\ldots, X_{i_p}]]\ldots]$, $i_j = 1, 2, \ldots, k$, span the tangent space $\mathbb{R}^n$ at every point of Ω.

For example the vector fields $X_1 = \partial/\partial x_1$, $X_2 = x_1\partial/\partial x_2$ do not span $\mathbb{R}^2$ along the line $x_1 = 0$. However X_1, X_2 and $[X_1, X_2] = \partial/\partial x_2$ span $\mathbb{R}^2$ everywhere. Hence X_1 and X_2 satisfy Hörmander's condition. We leave as an exercise verification that the vector fields $Y_1 = \partial/\partial x_1$ and $Y_2 = x_1^k \partial/\partial x_2$, k-positive integer, also satisfy Hörmander's condition (one has to consider commutators of length $k + 1$).

Celebrated theorem of Hörmander, [53], states that if the vector fields X_0, $X_1,\ldots,X_k$ satisfy Hörmander's condition, then the operator $L = X_0 + \sum_{i=1}^k X_i^2$ is hypoelliptic, that is u must be C^∞ function in every open set where Lu is a C^∞ function.

A huge amount of papers related to Hörmander's condition has been written after Hörmander's theorem, see [98], [39] and references therein.

Using the above example we get the hypoellipticity of the Grushin operator $L = Y_1^2 + Y_2^2 = \partial^2/\partial x_1^2 + x_1^{2k}\partial^2/\partial x_2^2$. We want to emphasize that Hörmander's theorem applies also to operators which are far from the elliptic type: we leave as an exercise verification that Hörmander's theorem applies to the heat operator $\partial/\partial t - \Delta$ and to the Kolmogorov operator $\partial^2/\partial x^2 + x\partial/\partial y - \partial/\partial t$.

The above class of operators is to large and we will restrict our attention to the so called sub-Laplacians which are of the form $\Delta_X = -\sum_{j=1}^k X_j^* X_j$, where the family $X_1,\ldots,X_k$ satisfy Hörmander's condition and X_j^* is a formal adjoint of X_j on L^2 i.e.,

$$\int_\Omega (X_j^* u)v\, dx = -\int_\Omega u X_j v\, dx,$$

for all $u, v \in C_0^\infty(\Omega)$. Note that X_j^* does not have to be a vector field, in general it is of the form $X_j^* = -X_j + f_j$, where f_j is a suitable smooth function. Thus $-\sum_{j=1}^k X_j^* X_j = \sum_{j=1}^k X_j^2 + Y$, where $Y = \sum_{j=1}^k f_j X_j$, and hence Δ_X is a Hörmander's operator.

Note that Grushin's operator $\partial^2/\partial X_1^2 + x_1^{2k}\partial^2/\partial x_2^2$ is a sub-Laplacian, while the heat operator is not (why?).

The results we shall discuss below can be generalized to the much more general class of operators, however we will be concerned with sub-Laplacians, just for simplicity sake. At the end of the section we will mention some of the further generalizations.

It seems, the first indication that many properties of non degenerated elliptic equations extend to sub-Laplacians, was the result of Bony [14], who proved the strong maximum principle: if $\Delta_X u = 0$ in Ω and u is continuous in $\overline{\Omega}$, then $u(x) < \max_{\partial\Omega} u$, for all $x \in \Omega$, unless u is identically constant.

As we already said, in the case of equation (10) one obtains s.i. Harnack's inequality (11) via Moser's iteration technique. Bony [14] used the above strong maximum principle to deduce the Harnack inequality with C depending on the radius of a ball B. This inequality is much weaker than its s.i. version. In particular one cannot deduce Hölder continuity from it.

In fact, in general, the s.i. Harnack's inequality does not hold for positive solutions to $\Delta_X u = 0$. Here is a very simple example. Function $u(x,y) = 6y^2 - x^4 + c$ satisfies $(\partial^2/\partial x^2 + x^2\partial^2/\partial y^2)u = 0$. For a suitable c, function u is positive in $Q_{2\varepsilon} = [-2\varepsilon, 2\varepsilon]^n$. Now it remains to make some elementary computations which we recommend to the reader.

However it appears that one can prove such s.i. Harnack's inequality for general sub-Laplacian, if he replaces Euclidean balls by balls with respect to the Carnot-Carathéodory metric which we shall define now.

Definition (cf. [98]). Let the family of C^∞, real valued vector fields $X_1,\ldots,X_k$, defined in the neighbourhood of $\overline{\Omega}$, satisfy Hörmander's condition. Let Γ_X denote the set of all Lipschitz paths $\gamma : [0,1] \to \Omega$ satisfying $\dot{\gamma}(t) = \sum_{i=1}^k a_i(t) X_i(\gamma(t))$, for almost every

$t \in [0,1]$. Define the "length" of γ as follows

$$|\gamma| = \int_0^1 \left(\sum_{i=1}^k a_i^2(t) \right)^{1/2} dt.$$

Now the Carnot–Carathéodory (C.-C.) distance between $x, y \in \Omega$ is defined as a "geodesic" distance

$$\rho(x,y) = \inf\{|\gamma| \mid \gamma \in \Gamma_X,\ \gamma(0) = x,\ \gamma(1) = y\}$$

($|\gamma|$ would be the length if the vector fields $X_1,\ldots,X_k$ were orthonormal in every point, but we do not require even that they are lineary independent.)

This distance is related to the paper of Carathéodory [19]. It is not clear whether every two points can be connected by a curve $\gamma \in \Gamma_X$. Fortunately we have.

Theorem 8 (Chow [21], [98, Theorem III.4.1]). *If Ω is a connected domain, and the family of vector fields satisfy Hörmander's condition, then every two points of Ω can be connected by a curve $\gamma \in \Gamma_X$, thus ρ is a metric. This metric induces standard topology.*

A weaker version of Chow's theorem was first proved by Carathéodory [19]. In order to learn what exactly Carathéodory and Chow proved, see [52, Chapter 18].

Now with given sub-Laplacian we can associate the C.-C. metric.

We want also to point out that applications of the C.-C. metric go far beyond the setting of Hörmander's theorem, see Gromov [39], Strichartz [95] and references therein. In fact as we could see, metric of this type had already been considered a long time before Hörmander's work. In the literature the C.-C. metric is sometimes called a sub-Riemannian, or singular Riemannian metric.

By $\tilde{B}$ we will denote a ball with respect to the C.-C. metric. Results of Nagel, Stein and Waigner [83] give quite precision estimates for the metric ρ. In particular they proved that *there is $0 < \lambda \leq 1$, such that for any compact $K \subset \Omega$, there is a constant C, such that*

$$|x - y| \leq \rho(x,y) \leq C|x - y|^\lambda \tag{12}$$

for all $x, y \in K$. Moreover for any compact $K \subset \Omega$, there is $r_0 > 0$, and $C \geq 1$, such that

$$|\tilde{B}(x,2r)| \leq C|\tilde{B}(x,r)| \tag{13}$$

for all $x \in K$, and all $r < r_0$. Here $|\tilde{B}|$ denotes the Lebesgue measure of the C.-C. ball.

It seems that it was Franchi and Lanconelli [32], who first realized that one can prove Hölder continuity of solutions to certain strongly degenerated elliptic equations via s.i. Harnack's inequality with respect to the C.-C. metric. In fact they were concerned with slightly different equations than sub-Laplacians: they worked with measurable coefficients, but on the other hand the algebraic structure of the equation was much simpler than that of general sub-Laplacian. Still they could define a variant of the C.-C. metric, and they proved a version of (12), and (13), for that particular metric. Then they proved the s.i. Harnack's inequality for the C.-C. balls, using Moser's technique. It was known (more or less) that in order to run Moser's technique, one had to check just a doubling property for C.-C. balls and a suitable version of Sobolev–Poincaré inequality on C.-C. balls (see notation introduced after (9)), and they proved it — in the particular case of the equations considered by them. Thus they obtained s.i. version of Harnack's inequality (11), where Euclidean balls were replaced by the C.-C. balls.

We could see above that the solutions $u_c(x,y) = 6y^2 - x^4 + c$ of the equation $(\partial^2/\partial x^2 + x^2\partial^2/\partial y^2)u = 0$ do not satisfy s.i. Harnack's inequality on Euclidean balls (or squares), with center at $(0,0)$. We suggest the reader to sketch the shape of the C.-C. balls with center at $(0,0)$, associated to this particular operator (i.e., defined by the vector fields $\partial/\partial x$, and $x\partial/\partial y$), and deduce (at least heuristically), that positive solutions of the form u_c satisfy s.i. Harnack's inequality on these C.-C. balls.

Jerison [55] extended the idea of Franchi and Lanconelli to general sub-Laplacian. He proved the following version of the Poincaré inequality

$$\left(\fint_{\tilde{B}(r)} |u - u_{\tilde{B}(r)}|^p \, dx\right)^{1/p} \leq Cr \left(\fint_{\tilde{B}(r)} \sum_{i=1}^{k} |X_i u|^p \, dx\right)^{1/p} \tag{14}$$

for all $1 \leq p < \infty$ (see also Jerison and Sanchez-Calle [56]). Here the integration is taken with respect to the Lebesgue measure. Since the related Sobolev inequality for compactly supported functions was known (see Theorem 13 and inequality (17.20) in Rothschild and Stein [86]), and the doubling property was also known (the above mentioned result of Nagel, Stein and Waigner), the Moser technique applied. It seems that the main problem in establishing the Poincaré inequality (14) was the problem with the boundary of the C.-C. balls. Jerison provided an example of a domain with smooth boundary for which Poincaré inequality fails! Note that the problem with boundary does not appear when one works with the compactly supported functions.

Saloff-Coste [87] proved that in a very general setting of subelliptic operators, which are formally self adjoint and positive with respect to a doubling measure, family of related Poincaré inequalities on the C.-C. balls, for $p = 2$, implies the family of Sobolev–Poincaré inequalities (with $p^* > p = 2$) on the C.-C. balls (in these inequalities the integration is taken with respect to the doubling measure). We want to emphasize that Saloff-Coste didn't prove the Poincaré inequalitry. He just proved the implication. He found a lot of important applications of this result in the context of parabolic version of Moser's iteration.

His proof, that the Poincaré inequalities imply Sobolev–Poincaré, uses the theory of submarkovian semigroups. This result was generalized by Biroli and Mosco [4], [5], [6] to the setting of metric spaces with doubling measure. They replaced the "gradient" $\sum_{i=1}^{k} |X_i u|^2$ by a suitable Dirichlet norm $a(u)$, and then they proved only a weak Sobolev inequality. In the paper [6] they also consider the case $1 \leq p < \infty$. Note that mentioned above result of the author and Koskela [44] (which states that (7) implies (9)) generalizes both results of Saloff-Coste and Biroli and Mosco. We do not require any relationship between u and the "gradient" g, beside the weak Poincaré inequality, we prove strong version of the Sobolev–Poincaré inequality, and we prove it for all $0 < p < \infty$. Because of the generality, there are chances that our result can be applied to Moser's iteration in much more general setting. We should also mention the papers of Saloff-Coste [88] and Maheux and Saloff-Coste [68], which contains further generalizations of the theorem of Saloff-Coste. Moreover the papers of Lu [67] and Franchi Lu and Wheeden [35] contains many Sobolev type imbedding theorems for Hörmander vector fields. Later we will mention further generalizations.

Once we have s.i. Harnack's inequality for the C.-C. balls, we obtain Hölder continuity (of solutions) with respect to the C.-C. metric, and then applying (12) we get Hölder continuity with respect to the Euclidean metric. Of course in the case of sub-Laplacians it is not necessary to use this argument in order to deduce Hölder continuity, since by Hörmander's theorem we already have C^∞ smoothness! But it appears that this scheme: s.i. Harnack's inequality with respect to C.-C. metric plus $d(x,y) \leq |x-y|^\lambda$ implies Hölder continuity, can be applied to obtain the Hölder continuity of solutions for much larger class of operators than sub-Laplacians. One such example was the above mentioned result of Franchi and Lanconelli [32].

The typical example of operators considered in [32] is the Grusin type operator $\Delta_x u + |x|^{2\sigma}\Delta_y u$, $\sigma > 0$, $(x,y) \in \mathbb{R}^{n+m}$. This operator is a sub-Laplacian only if σ is a positive integer. However still one can associate C.-C. metric to the vector fields ∇_x and $|x|^\sigma \nabla_y$, and prove suitable Sobolev–Poincaré inequality.

Following Franchi and Serapioni [33], one can consider even more degenerated operators for which the typical example is the following.

$$Lu = \mathrm{div}\,(\omega(x,y)(\nabla_x u + |x|^{2\sigma}\nabla_y u)), \tag{15}$$

with suitable $\omega \in L^1_{\mathrm{loc}}$, $\omega > 0$ a.e. Here $\sigma > 0$ and $(x,y) \in \mathbb{R}^{n+m}$. This operator mix both ways the operator can be degenerated: one considered in (10) and that for sub-Laplacians. In order to extend Moser's technique to operators of the form (15) one has to deal with the Sobolev–Poincaré inequalities with respect to suitable C.-C. metric and weighted doubling measure. Thus it is suitable setting to apply our metric approach to Sobolev spaces.

The far reaching generalization of (15) has been considered recently by Franchi, Gutierrez and Wheeden [34]. Also a very general form of quasilinear equations related to vector fields with Hörmander's condition is considered by Lu [67].

We want to emphasize that, because of the simplicity reason, we didn't mention the most general form of the quoted results. Also sometimes we omited some technical assumptions. Moreover we didn't mention all the contributors to this subject. In any case it is not possible to put the right chronological order in the above events, since similar results independently grown up in many papers.

3. Traces on fractal type sets. It was suggested to the author by Paweł Strzelecki that Sobolev spaces on metric spaces can be useful in description of traces of Sobolev functions. Classical trace theorem gives the description of traces on smooth submanifolds. We will see how to describe traces on much more general subsets. The results presented in this section announces the forthcoming paper of the author and Martio [46]. Here, for simplicity sake, we will be concerned with the most simple cases only, just to show the idea. The approach to traces of Besov spaces on fractal type subsets was developed by Jonsson and Wallin [59] and in a more general form by Jonsson [58]. Their results apply to the Sobolev space $W^{1,2}$. Although their approach involves different ideas, concerns Besov spaces rather than the Sobolev spaces, and is much more technical, their results are strongly related to that of ours.

Let's recall the trace theorem in the classical setting. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with the sufficiently regular boundary. It is well known theorem of Gagliardo [37], [64, Theorems 6.8.13 and 6.9.2] that there is a trace operator for $1 < p < \infty$

$$\mathrm{Tr}\,: W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega),$$

and the extension operator

$$\mathrm{Ext}\,: W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega),$$

where the space $W^{1-1/p,p}(\partial\Omega)$ is the Slobodeckiĭ space and is defined as follows

$$u \in W^{1-1/p,p}(\partial\Omega) \qquad \text{iff} \qquad \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p-2}}\,dx\,dy < \infty. \tag{16}$$

Thus the space $W^{1-1/p,p}(\partial\Omega)$ characterizes traces of the $W^{1,p}(\Omega)$ functions. We will see that using Sobolev spaces on metric spaces one can almost characterize the space of traces.

Theorem 9 (Hajłasz and Martio [46]).

$$W^{1-1/p,p}(\partial\Omega) \subset W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1}) \subset W^{1-1/(p-\varepsilon),p-\varepsilon}(\partial\Omega), \tag{17}$$

for any $\varepsilon > 0$.

Here the space $W^{1,p}$ is the Sobolev space on the metric space $\partial\Omega$ with the metric $d(x,y) = |x-y|^{1-1/p}$, and with respect to the measure H^{n-1}. Theorem 9 leads to the trace and extension operators

$$\mathrm{Tr}\,: W^{1,p}(\Omega) \to W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1}) \tag{18}$$

$$\mathrm{Ext}\,: W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1}) \to W^{1,p-\varepsilon}(\Omega) \tag{19}$$

for any $\varepsilon > 0$. Hence $W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1})$ almost characterizes traces of a Sobolev functions from $W^{1,p}(\Omega)$.

The space $W^{1-1/p,p}(\partial\Omega)$ gives the sharp characterization of traces, but its definition is of essentially different character than that of classical Sobolev space $W^{1,p}$. The "metric" approach is a unified approach to Sobolev spaces and trace spaces, but on the other hand it does not lead to a sharp characterization of traces — this is the price one has to pay.

To see how (18) works, we will apply the imbedding Theorem 6 to the right hand side of (18). First let's compute s with respect to which the space $(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1})$ is s–regular.

If $\tilde{B}(r)$ denotes a ball (subset of $\partial\Omega$) with respect to the metric $|\cdot|^{1-1/p}$, then $\tilde{B}(r) = B(r^{p/(p-1)})$, where the last ball is taken with respect to the Euclidean metric (induced from $\mathbb{R}^n$). Now $H^{n-1}(\tilde{B}(r)) = H^{n-1}(B(r^{p/(p-1)})) \approx r^{p(n-1)/(p-1)}$, which means the space is s regular for $s = p(n-1)/(p-1)$. Now applying imbedding theorem we get for $p < n$

$$W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1}) \subset L^{\frac{p(n-1)}{n-p}}(\partial\Omega),$$

and this is just a classical imbedding theorem for traces [64, Theorem 6.4.1].

One of the possible methods to obtain trace theorems on fractal type sets is the following. Assume that $u \in W^{1,p}(\Omega)$, $K \subset \Omega$ is a compact set, μ a Borel measure on K and $0 \le \lambda < 1$. If the operator

$$M^{\lambda}_{\operatorname{diam} K} : L^p(\mathbb{R}^n) \to L^q(K,\mu), \tag{20}$$

is bounded, then the pointwise inequality (2) immediately gives that $u|_K \in W^{1,q}(K, |\cdot|^{1-\lambda}, \mu)$ i.e., this leads to the following trace operator

$$\mathrm{Tr} : W^{1,p}(\Omega) \to W^{1,q}(K, |\cdot|^{1-\lambda}, \mu).$$

Using Marcinkiewicz interpolation theorem it is not difficult to prove theorems of the form (20). This leads, in particular, to the following theorem.

Theorem 10 (Hajłasz and Martio [46]). *If $u \in W^{1,p}(\mathbb{R}^n)$, $0 < \lambda < 1$, $1 \leq (n-d)/\lambda < p \leq n/\lambda$, and μ is a Borel measure with compact support, such that $\mu(B(x,r)) \leq Cr^d$ for all $x \in \mathbb{R}^n$ and all $r > 0$, then there is a continuous trace operator*

$$\mathrm{Tr} : W^{1,p}(\mathbb{R}^n) \to W^{1,\frac{dp}{n-\lambda p}}(\operatorname{supp}\mu, |\cdot|^{1-\lambda}, \mu).$$

Note that this theorem applies when $\operatorname{supp}\mu$ is a suitable d–dimensional fractal type set and μ is a d–dimensional Hausdorff measure restricted to that fractal. Note also that we can apply a Lusin type approximation (Theorem 5) to $W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1})$. This shows that we can approximate traces by a Hölder continuous functions in a Lusin sense. Argument of this type has been used by the author in the study of boundary behaviour of conformal, quasiconformal and, more generally, Sobolev mappings [42]. The main result was a generalization and simplification of the proof of Øksendal's theorem [85], [51] on harmonic measure. This is one of the example which shows how useful is the approach to traces via Sobolev spaces on metric spaces.

The paper [46] contains more sophisticated results than that cited above. Also an extension type results are studied in a general setting there.

4. Higher order derivatives. Now we present a "pointwise inequalities" for higher order derivatives. It will be a generalization of the inequality (1). For simplicity sake we will be concerned with the counterparts of (1) for $\lambda = 0$ only. Then we show some applications. Finally we make some comments on the pointwise inequalities for an arbitrary $0 \leq \lambda < 1$.

First some notation. $T_x^k u(y) = \sum_{|\alpha|\leq k} D^\alpha u(x)(y-x)^\alpha/\alpha!$ will stand for Taylor's polynomial. Here $u \in W^{k,p}_{\mathrm{loc}}$, and the derivatives $D^\alpha u$ are in the weak sense. By $\nabla^m u$ we denote the vector with components $D^\alpha u$, $|\alpha| = m$, and $|\nabla^m u|$ stands for its Euclidean length.

If $u \in W^{m,p}_{\mathrm{loc}}$, then we assume that u and *all* its derivatives up to order m are defined *everywhere* by the formula

$$D^\alpha u(x) = \limsup_{r\to 0} \fint_{B(x,r)} D^\alpha u(y)\, dy, \tag{21}$$

for $|\alpha| \leq m$. This extends the previous convention (3) to the higher order derivatives.

Since $W^{m,p}_{\mathrm{loc}} \subset W^{m,1}_{\mathrm{loc}}$, it suffices to consider, in many interesting cases, the space $W^{m,1}_{\mathrm{loc}}$ only. Main result — a higher order pointwise inequalities reads as follows.

Theorem 11 (Bojarski and Hajłasz [9]). *Let $u \in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and its derivatives be defined at every point by (21), moreover let $a = (a_\alpha)_{|\alpha|=m}$, $b = (b_\alpha)_{|\alpha|=m}$ be an arbitrary vectors with real components. Then the following pointwise inequalities*

$$|u(y) - T_x^{m-1}u(y)| \leq C|x-y|^m \left(M_{|x-y|}|\nabla^m u|(x) + M_{|x-y|}|\nabla^m u|(y)\right)$$

and

$$|u(y)-T_x^m u(y)| \le C|x-y|^m \left(M_{|x-y|}|\nabla^m u - a|(x) + M_{|x-y|}|\nabla^m u - b|(y)\right)$$

hold for all $x \neq y$. Both constants C depend on n and m only.

Some comments. Of course $|\nabla^m u - a|$ is the Euclidean length of the vector $\nabla^m u - a = (D^\alpha u - a_\alpha)_{|\alpha|=m}$. If $m = 1$, then the first inequality reduces to (1) with $\lambda = 0$, while the second inequality is new. Here we adopt the same convention as in Section 1: if the expression on the left hand side, in the one of the above inequalities, is of the indefinite form like, for example, $|\infty - \infty|$, then we set the left hand side equal to ∞. The inequality is still valid in such a case, since $D^\alpha u(z) = \pm\infty$, for certain $|\alpha| \le m$, implies $M_R|\nabla^m u|(z) = M_R|\nabla^m u - c|(z) = \infty$ for any $R > 0$ and any vector c.

In particular the second inequality of the above theorem holds with $a = \nabla^m u(x)$ and $b = \nabla^m u(y)$ (if $|a| = \infty$ or $|b| = \infty$, it is not dangerous, since then the right hand side is equal to infinity and the inequality is trivial). We can apply the above theorem to $D^\alpha u \in W^{m-|\alpha|,1}_{\rm loc}$ in place of u. This leads to the following corollary. Let $M^\sharp_R g(x) = \sup_{r<R} \fint_{B(x,r)} |g(z) - g(x)|\, dz$ for vector valued g, provided g is defined everywhere.

Corollary 2 (Bojarski and Hajłasz [9]). *If $u \in W^{m,1}_{\rm loc}(\mathbb{R}^n)$ and its derivatives are defined everywhere by (21), then for $|\alpha| \le m$ and all $x \neq y$*

$$|D^\alpha u(y) - T_x^{m-|\alpha|} D^\alpha u(y)| \le C|x-y|^{m-|\alpha|} \left(M^\sharp_{|x-y|}(\nabla^m u)(x) + M^\sharp_{|x-y|}(\nabla^m u)(y)\right).$$

This corollary gives the pointwise estimates for Taylor's remainder of u and of its derivatives. These estimates are very similar to that required in the statement of the following Whitney extension theorem.

Theorem 12 (Whitney [99], [69]). *Given a family $(u_\alpha)_{|\alpha|\le m}$ of continuous functions on a compact set $K \subset \mathbb{R}^n$. Then there exists a function $h \in C^m(\mathbb{R}^n)$, such that $D^\alpha h|_K = u_\alpha$ for all $|\alpha| \le m$ if and only if*

$$\sup_{|\alpha|\le m} \frac{|u_\alpha(y) - \sum_{|\beta|\le m-|\alpha|} u_{\alpha+\beta}(x)\frac{(y-x)^\beta}{\beta!}|}{|x-y|^{m-|\alpha|}} \longrightarrow 0$$

uniformly on K as $|x-y| \to 0$.

Now it is clear that if we proved $M^\sharp_R(\nabla^m u) \to 0$ uniformly on a compact set K, as $R \to 0$, we would have that there exists $h \in C^m(\mathbb{R}^n)$ such that $h|_K = u|_K$. It appears that it is not difficult to prove such convergence on "big" compact sets. This method was used by Bojarski and the author [9] to give an elementary proof of the following theorem of Michael and Ziemer.

Theorem 13 (Michael and Ziemer [77], [104, Theorem 3.11.6], [9]). *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set, $1 < p < \infty$, $1 \le m \le k$, m, k integers, and $u \in W^{k,p}_{\rm loc}(\Omega)$. Assume that u and its derivatives are defined everywhere by the formula (21). Then for every $\varepsilon > 0$, there exists a closed set $F \subset \Omega$ and a function $h \in C^m(\Omega)$ such that*

$$B_{k-m,p}(\Omega \setminus F) < \varepsilon,$$

$$D^\alpha u(x) = D^\alpha h(x), \ \textit{for } x \in F \textit{ and } |\alpha| \le m,$$
$$u - h \in W_0^{m,p}(\Omega),$$
$$||u-h||_{W^{m,p}} < \varepsilon.$$

If $k = m$, then we replace $B_{k-m,p}$ by the Lebesgue measure.

By $W_0^{m,p}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in the $W^{m,p}$ norm, hence in some sense it is a subspsce of $W^{m,p}(\Omega)$ consisting of functions which vanish on the boundary. Symbol $B_{k,p}$ is reserved for Bessel capacity. For definition we refer the reader to [104]. Here are some properties. Capacity $B_{k,p}$ is an outer measure, and it is finite on all bounded sets. If $kp < n$, then

$$H^{n-kp}(A) < \infty \quad \Rightarrow \quad B_{k,p}(A) = 0,$$
$$B_{k,p}(A) = 0 \quad \Rightarrow \quad \forall\, \varepsilon > 0 \ H^{n-kp+\varepsilon}(A) = 0.$$

If $kp > n$, then there exists a constant $C > 0$, such that

$$A \neq \emptyset \quad \Rightarrow B_{k,p}(A) > C. \tag{22}$$

This shows, $B_{k,p}$ has properties analogous, in some sense, to that of Hausdorff's content H_∞^{n-kp}.

If $(k-m)p > n$, the condition on the capacity of $\Omega \setminus F$ (in the above theorem) means $\Omega \setminus F$ is the empty (see (22)), thus $u \in C^m_{\rm loc}(\Omega)$. This particular case is also a direct consequence of the Sobolev imbedding theorem [104, Theorem 2.5.1]. However Sobolev theorem provides a stronger result in that case. Indeed, it gives the imbedding into $C^{m,\mu}_{\rm loc}$ functions (m-th derivatives are μ-Hölder continuous), for suitable $\mu > 0$. One can regard the theorem of Michael and Ziemer as an extension of the Sobolev imbedding into C^m, to the subcritical case $(k-m)p < n$. In a moment we will show how to obtain an analogous extension of the "full" Sobolev imbedding into $C^{m,\mu}$. Theorem of Michael and Ziemer generalizes former results of Calderón and Zygmund [18, Theorem 13] and Liu [66], see the discussion in Section 1, after Theorem 1.

The above presented simplified approach to Michael and Ziemer's theorem is based on Theorem 11, which generalizes (1) with $\lambda = 0$, to higher order derivatives. The already mentioned higher order generalization of (1), with an arbitrary $0 \le \lambda < 1$ leads to a generalization of Corollary 2, which in turn applies to $C^{m,1-\lambda}$-Whitney extension theorem. Hence as an application we obtain in the paper by Bojarski, Hajłasz and Strzelecki [11] a generalization of Theorem 13 with the approximation of u by $C^{m,1-\lambda}$ functions (instead of C^m). Note that we have already discussed, in details, result of this type, for $k = 1$, and $m = 0$ in Section 1 (see the neighborhood of Theorem 1). Bojarski and the author, [10], apply Theorem 13, and its generalization, [11], in the investigation of the geometric structure of the preimage of a point for a Sobolev mapping between manifolds.

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