Sobolev Mappings between Manifolds and Metric Spaces

Piotr Hajłasz

Abstract In connection with the theory of p-harmonic mappings, Eells and Lemaire raised a question about density of smooth mappings in the space of Sobolev mappings between manifolds. Recently Hang and Lin provided a complete solution to this problem. The theory of Sobolev mappings between manifolds has been extended to the case of Sobolev mappings with values into metric spaces. Finally analysis on metric spaces, the theory of Carnot–Carathéodory spaces, and the theory of quasiconformal mappings between metric spaces led to the theory of Sobolev mappings between metric spaces. The purpose of this paper is to provide a self-contained introduction to the theory of Sobolev spaces between manifolds and metric spaces. The paper also discusses new results of the author.

1 Introduction

For $\Omega \subset \mathbb{R}^n$ and $1 \leqslant p < \infty$ we denote by $W^{1,p}(\Omega)$ the usual Sobolev space of functions for which $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p < \infty$. This definition can easily be extended to the case of Riemannian manifolds $W^{1,p}(M)$. Let now M and N be compact Riemannian manifolds. We can always assume that N is isometrically embedded in the Euclidean space $\mathbb{R}^\nu$ (Nash's theorem). We also assume that the manifold N has no boundary, while M may have boundary. This allows one to define the class of Sobolev mappings between the two manifolds as follows:

$$W^{1,p}(M,N) = \{u \in W^{1,p}(M,\mathbb{R}^\nu) |\ u(x) \in N \text{ a.e.}\}$$

Piotr Hajłasz
University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260 USA,
e-mail: hajlasz@pitt.edu

$W^{1,p}(M,N)$ is equipped with the metric inherited from the norm $\varrho(u,v) = \|u-v\|_{1,p}$. The space $W^{1,p}(M,N)$ provides a natural setting for geometric variational problems like, for example, *weakly p-harmonic mappings* (called *weakly harmonic mappings* when $p=2$). Weakly p-harmonic mappings are stationary points of the functional

$$I(u) = \int_M |\nabla u|^p \quad \text{for } u \in W^{1,p}(M,N).$$

Because of the constrain in the image (manifold N) one has to clarify how the variation of this functional is defined. Let $\mathcal{U} \subset \mathbb{R}^\nu$ be a tubular neighborhood of N, and let $\pi : \mathcal{U} \to N$ be the smooth nearest point projection. For $\varphi \in C_0^\infty(M,\mathbb{R}^\nu)$, and $u \in W^{1,p}(M,N)$ the mapping $u + t\varphi$ takes on values into $\mathcal{U}$ provided that $|t|$ is sufficiently small. Then we say that u is *weakly p-harmonic* if

$$\frac{d}{dt}\Big|_{t=0} I(\pi(u+t\varphi)) = 0 \quad \text{for all } \varphi \in C_0^\infty(M,\mathbb{R}^\nu).$$

The condition that the mappings take values into the manifold N is a constrain that makes the corresponding Euler-Lagrange system

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2} A(u)(\nabla u, \nabla u), \tag{1.1}$$

very nonlinear and difficult to handle. Here, A is a second fundamental form of the embedding of N into the Euclidean space (see, for example, [4, 16, 34, 41, 42, 48, 65, 69, 79, 80, 83, 84]). There is a huge and growing literature on the subject, and it is impossible to list here all relevant papers, but the reader can easily find other papers following the references in the papers cited above.

Our main focus in this paper is the theory of Sobolev mappings between manifolds, and later, the theory of Sobolev mappings between metric spaces, rather than applications of this theory to variational problems, and the above example was just to illustrate one of many areas in which the theory applies.

In connection with the theory of p-harmonic mappings, Eells and Lemaire [18] raised a question about density of smooth mappings $C^\infty(M,N)$ in $W^{1,p}(M,N)$. If $p \geqslant n = \dim M$, then smooth mappings are dense in $W^{1,p}(M,N)$ [73, 74], but if $p < n$, the answer depends on the topology of manifolds M and N. Recently, Hang and Lin [39] found a necessary and sufficient condition for the density in terms of algebraic topology. Their result is a correction of an earlier result of Bethuel [3] and a generalization of a result of Hajłasz [26]. To emphasize the connection of the problem with algebraic topology, let us mention that it is possible to reformulate the Poincaré conjecture (now a theorem) in terms of approximability of Sobolev mappings [25]. The theory of Sobolev mappings between manifolds has been extended to the case of Sobolev mappings with values into metric spaces. The first papers on this subject include the work of Ambrosio [2] on limits of classical variational problems and the work of Gromov and Schoen [24] on Sobolev mappings into the Bruhat–Tits buildings, with applications to rigidity questions for discrete groups. Later, the theory of Sobolev mappings with values into metric spaces was developed in a more elaborated form by Korevaar and Schoen [55] in their approach to the theory of harmonic mappings into Alexandrov spaces of nonpositive curvature. Other papers on Sobolev mappings from a manifold into a metric space include [12, 17, 49, 50, 51, 52, 70, 76]. Finally, analysis on metric spaces, the theory of Carnot–Carathéodory spaces, and the theory of quasiconformal mappings between metric spaces led to the theory of Sobolev mappings between metric spaces [46, 47, 58, 81], among which the theory of Newtonian–Sobolev mappings $N^{1,p}(X,Y)$ is particularly important.

In Sect. 2, we discuss fundamental results concerning the density of smooth mappings in $W^{1,p}(M,N)$. Section 3 is devoted to a construction of the class of Sobolev mappings from a manifold into a metric space. We also show there that several natural questions to the density problem have negative answers when we consider mappings from a manifold into a metric space. In Sect. 4, we explain the construction and basic properties of Sobolev spaces on metric measure spaces and, in final Sect. 5, we discuss recent development of the theory of Sobolev mappings between metric spaces, including results about approximation of mappings.

The notation in the paper is fairly standard. We assume that all manifolds are compact (with or without boundary), smooth, and connected. We always assume that such a manifold is equipped with a Riemannian metric, but since all such metrics are equivalent, it is not important with which metric we work. By a closed manifold we mean a smooth compact manifold without boundary. The integral average of a function u over a set E is denoted by

$$u_E = \fint_E u\, d\mu = \mu(E)^{-1} \int_E u\, d\mu\,.$$

Balls are denoted by B and σB for $\sigma > 0$ denotes the ball concentric with B whose radius is σ times that of B. The symbol C stands for a general constant whose actual value may change within a single string of estimates. We write $A \approx B$ if there is a constant $C \geqslant 1$ such that $C^{-1}A \leqslant B \leqslant CA$.

2 Sobolev Mappings between Manifolds

It is easy to see and is well known that smooth functions are dense in the Sobolev space $W^{1,p}(M)$. Thus, if N is isometrically embedded into $\mathbb{R}^\nu$, it follows that every $W^{1,p}(M,N)$ mapping can be approximated by $C^\infty(M,\mathbb{R}^\nu)$ mappings and the question is whether we can approximate $W^{1,p}(M,N)$ by $C^\infty(M,N)$ mappings. It was answered in the affirmative by Schoen and Uhlenbeck [73, 74] in the case $p \geqslant n = \dim M$.

Theorem 2.1. *If $p \geqslant n = \dim M$, then the class of smooth mappings $C^\infty(M,N)$ is dense in the Sobolev space $W^{1,p}(M,N)$.*

Proof. [1] Assume that N is isometrically embedded in some Euclidean space $\mathbb{R}^\nu$. If $p > n$, then the result is very easy. Indeed, let $u_k \in C^\infty(M,\mathbb{R}^\nu)$ be a sequence of smooth mappings that converge to u in the $W^{1,p}$ norm. Since $p > n$, the Sobolev embedding theorem implies that u_k converges uniformly to u. Hence for $k \geqslant k_0$ values of the mappings u_k belong to a tubular neighborhood $\mathcal{U} \subset \mathbb{R}^\nu$ of N from which there is a smooth nearest point projection $\pi : \mathcal{U} \to N$. Now $\pi \circ u_k \in C^\infty(M,N)$ and $\pi \circ u_k \to \pi \circ u = u$ in the $W^{1,p}$ norm. If $p = n$, then we do not have uniform convergence, but one still can prove that the values of the approximating sequence u_k whose construction is based locally on the convolution approximation belong to the tubular neighborhood of N for all sufficiently large k. This follows from the Poincaré inequality. To see this, it suffices to consider the localized problem where the mappings are defined on an Euclidean ball. Let $u \in W^{1,n}(B^n(0,1),N)$, and let $\overline{u}$ be the extension of u to a neighborhood of the ball (by reflection). We define $u_\varepsilon = \overline{u} * \varphi_\varepsilon$, where φ_ε is a standard mollifying kernel. The Poincaré inequality yields

$$\left(\fint_{B^n(x,\varepsilon)} |\overline{u}(y) - u_\varepsilon(x)|^n \, dy \right)^{1/n} \leqslant Cr \left(\fint_{B^n(x,\varepsilon)} |\nabla \overline{u}|^n \right)^{1/n}$$

$$= C' \left(\int_{B^n(x,\varepsilon)} |\nabla \overline{u}|^n \right)^{1/n} . \tag{2.1}$$

The right-hand side (as a function of x) converges to 0 as $\varepsilon \to 0$ uniformly on $B^n(0,1)$. Since

$$\operatorname{dist}(u_\varepsilon(x), N) \leqslant |\overline{u}(y) - u_\varepsilon(x)|$$

for all y, from (2.1) we conclude that

$$\operatorname{dist}(u_\varepsilon(x), N) \to 0 \quad \text{as } \varepsilon \to 0$$

uniformly on $B^n(0,1)$. Hence for $\varepsilon < \varepsilon_0$ values of the smooth mappings u_ε belong to $\mathcal{U}$ and thus $\pi \circ u_\varepsilon \to \pi \circ u = u$ as $\varepsilon \to 0$. □

Arguments used in the above proof lead to the following result.

[1] See also Theorems 3.7 and 5.5.

Proposition 2.2. *If $u \in W^{1,p}(M,N)$ can be approximated by continuous Sobolev mappings $C^0 \cap W^{1,p}(M,N)$, then it can be approximated by smooth $C^\infty(M,N)$ mappings.*

Proof. Indeed, if $v \in C^0 \cap W^{1,p}(M,N)$, then $v_\varepsilon \rightrightarrows v$ uniformly and hence $\pi \circ v_\varepsilon \to \pi \circ v = v$ in $W^{1,p}$. □

A basic tool in the study of Sobolev mappings between manifolds is a variant of the Fubini theorem for Sobolev functions. Let us illustrate it in a simplest setting. Suppose that $u, u_i \in W^{1,p}([0,1]^n)$, $\|u - u_i\|_{1,p} \to 0$ as $i \to \infty$. Denote by (t,x), where $t \in [0,1]$, $x \in [0,1]^{n-1}$, points in the cube. Then

$$\int\limits_{[0,1]^n} |u - u_i|^p + |\nabla u - \nabla u_i|^p$$

$$= \int\limits_0^1 \left(\int\limits_{[0,1]^{n-1}} |u - u_i|^p + |\nabla u - \nabla u_i|^p \, dx \right) dt$$

$$= \int\limits_0^1 F_i(t)\, dt \overset{i\to\infty}{\longrightarrow} 0\,.$$

Hence $F_i \to 0$ in $L^1(0,1)$ and so there is a subsequence u_{i_j} such that $F_{i_j}(t) \to 0$ for almost all $t \in (0,1)$. That means that for almost all $t \in [0,1]$ we have $u(t,\cdot), u_{i_j}(t,\cdot) \in W^{1,p}([0,1]^{n-1})$ and $u_{i_j}(t,\cdot) \to u(t,\cdot)$ in $W^{1,p}([0,1]^{n-1})$. Clearly, the same argument applies to lower dimensional slices of the cube.

As was already pointed out, if $p < n = \dim M$, then density of smooth mappings does not always hold. The first example of this type was provided by Schoen and Uhlenbeck, and it is actually quite simple. A direct computation shows that the radial projection

$$u_0(x) = x/|x| : B^n(0,1) \setminus \{0\} \to S^{n-1}$$

belongs to the Sobolev space $W^{1,p}(B^n, S^{n-1})$ for all $1 \leqslant p < n$. Shoen and Uhlenbeck [73, 74] proved the following assertion.

Theorem 2.3. *If $n-1 \leqslant p < n$, then the mapping u_0 cannot be approximated by smooth mappings $C^\infty(B^n, S^{n-1})$ in the $W^{1,p}$ norm.*

Proof. Suppose that there is a sequence $u_k \in C^\infty(B^n, S^{n-1})$ such that $\|u_k - u_0\|_{1,p} \to 0$ as $k \to \infty$ for some $n-1 \leqslant p < n$. Then from the Fubini theorem it follows that there is a subsequence (still denoted by u_k) such that for almost every $0 < r < 1$

$$u_k|_{S^{n-1}(0,r)} \to u_0|_{S^{n-1}(0,r)}$$

in the $W^{1,p}(S^{n-1}(0,r))$ norm. If $n-1 < p < n$, then the Sobolev embedding theorem into Hölder continuous functions implies that u_k restricted to such spheres converges uniformly to u_0

$$u_k|_{S^{n-1}(0,r)} \rightrightarrows u_0|_{S^{n-1}(0,r)},$$

which is impossible because the Brouwer degree[2] of $u_k|_{S^{n-1}(0,r)}$ is 0 and the degree of $u_0|_{S^{n-1}(0,r)}$ is 1. The case $p = n-1$ needs a different, but related argument. The degree of a mapping $v : M \to N$ between two oriented $(n-1)$-dimensional compact manifolds without boundary can be defined by the integral formula

$$\deg v = \int_M \det Dv/\operatorname{vol} N,$$

and from the Hölder inequality it follows that the degree is continuous in the $W^{1,n-1}$ norm. This implies that if $u_k \to u_0$ in $W^{1,n-1}(S^{n-1}(0,r))$, then the degree of $u_k|_{S^{n-1}(0,r)}$ which is 0 converges to the degree of $u_0|_{S^{n-1}(0,r)}$ is 1. Again we obtain a contradiction. □

It turns out, however, that for $1 \leqslant p < n-1$ smooth maps are dense in $W^{1,p}(B^n, S^{n-1})$. Indeed, the following result was proved by Bethuel and Zheng [5].

Theorem 2.4. *For $1 \leqslant p < k$ smooth mappings $C^\infty(M, S^k)$ are dense in $W^{1,p}(M, S^k)$.*

Proof. Let $u \in W^{1,p}(M, S^k)$. It is easy to see that for every $x \in S^k$ and $\delta > 0$ there is a Lipschitz retraction $\pi_{x,\delta} : S^k \to S^k \setminus B(x,\delta)$, i.e., $\pi_{x,\delta} \circ \pi_{x,\delta} = \pi_{x,\delta}$, with the Lipschitz constant bounded by $C\delta^{-1}$. Now we consider the mapping $u_{x,\delta} = \pi_{x,\delta} \circ u$. Since $u_{x,\delta}$ maps M into the set $S^k \setminus B(x,\delta)$ which is diffeomorphic with a closed k dimensional ball, it is easy to see that $u_{x,\delta}$ can be approximated by smooth maps from M to $S^k \setminus B(x,\delta) \subset S^k$. Thus, it remains to prove that for every $\varepsilon > 0$ there is $\delta > 0$ and $x \in S^k$ such that $\|u - u_{x,\delta}\|_{1,p} < \varepsilon$.

There are $C\delta^{-k}$ disjoint balls of radius δ on S^k. Such a family of balls is denoted by $B(x_i,\delta)$, $i = 1, 2, \ldots, N_\delta$, where $N_\delta \approx \delta^{-k}$. Note that the mapping $u_{x_i,\delta}$ differs from u on the set $u^{-1}(B(x_i,\delta))$ and this is a family of $N_\delta \approx \delta^{-k}$ disjoint subset of M. Therefore, there is i such that

$$\int_{u^{-1}(B(x_i,\delta))} |u|^p + |\nabla u|^p \leqslant C\delta^k \|u\|_{1,p}^p.$$

[2] The degree is 0 because u_k has continuous (actually smooth) extension to the entire ball.

Using the fact that the Lipschitz constant of $\pi_{x_i,\delta}$ is bounded by $C\delta^{-1}$, it is easy to see that

$$\int\limits_{u^{-1}(B(x_i,\delta))} |\nabla u_{x_i,\delta}|^p \leqslant C\delta^{-p} \int\limits_{u^{-1}(B(x_i,\delta))} |\nabla u|^p \leqslant C\delta^{k-p}\|u\|^p_{1,p}$$

Since $u = u_{x_i,\delta}$ on the complement of the set $u^{-1}(B(x_i,\delta))$, we have

$$\|\nabla u - \nabla u_{x_i,\delta}\|_p = \Bigg(\int\limits_{u^{-1}(B(x_i,\delta))} |\nabla u - \nabla u_{x_i,\delta}|^p \Bigg)^{1/p}$$

$$\leqslant \Bigg(\int\limits_{u^{-1}(B(x_i,\delta))} |\nabla u|^p \Bigg)^{1/p} + \Bigg(\int\limits_{u^{-1}(B(x_i,\delta))} |\nabla u_{x_i,\delta}|^p \Bigg)^{1/p}$$

$$\leqslant C(\delta^{k/p} + \delta^{(k-p)/p})\|u\|_{1,p}\,.$$

Since $k - p > 0$, this implies that for given $\varepsilon > 0$ there is $\delta > 0$ and $x \in S^k$ such that $\|\nabla u - \nabla u_{x,\delta}\|_{1,p} < \varepsilon$. It remains to note that the mappings u and $u_{x,\delta}$ are also close in the L^p norm. Indeed, they are both uniformly bounded (as mappings into the unit sphere) and they coincide outside a set of very small measure. □

The above two results show that the answer to the problem of density of smooth mappings in the Sobolev space $W^{1,p}(M,N)$ depends of the topology of the manifold N and perhaps also on the topology of the manifold M. We find now necessary conditions for the density of $C^\infty(M,N)$ in $W^{1,p}(M,N)$.

The density result (Theorem 2.1) implies that if M and N are two smooth oriented compact manifolds without boundary, both of dimension n, then we can define the degree of mappings in the class $W^{1,n}(M,N)$. Indeed, if $u \in C^\infty(M,N)$, then the degree is defined in terms of the integral of the Jacobian and then it can be extended to the entire space $W^{1,n}(M,N)$ by the density of smooth mappings. Thus,

$$\deg : W^{1,n}(M,N) \to \mathbb{Z}$$

is a continuous function and it coincides with the classical degree on the subclass of smooth mappings. It turns out, however, that not only degree, but also homotopy classes can be defined. This follows from the result of White [85].

Theorem 2.5. *Let M and N be closed manifolds, and let $n = \dim M$. Then for every $f \in W^{1,n}(M,N)$ there is $\varepsilon > 0$ such that any two smooth mappings $g_1, g_2 : M \to N$ satisfying $\|f - g_i\|_{1,n} < \varepsilon$ for $i = 1, 2$ are homotopic.*

Note that Theorem 2.5 is also a special case of Theorem 2.8 and Theorem 5.5 below.

We use the above result to find the first necessary condition for the density of smooth mappings in the Sobolev space. The following result is due to Bethuel and Zheng [5] and Bethuel [3]. A simplified proof provided below is taken from [25]. Let $[p]$ denote the largest integer less than or equal to p. In the following theorem, π_k stands for the homotopy group.

Theorem 2.6. *If $\pi_{[p]}(N) \neq 0$ and $1 \leqslant p < n = \dim M$, then the smooth mappings $C^\infty(M,N)$ are not dense in $W^{1,p}(M,N)$.*

Proof. It is easy to construct a smooth mapping $f : B^{[p]+1} \to S^{[p]}$ with two singular points such that f restricted to small spheres centered at the singularities have degree $+1$ and -1 respectively and f maps a neighborhood of the boundary of the ball $B^{[p]+1}$ into a point. We can model the singularities on the radial projection mapping as in Theorem 2.3 so the mapping f belongs to $W^{1,p}$. Let now $g : B^{[p]+1} \times S^{n-[p]-1} \to S^{[p]}$ be defined by $g(b,s) = f(b)$. We can embed the torus $B^{[p]+1} \times S^{n-[p]-1}$ into the manifold M and extend the mapping on the completion of this torus as a mapping into a point. Clearly, $g \in W^{1,p}(M, S^{[p]})$. Let $\varphi : S^{[p]} \to N$ be a smooth representative of a nontrivial homotopy class. We prove that the mapping $\varphi \circ g \in W^{1,p}(M,N)$ cannot be approximated by smooth mappings from $C^\infty(M,N)$. By contrary, suppose that $u_k \in C^\infty(M,N)$ converges to $\varphi \circ f$ in the $W^{1,p}$ norm. In particular, $u_k \to \varphi \circ g$ in $W^{1,p}(B^{[p]+1} \times S^{n-[p]-1}, N)$. From the Fubini theorem it follows that there is a subsequence of u_k (still denoted by u_k) such that for almost every $s \in S^{n-[p]-1}$, u_k restricted to the slice $B^{[p]+1} \times \{s\}$ converges to the corresponding restriction of $\varphi \circ g$ in the Sobolev norm. Take such a slice and denote it simply by $B^{[p]+1}$. Again, by the Fubini theorem, u_k restricted to almost every sphere centered at the $+1$ singularity of f converges to the corresponding restriction of $\varphi \circ g$ in the Sobolev norm. Denote such a sphere by $S^{[p]}$. Hence $u_k|_{S^{[p]}} \to \varphi \circ g|_{S^{[p]}}$ in the space $W^{1,p}(S^{[p]}, N)$. Now the mapping $u_k|_{S^{[p]}} : S^{[p]} \to N$ is contractible (because it has a smooth extension to the ball), while $\varphi \circ g|_{S^{[p]}} : S^{[p]} \to N$ is a smooth representative of a nontrivial homotopy class $\pi_{[p]}(N)$, so $u_k|_{S^{[p]}}$ cannot be homotopic to $\varphi \circ g|_{S^{[p]}}$, which contradicts Theorem 2.5. □

It turns out that, in some cases, the condition $\pi_{[p]}(N) = 0$ is also sufficient for the density of smooth mappings. The following statement is due to Bethuel [3].

Theorem 2.7. *If $1 \leqslant p < n$, then smooth mappings $C^\infty(B^n, N)$ are dense in $W^{1,p}(B^n, N)$ if and only if $\pi_{[p]}(N) = 0$.*

Actually, Bethuel [3] claimed a stronger result that $\pi_{[p]}(N) = 0$ is a necessary and sufficient condition for the density of $C^\infty(M, N)$ mappings in $W^{1,p}(M, N)$ for any compact manifold M of dimension $\dim M = n > p$. This, however, turned out to be false: Hang and Lin [38] provided a counterexample to Bethuel's claim by demonstrating that despite the equality $\pi_3(\mathbb{CP}^2) = 0$, $C^\infty(\mathbb{CP}^3, \mathbb{CP}^2)$ is *not* dense in $W^{1,3}(\mathbb{CP}^3, \mathbb{CP}^2)$. Bethuel's claim made people believe that the problem of density of smooth mappings in the Sobolev space has a local nature. However the example of Hang and Lin and Theorem 2.7 shows that there might be global obstacles. Indeed, the mapping constructed by Hang and Lin cannot be approximated by smooth mappings $C^\infty(\mathbb{CP}^3, \mathbb{CP}^2)$, however, since $\pi_3(\mathbb{CP}^2) = 0$, Theorem 2.7 shows that this mapping can be smoothly approximated in a neighborhood of any point in $\mathbb{CP}^3$.

Therefore, searching for a necessary and sufficient condition for the density of smooth mappings, one has to take into account the topology of both manifolds M and N, or rather the interplay between the topology of M and the topology of N. Now we find such a necessary condition for the density of smooth mappings. Before we start, we need to say a few words about the behavior of Sobolev mappings on k-dimensional skeletons of generic smooth triangulations.

Let the manifold M be equipped with a smooth triangulation M^k, $k = 0, 1, 2, \ldots, n = \dim M$. Since the skeletons of the triangulation are piecewise smooth, it is not difficult to define the Sobolev space on skeletons $W^{1,p}(M^k)$. There is no problem with the definition of Sobolev functions in the interiors of the simplexes, but one needs to clarify how the Sobolev functions meet at the boundaries, so that the function belongs to the Sobolev space not only in each of the simplexes, but on the whole skeleton M^k. One possibility is to define the Sobolev norm $\|u\|_{1,p}$ for functions u that are Lipschitz continuous on M^k and then define $W^{1,p}$ by completion. Suppose now that $u \in W^{1,p}(M)$. If $v \in W^{1,p}([0, 1]^n)$, then, in general, it is not true that the function v restricted to *each* slice $\{t\} \times [0, 1]^{n-1}$ belongs to the Sobolev space $W^{1,p}([0, 1]^{n-1})$, but it is true for almost all $t \in [0, 1]$. By the same reason, u restricted to M^k does not necessarily belong to the Sobolev space $W^{1,p}(M^k)$. This problem can, however, be handled. Indeed, faces of the k dimensional skeleton M^k can be translated in the remaining directions which form an $n - k$ dimensional space. Hence, roughly speaking, with each skeleton M^k we can associate an $n - k$ dimensional family of skeletons.[3] Now u restricted to almost every skeleton in this family belongs to the Sobolev space $W^{1,p}$ on that skeleton by the Fubini theorem. We briefly summarize this construction by saying that if $u \in W^{1,p}(M)$, then u restricted to a generic k dimensional skeleton M^k belongs to the Sobolev space $W^{1,p}(M^k)$. Moreover, if $u, u_i \in W^{1,p}(M)$, $\|u - u_i\|_{1,p} \to 0$, then there is a subsequence u_{i_j} such that $u_{i_j} \to u$ in

[3] This is not entirely obvious because we translate different faces in different directions and we have to make sure that after all the faces glue together, so that we still have a k-dimensional skeleton. This, however, can be done and there are no unexpected surprises.

$W^{1,p}(M^k)$ on generic k-dimensional skeletons. This follows from the Fubini theorem argument explained above.

We say that two continuous mappings $f, g : M \to N$ are *k-homotopic*, $0 \leqslant k \leqslant n = \dim M$, if the restrictions of both mappings to the k-dimensional skeleton of a triangulation of M are homotopic. Using elementary topology, one can prove that the above definition does not depend on the choice of a triangulation of M (see [39, Lemma 2.1]). Theorem 2.5 is a special case of a more general result of White [85].

Theorem 2.8. *Let M and N be closed manifolds, and let $n = \dim M$. Then for every $f \in W^{1,p}(M,N)$, $1 \leqslant p \leqslant n$, there is $\varepsilon > 0$ such that any two Lipschitz mappings $g_1, g_2 : M \to N$ satisfying $\|f - g_i\|_{1,p} < \varepsilon$, $i = 1,2$ are $[p]$-homotopic.*

Another result that we frequently use is the homotopy extension theorem. We state it only in a special case.

Theorem 2.9. *Let M be a smooth compact manifold equipped with a smooth trinagulation M^k, $k = 0,1,2,\ldots,n = \dim M$. Then for any topological space X every continuous mapping*

$$H : (M \times \{0\}) \cup (M^k \times [0,1]) \to X$$

has a continuous extension to $\widetilde{H} : M \times [0,1] \to X$.

In particular, the theorem implies that if $f : M \to N$ is continuous and $g : M^k \to N$ is homotopic to $f|_{M^k}$, then g admits a continuous extension to $\widetilde{g} : M \to N$. We apply this observation below.

In the proof of the necessity of the condition $\pi_{[p]}(N) = 0$, we constructed a map with the $(n - [p] - 1)$-dimensional singularity. The condition we will present now will actually imply $\pi_{[p]}(N) = 0$ and, not surprisingly, our argument will also involve a construction of a map with the $(n - [p] - 1)$-dimensional singularity.

Let $1 \leqslant p < n = \dim M$. Suppose that smooth mappings $C^\infty(M,N)$ are dense in $W^{1,p}(M,N)$. Assume that M is endowed with a smooth triangulation. Let $h : M^{[p]} \to N$ be a Lipschitz mapping. Observe that if $f \in W^{1,p}(S^k,N)$, then the integration in spherical coordinates easily implies that the mapping $\overline{f}(x) = f(x/|x|)$ belongs to $W^{1,p}(B^{k+1},N)$ provided that $p < k+1$. Clearly, the ball B^{k+1} can be replaced by a $(k+1)$-dimensional simplex and S^k by its boundary. By this reason, the mapping $h : M^{[p]} \to N$ can be extended to a mapping in $W^{1,p}(M^{[p]+1},N)$. The extension will have singularity consisting of one point in each $([p]+1)$-dimensional simplex in $M^{[p]+1}$. Next, we can extend the mapping to $W^{1,p}(M^{[p]+2},N)$. Now, the singularity is one dimensional. We can continue this process by extending the mapping to higher dimensional skeletons. Eventually, we obtain a mapping $\overline{h} \in W^{1,p}(M,N)$ with the $(n - [p] - 1)$-dimensional singularity located on a dual skeleton to $M^{[p]}$.

Let $u_i \in C^\infty(M,N)$ be such that $\|\overline{h} - u_i\|_{1,p} \to 0$ as $i \to \infty$. From the Fubini theorem it follows that there is a subsequence u_{i_j} such that $u_{i_j} \to \overline{h}$ in $W^{1,p}$ on generic $[p]$-dimensional skeletons, so

$$u_{i_j} \to \overline{h} \quad \text{in } W^{1,p}(\widetilde{M}^{[p]}, N)\,,$$

where $\widetilde{M}^{[p]}$ is a "tilt" of $M^{[p]}$. Since $\overline{h}$ and u_{i_j} are Lipschitz, from Theorem 2.8 it follows that u_{i_j} is homotopic to $\overline{h}$ on $\widetilde{M}^{[p]}$ for all $j \geqslant j_0$. Now, from the homotopy extension theorem (see Theorem 2.9) it follows that the mapping $\overline{h}|_{\widetilde{M}^{[p]}}$ admits an extension to a continuous mapping $\overline{\overline{h}} : M \to N$. Hence also $h : M^{[p]} \to N$ can be extended to a continuous mapping $h' : M \to N$.

We proved that every Lipschitz mapping $h : M^{[p]} \to N$ admits a continuous extension $h' : M \to N$. Since every continuous mapping $f : M^{[p]} \to N$ is homotopic to a Lipschitz mapping, another application of the homotopy extension theorem implies that also f has continuous extension. We proved the following assertion.

Proposition 2.10. *If $1 \leqslant p < n = \dim M$ and $C^\infty(M,N)$ is dense in $W^{1,p}(M,N)$, then every continuous mapping $f : M^{[p]} \to N$ can be extended to a continuous mapping $f' : M \to N$.*

The following result provides a characterization of the property described in the above proposition.

We say that M has $(k-1)$*-extension property with respect to* N if for every continuous mapping $f \in C(M^k, N)$, $f|_{M^{k-1}}$ has a continuous extension to $\widetilde{f} \in C(M,N)$.

Proposition 2.11. *If $1 \leqslant k < n = \dim M$, then every continuous mapping $f : M^k \to N$ can be extended to a continuous mapping $f' : M \to N$ if and only if $\pi_k(N) = 0$ and M has the $(k-1)$-extension property with respect to N.*

Proof. Suppose that every continuous mapping $f : M^k \to N$ has a continuous extension to $f' : M \to N$. Then it is obvious that M has the $(k-1)$-extension property with respect to N. We need to prove that $\pi_k(N) = 0$. Suppose that $\pi_k(N) \neq 0$. Let Δ be a $(k+1)$-dimensional simplex in M^{k+1}, and let $\partial\Delta$ be its boundary. It is easy to see that there is a continuous retraction $\pi : M^k \to \partial\Delta$. Let $\varphi : \partial\Delta \to N$ be a representative of a nontrivial element in the homotopy group $\pi_k(N)$. Then φ cannot be extended to Δ. Hence $f = \varphi \circ \pi : M^k \to N$ has no continuous extension to M. We obtain a contradiction.

Now, suppose that $\pi_k(N) = 0$ and M has the $(k-1)$-extension property with respect to N. Let $f : M^k \to N$ be continuous. We need to show that f can be continuously extended to M. Let $\widetilde{f} : M \to N$ be a continuous extension of $f|_{M^{k-1}}$.

The set $M^k \times [0,1]$ is the union of $(k+1)$-dimensional cells $\Delta \times [0,1]$, where Δ is a k-dimensional simplex in M^k. Denote by (x,t) the points in $\Delta \times [0,1]$

and define the mapping on the boundary on each cell as follows:

$$H(x,0) = \widetilde{f}(x) \qquad \text{for } x \in \Delta,$$

$$H(x,1) = f(x) \qquad \text{for } x \in \Delta,$$

$$H(x,t) = \widetilde{f}(x) = f(x) \text{ for } x \in \partial\Delta.$$

Because $\pi_k(N) = 0$, H can be continuously extended to the interior of each cell. Denote by $H : M^k \times [0,1] \to N$ the extension. Now, from the homotopy extension theorem it follows that $f : M^k \to N$ admits a continuous extension $f' : M \to N$. □

Thus, if $f \in W^{1,p}(M,N)$, $1 \leqslant p < n = \dim M$, can be approximated by smooth mappings, then $\pi_{[p]}(N) = 0$ and for every continuous mapping $g : M^{[p]} \to N$, $g|_{M^{[p]-1}}$ has a continuous extension to M.

Actually, this property was used by Hang and Lin [38] to demonstrate that $C^\infty(\mathbb{CP}^3, \mathbb{CP}^2)$ mappings are not dense in $W^{1,3}(\mathbb{CP}^3, \mathbb{CP}^2)$ (despite the fact that $\pi_3(\mathbb{CP}^2) = 0$).

Since the extension property is of topological nature, it is easier to work with the natural CW structure of $\mathbb{CP}^n$ rater than with the triangulation and the extension property can be equivalently formulated for CW structures.

It is well known that $\mathbb{CP}^n$ has a natural CW structure

$$\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \ldots \subset \mathbb{CP}^n.$$

If $M = \mathbb{CP}^3$, then $M^2 = M^3 = \mathbb{CP}^1$. Now, from the elementary algebraic topology it follows that the identity mapping

$$i : M^3 = \mathbb{CP}^1 \subset \mathbb{CP}^2$$

cannot be continuously extended to $\widetilde{i} : \mathbb{CP}^3 \to \mathbb{CP}^2$ and since $M^2 = M^3$ we also have that $i|_{M^2}$ has no continuous extension. Thus, $C^\infty(\mathbb{CP}^3, \mathbb{CP}^2)$ mappings are not dense in $W^{1,p}(\mathbb{CP}^3, \mathbb{CP}^2)$ (see [38, pp. 327-328] for more details).

It turns out that the above necessary condition for density is also sufficient. Namely, the following result was proved by Hang and Lin [39].

Theorem 2.12. *Assume that M and N are compact smooth Riemannian manifolds without boundary. If $1 \leqslant p < \dim M$, then smooth mappings $C^\infty(M,N)$ are dense in $W^{1,p}(M,N)$ if and only if $\pi_{[p]}(N) = 0$ and M has the $([p]-1)$-extension property with respect to N.*

The following two corollaries easily follow from the theorem (see [39]).

Corollary 2.13. *If $1 \leqslant p < n = \dim M$, k is an integer such that $0 \leqslant k \leqslant [p]-1$, $\pi_i(M) = 0$ for $1 \leqslant i \leqslant k$, and $\pi_i(N) = 0$ for $k+1 \leqslant i \leqslant [p]$, then $C^\infty(M,N)$ is dense in $W^{1,p}(M,N)$.*

Corollary 2.14. *If* $1 \leqslant p < n = \dim M$, $\pi_i(N) = 0$ *for* $[p] \leqslant i \leqslant n-1$, *then* $C^\infty(M,N)$ *is dense in* $W^{1,p}(M,N)$.

In particular, Corollary 2.13 with $k = 0$ gives the following result that was previously proved in [26].

Corollary 2.15. *If* $1 \leqslant p < n = \dim M$ *and* $\pi_1(N) = \pi_2(N) = \ldots = \pi_{[p]}(N) = 0$, *then* $C^\infty(M,N)$ *is dense in* $W^{1,p}(M,N)$.

The reason why we stated this corollary in addition to Corollary 2.13 is that, in the case of Sobolev mappings from metric spaces supporting Poincaré inequalities into Lipschitz polyhedra, the homotopy condition from Corollary 2.15 turns out to be necessary and sufficient for density (see Theorem 5.6).

Another interesting question regarding density of smooth mappings is the question about the density in the sequential weak topology. We do not discuss this topic here and refer the reader to [26, 36, 37, 39, 40, 66, 67].

3 Sobolev Mappings into Metric Spaces

There were several approaches to the definition of the class of Sobolev mappings from a manifold, or just an open set in $\mathbb{R}^n$ into a metric space (see, for example, [2, 24, 49, 55, 70]). The approach presented here is taken from [35] and it is an elaboration of ideas of Ambrosio [2] and Reshetnyak [70]. One of the benefits of the construction presented here is that the Sobolev space of mappings into a metric space is equipped in a natural way with a metric, so one can ask whether the class of Lipschitz mappings is dense. In the case of mappings into metric spaces, it does not make sense to talk about smooth mappings, so we need to consider Lipschitz mappings instead.

Since every metric space X admits an isometric embedding into a Banach space[4] V, the idea is to define the Sobolev space of functions with values into a Banach space V and then define the Sobolev space of mappings with values into X as

$$W^{1,p}(M,X) = \{f \in W^{1,p}(M,V) |\, f(M) \subset X\}\,.$$

Since $W^{1,p}(M,V)$ is a Banach space, this approach equips $W^{1,p}(M,X)$ with a natural metric inherited from the norm of $W^{1,p}(M,V)$, just like in the case of Sobolev mappings between manifolds. With this metric at hand, we can ask under what conditions the class of Lipschitz mappings $\mathrm{Lip}\,(M,X)$ is dense in $W^{1,p}(M,X)$.

On the other hand, the approach described above depends on the isometric embedding of X into V, so it is useful to find another, equivalent and intrinsic

[4] Every metric space admits an isometric embedding into the Banach space $V = \ell^\infty(X)$ of bounded functions on X. If, in addition, X is separable, then X admits an isometric embedding into ℓ^∞.

approach independent of the embedding. In this section, we describe both such approaches. In our approach, we follow [35], where the reader can find detailed proofs of results stated here.

For the sake of simplicity, we consider Sobolev functions defined on a domain in $\mathbb{R}^n$ rather than on a manifold, but all the statements can easily be generalized to the case of Sobolev functions defined on manifolds.

Before we define the Sobolev space of functions with values into a Banach space, we need briefly recall the notion of the Bochner integral (see [15]).

Let V be a Banach space, $E \subset \mathbb{R}^n$ a measurable set, and $1 \leqslant p \leqslant \infty$. We say that $f \in L^p(E,V)$ if

(1) f is *essentially separable valued*, i.e., $f(E \setminus Z)$ is a separable subset of V for some set Z of Lebesgue measure zero,

(2) f is *weakly measurable*, i.e., for every $v^* \in V^*$, $\langle v^*, f\rangle$ is measurable;

(3) $\|f\| \in L^p(E)$.

If $f = \sum_{i=1}^k a_i \chi_{E_i} : E \to V$ is a simple function, then the Bochner integral is defined by the formula

$$\int_E f(x)\,dx = \sum_{i=1}^k a_i |E_i|$$

and for $f \in L^1(E,V)$ the Bochner integral is defined as the limit of integrals of simple functions that converge to f almost everywhere. The following two properties of the Bochner integral are well known:

$$\left\| \int_E f(x)\,dx \right\| \leqslant \int_E \|f(x)\|\,dx$$

and

$$\left\langle v^*, \int_E f(x)\,dx \right\rangle = \int_E \langle v^*, f(x)\rangle\,dx \quad \text{for all } v^* \in V^*. \tag{3.1}$$

In the theory of the Bochner integral, a measurable set $E \subset \mathbb{R}^n$ can be replaced by a more general measure space. We need such a more general setting later, in Sect. 5.

Let now $\Omega \subset \mathbb{R}^n$ be an open set, and let V be a Banach space. It is natural to define the Sobolev space $W^{1,p}(\Omega,V)$ using the notion of weak derivative, just like in the case of real valued functions. We say that $f \in W^{1,p}(\Omega,V)$ if $f \in L^p(\Omega,V)$ and for $i = 1,2,\ldots,n$ there are functions $f_i \in L^p(\Omega,V)$ such that

$$\int_\Omega \frac{\partial\varphi}{\partial x_i}(x) f(x)\, dx = -\int_\Omega \varphi(x) f_i(x)\, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

We denote $f_i = \partial f/\partial x_i$ and call these functions *weak partial derivatives*. We also write $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ and

$$|\nabla f| = \left(\sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|^2 \right)^{1/2}.$$

The space $W^{1,p}(\Omega, V)$ is equipped with the norm

$$\|f\|_{1,p} = \left(\int_\Omega \|f\|^p \right)^{1/p} + \left(\int_\Omega |\nabla f|^p \right)^{1/p}.$$

It is an easy exercise to show that $W^{1,p}(\Omega, V)$ is a Banach space.

The problem with this definition is that it is not clear what conditions are needed to guarantee that Lipschitz functions belong to the Sobolev space. Indeed, a Lipschitz function $f : [0,1] \to V$ need not be differentiable in the Fréchet sense at any point, unless V has the Radon–Nikodym property (see [61, p. 259]). Since we want to work with Sobolev mappings from the geometric point of view, it is a very unpleasant situation.

There is another, more geometric, definition of the Sobolev space of functions with values in Banach spaces which we describe now. The definition below is motivated by the work of Ambrosio [2] and Reshetnyak [70].

Let $\Omega \subset \mathbb{R}^n$ be an open set, V a Banach space, and $1 \leqslant p < \infty$. The space $R^{1,p}(\Omega, V)$ is the class of all functions $f \in L^p(\Omega, V)$ such that

(1) for every $v^* \in V^*$, $\|v^*\| \leqslant 1$ we have $\langle v^*, f\rangle \in W^{1,p}(\Omega)$;

(2) there is a nonnegative function $g \in L^p(\Omega)$ such that

$$|\nabla \langle v^*, f\rangle| \leqslant g \quad \text{a.e.} \tag{3.2}$$

for every $v^* \in V^*$ with $\|v^*\| \leqslant 1$.

Using arguments similar to those in the proof of the completeness of L^p, one can easily show that $R^{1,p}(\Omega, V)$ is a Banach space with respect to the norm

$$\|f\|_{R^{1,p}} = \|f\|_p + \inf \|g\|_p$$

where the infimum is over the class of all functions g satisfying the inequality (3.2). Using the definitions and the property (3.1), one can easily prove the following result (see [35]).

Proposition 3.1. *If $\Omega \subset \mathbb{R}^n$ is open and V is a Banach space, then $W^{1,p}(\Omega, V) \subset R^{1,p}(\Omega, V)$ and $\|f\|_{R^{1,p}} \leqslant \|f\|_{1,p}$ for all $f \in W^{1,p}(\Omega, V)$.*

However, we can prove the opposite inclusion only under additional assumptions about the space V (see [35]).

Theorem 3.2. *If $\Omega \subset \mathbb{R}^n$ is open, $V = Y^*$ is dual to a separable Banach space Y, and $1 \leqslant p < \infty$, then $W^{1,p}(\Omega, V) = R^{1,p}(\Omega, V)$ and $\|f\|_{R^{1,p}} \leqslant \|f\|_{1,p} \leqslant \sqrt{n}\|f\|_{R^{1,p}}$.*

Idea of the proof. One only needs to prove the inclusion $R^{1,p} \subset W^{1,p}$ along with the estimate for the norm. Actually, the proof of this inclusion is quite long and it consists of several steps. In the sketch provided below, many delicate steps are omitted.

By the canonical embedding $Y \subset Y^{**} = V^*$, elements of the Banach space Y can be interpreted as functionals on V. Observe that if $u : [0,1] \to V$ is absolutely continuous, then for every $v^* \in Y$ the function $\langle v^*, u\rangle$ is absolutely continuous, so it is differentiable almost everywhere and satisfies the integration by parts formula. Since the space Y is separable, we have almost everywhere the differentiability of $\langle v^*, u\rangle$ and the integration by parts for all v^* from a countable and dense subset of Y. This implies that the function $u : [0,1] \to V$ is differentiable in a certain weak sense known as the w^*-differentiability. Moreover, the w^*-derivative $u' : [0,1] \to V$ satisfies the integration by parts formula

$$\int_0^1 \varphi'(t)u(t)\,dt = -\int_0^1 \varphi(t)u'(t)\,dt.$$

Using this fact and the Fubini theorem, one can prove that a function $f \in L^p(\Omega, V)$ that is absolutely continuous on almost all lines parallel to coordinate axes and such that the w^*-partial derivatives of f satisfy $\|\partial f/\partial x_i\| \leqslant g$ almost everywhere for some $g \in L^p(\Omega)$ belongs to the Sobolev space $W^{1,p}(\Omega, V)$, $\|f\|_{1,p} \leqslant \|f\|_p + \sqrt{n}\|g\|_p$. This fact is similar to the well-known characterization of the Sobolev space $W^{1,p}(\Omega)$ by absolute continuity on lines.

At the last step, one proves that if $f \in R^{1,p}(\Omega, V)$, then f is absolutely continuous on almost all lines parallel to the coordinate axes and the w^*-partial derivatives satisfy $\|\partial f/\partial x_i\| \leqslant g$, where the function $g \in L^p(\Omega)$ satisfies (3.2).

The above facts put together easily imply the result. □

One can prove the following more geometric characterization of the space $R^{1,p}(\Omega, V)$ which is very useful (see [35]).

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be open, V a Banach space and $1 \leqslant p < \infty$. Then $f \in R^{1,p}(\Omega, V)$ if and only if $f \in L^p(\Omega, V)$ and there is a nonnegative function $g \in L^p(\Omega)$ such that for every Lipschitz continuous function $\varphi : V \to \mathbb{R}$, $\varphi \circ f \in W^{1,p}(\Omega)$ and $|\nabla(\varphi \circ f)| \leqslant \mathrm{Lip}\,(\varphi)g$ almost everywhere.*

Idea of the proof. One implication is obvious. Indeed, if a function f satisfies the condition described in the above theorem, then it belongs to the space $R^{1,p}(\Omega, V)$ because for $v^* \in V^*$, $\|v^*\| \leqslant 1$, $\varphi(v) = \langle v^*, v\rangle$ is 1-Lipschitz continuous and hence $\langle v^*, f\rangle \in W^{1,p}(\Omega)$ with $|\nabla\langle v^*, f\rangle| \leqslant g$ almost everywhere.

In the other implication, we use the fact that $R^{1,p}(\Omega, V)$ functions are absolutely continuous on almost all lines parallel to coordinate axes. This implies that if $\varphi : V \to \mathbb{R}$ is Lipschitz continuous, then also $\varphi \circ f$ is absolutely continuous on almost all lines and hence $\varphi \circ f \in W^{1,p}(\Omega)$ by the characterization of $W^{1,p}(\Omega)$ in terms of absolute continuity on lines. □

Now we are ready to define the Sobolev space of mappings with values into an arbitrary metric space. Let $\Omega \subset \mathbb{R}^n$ be open, and let X be a metric space. We can assume that X is isometrically embedded into a Banach space V. We have now two natural definitions

$$W^{1,p}(\Omega, X) = \{f \in W^{1,p}(\Omega, V) |\, f(\Omega) \subset X\}$$

and

$$R^{1,p}(\Omega, X) = \{f \in R^{1,p}(\Omega, V) |\, f(\Omega) \subset X\}$$

Both spaces $W^{1,p}(\Omega, X)$ and $R^{1,p}(\Omega, X)$ are endowed with the norm metric.

Since every Lipschitz function $\varphi : X \to \mathbb{R}$ can be extended to a Lipschitz function $\widetilde{\varphi} : V \to \mathbb{R}$ with the same Lipschitz constant (McShane extension), we easily see that *if X is compact and Ω is bounded, then $f \in R^{1,p}(\Omega, X)$ if and only if there is a nonnegative function $g \in L^p(\Omega)$ such that for every Lipschitz continuous function $\varphi : X \to \mathbb{R}$ we have $\varphi \circ f \in W^{1,p}(\Omega)$ and $|\nabla(\varphi \circ f)| \leqslant \mathrm{Lip}\,(\varphi) g$ almost everywhere.*

We assume here the compactness of X and boundedness of Ω to avoid problems with the L^p integrability of f.

Observe that the last characterization of the space $R^{1,p}(\Omega, X)$ is independent of the isometric embedding of X into a Banach space.

As a direct application of Theorem 3.2, we have

Theorem 3.4. *If $\Omega \subset \mathbb{R}^n$ is open, $V = Y^*$ is dual to a separable Banach space Y, $1 \leqslant p < \infty$, and $X \subset V$, then $W^{1,p}(\Omega, X) = R^{1,p}(\Omega, X)$.*

The most interesting case is that where the space X is separable. In this case, X admits an isometric embedding to $V = \ell^\infty$ which is dual to a separable Banach space, $\ell^\infty = (\ell^1)^*$ and hence Theorem 3.4 applies.

With a minor effort one can extend the above arguments to the case of Sobolev spaces defined on a manifold, which leads to the spaces $W^{1,p}(M, X)$ and $R^{1,p}(M, X)$.

The following theorem is the main result in [35].

Suppose that any two points $x, y \in X$ can be connected by a curve of finite length. Then $d_\ell(x, y)$ defined as the infimum of lengths of curves connecting x to y is a metric. We call it the *length metric*. Since $d_\ell(x, y) \geqslant d(x, y)$, it easily follows that if X is compact with respect to d_ℓ, then X is compact with respect to d.

Theorem 3.5. *Let X be a metric space, compact with respect to the length metric. If $n \geqslant 2$, then there is a continuous Sobolev mapping $f \in C^0 \cap W^{1,n}([0,1]^n, X)$ such that $f([0,1]^n) = X$.*

3.1 Density

Once the space of Sobolev mappings with values into metric spaces has been defined, we can ask under what conditions Lipschitz mappings $\mathrm{Lip}\,(M, X)$ are dense in $W^{1,p}(M, X)$ or in $R^{1,p}(M, X)$. In this section, we follow [30] and provide several counterexamples to natural questions and very few positive results. For the sake of simplicity, we assume that the metric space X is compact and admits an isometric embedding into the Euclidean space. Thus, $X \subset \mathbb{R}^\nu$ and we simply define

$$W^{1,p}(M, X) = \{f \in W^{1,p}(M, \mathbb{R}^\nu) |\, f(M) \subset X\}.$$

If M and N are smooth compact manifolds, $\dim M = n$, then, as we know (Theorem 2.1), smooth mappings are dense in $W^{1,n}(M, N)$. The key property of N used in the proof was the existence of a smooth nearest point projection from a tubular neighborhood of N. The proof employed the fact that the composition with the smooth nearest point projection is continuous in the Sobolev norm. It turns out that the composition with a Lipschitz mapping need not be continuous in the Sobolev norm [30].

Theorem 3.6. *There is a Lipschitz function $\varphi \in \mathrm{Lip}\,(\mathbb{R}^2)$ with compact support such that the operator $\Phi : W^{1,p}([0,1], \mathbb{R}^2) \to W^{1,p}([0,1])$ defined as composition $\Phi(f) = \varphi \circ f$ is not continuous for any $1 \leqslant p < \infty$.*

The proof of the continuity of composition with a *smooth* function φ is based on the chain rule and continuity of the derivative $\nabla\varphi$. If φ is just Lipschitz continuous, then $\nabla\varphi$ is only measurable, so the proof does not work and the existence of the example as in the theorem above is not surprising after all (see, however, [64]).

Although the composition with a Lipschitz mapping is not continuous in the Sobolev norm, we can still prove that Theorem 2.1 is true if we replace N by a compact Lipschitz neighborhood retract.

We say that a closed set $X \subset \mathbb{R}^\nu$ is a *Lipschitz neighborhood retract* if there is an open neighborhood $\mathcal{U} \subset \mathbb{R}^\nu$ of X, $X \subset \mathcal{U}$, and a Lipschitz retraction $\pi : \mathcal{U} \to X$, $\pi \circ \pi = \pi$.

The following result was proved in [30].

Theorem 3.7. *Let $X \subset \mathbb{R}^\nu$ be a compact Lipschitz neighborhood retract. Then for every smooth compact n-dimensional manifold M Lipschitz mappings* $\operatorname{Lip}(M, X)$ *are dense in* $W^{1,p}(M, X)$ *for* $p \geqslant n$.

Sketch of the proof. If $f \in W^{1,p}(M, X)$ and $f_i \in C^\infty(M, \mathbb{R}^\nu)$ is a smooth approximation based on the mollification, then $\|f_i - f\|_{1,p} \to 0$ and for all sufficiently large i the values of f_i belong to $\mathcal{U}$ (Sobolev embedding for $p > n$ and Poincaré inequality for $p = n$), but there is no reason to claim that $\pi \circ f_i \to \pi \circ f = f$. To overcome this problem, one needs to construct another approximation $f_t \in \operatorname{Lip}(M, \mathbb{R}^\nu)$ such that

(1) Lipschitz constant of f_t is bounded by Ct;

(2) $t^p |\{f \neq f_t\}| \to 0$ as $t \to \infty$;

(3) $\sup_{x \in M} \operatorname{dist}(f_t(x), X) \to 0$ as $t \to \infty$.

The construction of such an approximation is not easy, but once we have it, a routine calculation shows that $\|f - \pi \circ f_t\|_{1,p} \to 0$ as $t \to \infty$. Indeed,

$$\Big(\int_M |\nabla f - \nabla(\pi \circ f_t)|^p\Big)^{1/p}$$

$$\leqslant \Big(\int_{\{f \neq f_t\}} |\nabla f|^p\Big)^{1/p} + \Big(\int_{\{f \neq f_t\}} |\nabla(\pi \circ f_t)|^p\Big)^{1/p}$$

$$\leqslant \Big(\int_{\{f \neq f_t\}} |\nabla f|^p\Big)^{1/p} + Ct|\{f \neq f_t\}|^{1/p} \to 0 \quad \text{as } t \to \infty.$$

The proof is complete. □

The class of Lipschitz neighborhood retracts contains Lipschitz submanifolds of $\mathbb{R}^\nu$ [63, Theorem 5.13].

In the following example, X is replaced by an n-dimensional submanifold of the Euclidean space such that it is smooth except for a just one point, and we no longer have the density of Lipschitz mappings [30].

Theorem 3.8. *Let $M \subset \mathbb{R}^\nu$ be a closed n-dimensional manifold. Then there is a homeomorphism $\varPhi \in C^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu)$ which is a diffeomorphism in $\mathbb{R}^\nu \setminus \{0\}$ which is identity outside a sufficiently large ball and has the property that Lipschitz mappings* $\operatorname{Lip}(M, \widetilde{M})$ *are not dense in* $W^{1,n}(M, \widetilde{M})$, *where* $\widetilde{M} = \varPhi^{-1}(M)$.

Clearly, $\widetilde{M}$ cannot be a Lipschitz neighborhood retract. The derivative of the mapping Φ is zero at 0 and hence derivative of Φ^{-1} is unbounded in a neighborhood of 0. This causes $\widetilde{M}$ to have highly oscillating smooth "wrinkles" which accumulate at one point. In a neighborhood of that point, $\widetilde{M}$ is the graph of a continuous function which is smooth everywhere except for this point. Actually, the construction is done in such a way that $\widetilde{M}$ is $W^{1,n}$-homeomorphic to M, but, due to high oscillations, there is no Lipschitz mapping from M onto $\widetilde{M}$, and one proves that this $W^{1,n}$-homeomorphism cannot be approximated by Lipschitz mappings. This actually shows that there is a continuous Sobolev mapping from M onto $\widetilde{M}$ which cannot be approximated by Lipschitz mappings, a situation which never occurs in the case of approximation of mappings between smooth manifolds (see Proposition 2.2).

Another interesting question is the stability of density of Lipschitz mappings with respect to bi-Lipschitz modifications of the target.

Assume that X and Y are compact subsets of $\mathbb{R}^\nu$ that are bi-Lipschitz homeomorphic. Assume that M is a closed n-dimensional manifold and Lipschitz mappings $\mathrm{Lip}\,(M,X)$ *are dense in* $W^{1,p}(M,X)$ *for some* $1 \leqslant p < \infty$. *Are the Lipschitz mappings* $\mathrm{Lip}\,(M,Y)$ *dense in* $W^{1,p}(M,Y)$*?*

Since bi-Lipschitz invariance is a fundamental principle in geometric analysis on metric spaces, one expects basic theorems and definitions to remain unchanged when the ambient space is subject to a bi-Lipschitz transformation. Although the composition with a Lipschitz mapping is not continuous in the Sobolev norm, there are several reasons to expect a positive answer to remain in accordance with the principle.

First, if $\Phi : X \to Y$ is a bi-Lipschitz mapping, then $T(f) = \Phi \circ f$ induces bijections

$$T : W^{1,p}(M,X) \to W^{1,p}(M,Y), \quad T : \mathrm{Lip}\,(M,X) \to \mathrm{Lip}\,(M,Y).$$

Second, have the following positive result [31].

Theorem 3.9. *If Lipschitz mappings* $\mathrm{Lip}\,(M,X)$ *are dense in* $W^{1,p}(M,X)$ *in the following strong sense: for every* $\varepsilon > 0$ *there is* $f_\varepsilon \in \mathrm{Lip}\,(M,X)$ *such that* $|\{x|\, f_\varepsilon(x) \neq f(x)\}| < \varepsilon$ *and* $\|f - f_\varepsilon\|_{1,p} < \varepsilon$, *then Lipschitz mappings are dense in* $W^{1,p}(M,Y)$.

The strong approximation property described in the theorem is quite natural because if $f \in W^{1,p}(M,\mathbb{R}^\nu)$, then for every $\varepsilon > 0$ there is a Lipschitz mapping $f_\varepsilon \in \mathrm{Lip}\,(M,\mathbb{R}^\nu)$ such that $|\{x|\, f_\varepsilon(x) \neq f(x)\}| < \varepsilon$ and $\|f - f_\varepsilon\|_{1,p} < \varepsilon$. Such an approximation argument was employed in the proof of Theorem 3.7.

The above facts are convincing reasons to believe that the answer to the stability question should be positive. Surprisingly it is not. The following counterexample was constructed in [30].

Theorem 3.10. *Fix an integer* $n \geqslant 2$. *There is a compact and connected set* $X \subset \mathbb{R}^{n+2}$ *and a global bi-Lipschitz homeomorphism* $\Phi : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ *with* *the property that for any closed n-dimensional manifold M smooth mappings $C^\infty(M,X)$ are dense in $W^{1,n}(M,X)$, but Lipschitz mappings* $\mathrm{Lip}\,(M,Y)$ *are not dense in $W^{1,n}(M,Y)$, where $Y=\Phi(X)$.*

By smooth mappings $C^\infty(M,X)$ we mean smooth mappings from M to $\mathbb{R}^{n+2}$ with the image contained in X.

The space X constructed in the proof is quite irregular: it is the closure of a carefully constructed sequence of smooth submanifolds that converges to a manifold with a point singularity and all the manifolds are connected by a fractal curve. The space X looks like a stack of pancakes. The proof involves also a construction of a mapping $f \in W^{1,p}(M,X)$ which can be approximated by Lipschitz mappings, but the mappings that approximate f do not coincide with f at any point, so the strong approximation property from Theorem 3.9 is not satisfied.

4 Sobolev Spaces on Metric Measure Spaces

In order to define the space of Sobolev mappings between metric spaces, we need first define Sobolev spaces on metric spaces equipped with so-called doubling measures. By the end of the 1970s, it was discovered that a substantial part of harmonic analysis could be generalized such spaces [14]. This included the study of maximal functions, Hardy spaces and BMO, but it was only the zeroth order analysis in the sense that no derivatives were involved. The study of the first order analysis with suitable generalizations of derivatives, fundamental theorem of calculus, and Sobolev spaces, in the setting of metric spaces with a doubling measure was developed since the 1990s. This area is growing and plays an important role in many areas of the contemporary mathematics [43].

We recommend the reader a beautiful expository paper of Heinonen [44], where the significance and broad scope of applications of the first order analysis on metric spaces is carefully explained.

We precede the definition of the Sobolev space with auxiliary definitions and results. The material of Sects. 4.1–4.5 is standard by now. In our presentation, we follow [29], where the reader can find detailed proofs.

4.1 Integration on rectifiable curves

Let (X,d) be a metric space. By a *curve* in X we mean any continuous mapping $\gamma:[a,b]\to X$. The *image of the curve* is denoted by $|\gamma|=\gamma([a,b])$. The *length* of γ is defined by

$$\ell(\gamma) = \sup\left\{\sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))\right\},$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \ldots < t_n = b$. We say that the curve is *rectifiable* if $\ell(\gamma) < \infty$. The *length function* associated with a rectifiable curve $\gamma : [a, b] \to X$ is $s_\gamma : [a, b] \to [0, \ell(\gamma)]$ given by $s_\gamma(t) = \ell(\gamma|_{[a,t]})$. Not surprisingly, the length function is nondecreasing and continuous.

It turns out that every rectifiable curve admits the *arc-length* parametrization.

Theorem 4.1. *If $\gamma : [a, b] \to X$ is a rectifiable curve, then there is a unique curve $\widetilde{\gamma} : [0, \ell(\gamma)] \to X$ such that*

$$\gamma = \widetilde{\gamma} \circ s_\gamma. \tag{4.1}$$

Moreover, $\ell(\widetilde{\gamma}|_{[0,t]}) = t$ for every $t \in [0, \ell(\gamma)]$. In particular, $\widetilde{\gamma} : [0, \ell(\gamma)] \to X$ is a 1-Lipschitz mapping.

We call $\widetilde{\gamma}$ parametrized by the *arc-length* because $\ell(\widetilde{\gamma}|_{[0,t]}) = t$ for $t \in [0, \ell(\gamma)]$.

Now we are ready to define the integrals along the rectifiable curves. Let $\gamma : [a, b] \to X$ be a rectifiable curve, and let $\varrho : |\gamma| \to [0, \infty]$ be a Borel measurable function, where $|\gamma| = \gamma([a, b])$. Then we define

$$\int_\gamma \varrho := \int_0^{\ell(\gamma)} \varrho(\widetilde{\gamma}(t))\, dt,$$

where $\widetilde{\gamma} : [0, \ell(\gamma)] \to X$ is the arc-length parametrization of γ.

It turns out that we can nicely express this integral in any Lipschitz parametrization of γ.

Theorem 4.2. *For every Lipschitz curve $\gamma : [a, b] \to X$ the speed*

$$|\dot{\gamma}|(t) := \lim_{h\to 0} \frac{d(\gamma(t + h), \gamma(t))}{|h|},$$

exists almost everywhere and

$$\ell(\gamma) = \int_a^b |\dot{\gamma}|(t)\, dt. \tag{4.2}$$

Theorem 4.3. *Let $\gamma : [a, b] \to X$ be a Lipschitz curve, and let $\varrho : |\gamma| \to [0, \infty]$ be Borel measurable. Then*

$$\int_\gamma \varrho = \int_a^b \varrho(\gamma(t))|\dot{\gamma}|(t)\,dt.$$

4.2 Modulus

In the study of geometric properties of Sobolev functions on Euclidean spaces, the absolute continuity on almost all lines plays a crucial role. Thus, there is a need to define a notion of almost all curves also in the setting of metric spaces. This leads to the notion of the modulus of the family of rectifiable curves, which is a kind of a measure in the space of all rectifiable curves.

Let (X, d, μ) be a *metric measure space*, i.e., a metric space with a Borel measure that is positive and finite on every ball.

Let $\mathfrak{M}$ denote the family of all nonconstant rectifiable curves in X. It may happen that $\mathfrak{M} = \varnothing$, but we are interested in metric spaces for which the space $\mathfrak{M}$ is sufficiently large.

For $\Gamma \subset \mathfrak{M}$, let $F(\Gamma)$ be the family of all Borel measurable functions $\varrho : X \to [0, \infty]$ such that

$$\int_\gamma \varrho \geqslant 1 \quad \text{for every } \gamma \in \Gamma.$$

Now for each $1 \leqslant p < \infty$ we define

$$\operatorname{Mod}_p(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_X \varrho^p \, d\mu.$$

The number $\operatorname{Mod}_p(\Gamma)$ is called *p-modulus* of the family Γ.

The following result is easy to prove.

Theorem 4.4. Mod_p *is an outer measure on* $\mathfrak{M}$.

If some property holds for all curves $\gamma \in \mathfrak{M} \setminus \Gamma$, where $\operatorname{Mod}_p(\Gamma) = 0$, then we say that the property holds for *p-a.e. curve*.

The notion of p-a.e.curve is consistent with the notion of almost every line parallel to a coordinate axis. Indeed, if $E \subset [0, 1]^{n-1}$ is Borel measurable and we consider straight segments passing through E

$$\Gamma_E = \{\gamma_{x'} : [0, 1] \to [0, 1]^n : \gamma_{x'}(t) = (t, x'),\ x' \in E\}$$

then $\operatorname{Mod}_p(\Gamma_E) = 0$ if and only if the $(n-1)$-dimensional Lebesgue measure of E is zero. This fact easily follows from the definition of the modulus and the Fubini theorem.

4.3 Upper gradient

As before, we assume that (X, d, μ) is a metric measure space. Let $u : X \to \mathbb{R}$ be a Borel function. Following [46], we say that a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if

$$|u(\gamma(a)) - u(\gamma(b))| \leqslant \int_\gamma g \tag{4.3}$$

for *every* rectifiable curve $\gamma : [a, b] \to X$. We say that g is a *p-weak upper gradient* of u if (4.3) holds on p-a.e. curve $\gamma \in \mathfrak{M}$.

If g is an upper gradient of u and $\widetilde{g} = g$, μ-a.e., is another nonnegative Borel function, then it may be that $\widetilde{g}$ is no longer upper gradient of u. However, we have the following assertion.

Lemma 4.5. *If g is a p-weak upper gradient of u and $\widetilde{g}$ is another nonnegative Borel function such that $\widetilde{g} = g$ μ-a.e., then $\widetilde{g}$ is a p-weak upper gradient of u too.*

It turns out that p-weak upper gradients can be approximated in the L^p norm by upper gradients.

Lemma 4.6. *If g is a p-weak upper gradient of u which is finite almost everywhere, then for every $\varepsilon > 0$ there is an upper gradient g_ε of u such that*

$$g_\varepsilon \geqslant g \text{ everywhere} \qquad \text{and} \qquad \|g_\varepsilon - g\|_{L^p} < \varepsilon.$$

We do not require here that $g \in L^p$.

The following result shows that the notion of an upper gradient is a natural generalization of the length of the gradient to the setting of metric spaces (see also Theorem 4.10).

Proposition 4.7. *If $u \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$, then $|\nabla u|$ is an upper gradient of u. This upper gradient is the least one in the sense that if $g \in L^1_{\mathrm{loc}}(\Omega)$ is another upper gradient of u, then $g \geqslant |\nabla u|$ almost everywhere.*

4.4 Sobolev spaces $N^{1,p}$

Let $\widetilde{N}^{1,p}(X, d, \mu)$, $1 \leqslant p < \infty$, be the class of all L^p integrable Borel functions on X for which there exists a p-weak upper gradient in L^p. For $u \in \widetilde{N}^{1,p}(X, d, \mu)$ we define

$$\|u\|_{\widetilde{N}^{1,p}} = \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all p-weak upper gradients g of u.

Lemma 4.6 shows that in the definition of $\widetilde{N}^{1,p}$ and $\|\cdot\|_{\widetilde{N}^{1,p}}$, p-weak upper gradients can be replaced by upper gradients.

We define the equivalence relation in $\widetilde{N}^{1,p}$ as follows: $u \sim v$ if and only if $\|u - v\|_{\widetilde{N}^{1,p}} = 0$. Then the space $N^{1,p}(X, d, \mu)$ is defined as the quotient $\widetilde{N}^{1,p}(X, d, \mu)/\sim$ and is equipped with the norm

$$\|u\|_{N^{1,p}} := \|u\|_{\widetilde{N}^{1,p}}.$$

The space $N^{1,p}$ was introduced by Shanmugalingam [77].

Theorem 4.8. *$N^{1,p}(X, d, \mu)$, $1 \leqslant p < \infty$, is a Banach space.*

One can prove that functions $u \in N^{1,p}(X, d, \mu)$ are absolutely continuous on almost all curves in the sense that for p-a.e. $\gamma \in \mathfrak{M}$, $u \circ \widetilde{\gamma}$ is absolutely continuous, where $\widetilde{\gamma}$ is the arc-length parametrization of γ. This fact, Proposition 4.7, and the characterization of the classical Sobolev space $W^{1,p}(\Omega)$, by the absolute continuity on lines, lead to the following result.

Theorem 4.9. *If $\Omega \subset \mathbb{R}^n$ is open and $1 \leqslant p < \infty$, then*

$$N^{1,p}(\Omega, |\cdot|, \mathcal{L}^n) = W^{1,p}(\Omega)$$

and the norms are equal.

Here, we consider the space $N^{1,p}$ on Ω regarded as a metric space with respect to the Euclidean metric $|\cdot|$ and the Lebesgue measure $\mathcal{L}^n$. The following result supplements the above theorem.

Theorem 4.10. *Any function $u \in W^{1,p}(\Omega)$, $1 \leqslant p < \infty$, has a representative for which $|\nabla u|$ is a p-weak upper gradient. On the other hand, if $g \in L^1_{\rm loc}$ is a p-weak upper gradient of u, then $g \geqslant |\nabla u|$ almost everywhere.*

Both above theorems hold also when Ω is replaced by a Riemannian manifold, and also, in this case, $|\nabla u|$ is the least p-weak upper gradient of $u \in W^{1,p}$. Actually, one can prove that there always exists a minimal p-weak upper gradient.

Theorem 4.11. *For any $u \in N^{1,p}(X, d, \mu)$ and $1 \leqslant p < \infty$ there exists the least p-weak upper gradient $g_u \in L^p$ of u. It is smallest in the sense that if $g \in L^p$ is another p-weak upper gradient of u, then $g \geqslant g_u$ μ-a.e.*

4.5 Doubling measures

We say that a measure μ is *doubling* if there is a constant $C_d \geqslant 1$ (called *doubling constant*) such that $0 < \mu(2B) \leqslant C_d\mu(B) < \infty$ for every ball $B \subset X$.

We say that a metric space X is *metric doubling* if there is a constant $M > 0$ such that every ball in X can be covered by at most M balls of half the radius.

If μ is a doubling measure on X, then it easily follows that X is metric doubling. In particular, bounded sets in X are totally bounded. Hence, if X is a complete metric space equipped with a doubling measure, then bounded and closed sets are compact.

The following beautiful characterization of metric spaces supporting doubling measures was proved by Volberg and Konyagin [62, 82].

Theorem 4.12. *Let X be a complete metric space. Then there is a doubling measure on X if and only if X is metric doubling.*

The doubling condition implies a lower bound for the measure of a ball.

Lemma 4.13. *If the measure μ is doubling with the doubling constant C_d and $s = \log_2 C_d$, then*

$$\frac{\mu(B(x,r))}{\mu(B_0)} \geqslant 4^{-s}\left(\frac{r}{r_0}\right)^s \tag{4.4}$$

whenever B_0 is a ball of radius r_0, $x \in B_0$ and $r \leqslant r_0$.

The lemma easily follows from the iteration of the doubling inequality. The exponent s is sharp as the example of the Lebesgue measure shows.

Metric spaces equipped with a doubling measure are called *spaces of homogeneous type* and $s = \log_2 C_d = \log C_d / \log 2$ is called *homogeneous dimension.*

An important class of doubling measures is formed by the so-called *n-regular* measures[5], which are measures for which there are constants $C \geqslant 1$ and $s > 0$ such that $C^{-1}r^s \leqslant \mu(B(x,r)) \leqslant Cr^s$ for all $x \in X$ and $0 < r < \operatorname{diam} X$. The s-regular measures are closely related to the Hausdorff measure $\mathcal{H}^s$ since we have the following assertion.

Theorem 4.14. *If μ is an s-regular measure, then there is a constant $C \geqslant 1$ such that $C^{-1}\mu(E) \leqslant \mathcal{H}^s(E) \leqslant C\mu(E)$ for every Borel set $E \subset X$. In particular, $\mathcal{H}^s$ is s-regular too.*

The proof is based on the so-called $5r$-covering lemma.

For a locally integrable function $g \in L^1_{\mathrm{loc}}(\mu)$ we define the *Hardy–Littlewood maximal function*

$$\mathcal{M}g(x) = \sup_{r>0} \fint_{B(x,r)} |g|\, d\mu\,.$$

[5] Called also *Ahlfors–David* regular measures.

Theorem 4.15. *If μ is doubling, then*

1) $\mu(\{x : \mathcal{M}g(x) > t\}) \leqslant Ct^{-1} \int_X |g| d\mu$ *for every* $t > 0$;

2) $\|\mathcal{M}g\|_{L^p} \leqslant C\|g\|_{L^p}$, *for* $1 < p < \infty$.

4.6 Other spaces of Sobolev type

There are many other definitions of Sobolev type spaces on metric spaces that we describe now (see [19, 27, 29, 32, 33]). Let (X, d, μ) be a metric measure space with a doubling measure.

Following [27], for $0 < p < \infty$ we define $M^{1,p}(X, d, \mu)$ to be the set of all functions $u \in L^p(\mu)$ for which there is $0 \leqslant g \in L^p(\mu)$ such that

$$|u(x) - u(y)| \leqslant d(x, y)(g(x) + g(y)) \quad \mu\text{-a.e.} \tag{4.5}$$

Then we set

$$\|u\|_{M^{1,p}} = \|u\|_p + \inf_g \|g\|_p$$

where the infimum is taken over the class of all g satisfying (4.5). For $p \geqslant 1$, $\|\cdot\|_{1,p}$ is a norm and $M^{1,p}(X, d, \mu)$ is a Banach space.

For a locally integrable function u we define the *Calderón maximal function*

$$u_1^{\#}(x) = \sup_{r>0} \ r^{-1} \fint_{B(x,r)} |u - u_B| \, d\mu \, .$$

Following [32], we define $C^{1,p}(X, d, \mu)$ to be the class of all $u \in L^p(\mu)$ such that $u_1^{\#} \in L^p(\mu)$. Again, for $p \geqslant 1$, $C^{1,p}(X, d, \mu)$ is a Banach space with respect to the norm

$$\|u\|_{C^{1,p}} = \|u\|_p + \|u_1^{\#}\|_p \, .$$

Following [33], for $0 < p < \infty$ we say that a locally integrable function $u \in L^1_{\text{loc}}$ belongs to the space $P^{1,p}(X, d, \mu)$ if there are $\sigma \geqslant 1$ and $0 \leqslant g \in L^p(\mu)$ such that

$$\fint_B |u - u_B| \, d\mu \leqslant r \left(\fint_{\sigma B} g^p \, d\mu \right)^{1/p} \quad \text{for every ball } B \text{ of radius } r. \tag{4.6}$$

We do not equip the space $P^{1,p}$ with a norm.

To motivate the above definitions, we observe that $u \in W^{1,p}(\mathbb{R}^n)$ satisfies the pointwise inequality

$$|u(x) - u(y)| \leqslant C|x - y|(\mathcal{M}|\nabla u|(x) + \mathcal{M}|\nabla u|(y)) \quad \text{a.e.,}$$

where $\mathcal{M}|\nabla u|$ is the Hardy–Littlewood maximal function, so $g = \mathcal{M}|\nabla u| \in L^p$ for $p > 1$, and actually one can prove [27] that $u \in W^{1,p}(\mathbb{R}^n)$, $p > 1$, if and only if $u \in L^p$ and there is $0 \leqslant g \in L^p$ such that $|u(x) - u(y)| \leqslant |x-y|(g(x)+g(y))$ almost everywhere. Moreover, $\|u\|_{1,p} \approx \|u\|_p + \inf_g \|g\|_p$. Thus, for $p > 1$, $W^{1,p}(\mathbb{R}^n) = M^{1,p}(\mathbb{R}^n)$. In the case $p = 1$, $M^{1,1}(\mathbb{R}^n)$ is not equivalent with $W^{1,1}(\mathbb{R}^n)$ [28] (see, however, [57] and Theorem 4.16 below).

The classical Poincaré inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \leqslant Cr \fint_{B(x,r)} |\nabla u| \tag{4.7}$$

implies that for $u \in W^{1,p}(\mathbb{R}^n)$ the Calderón maximal function is bounded by the maximal function of $|\nabla u|$ and hence it belongs to L^p for $p > 1$. Calderón [10] proved that for $p > 1$, $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p$ and $u_1^{\#} \in L^p$. Moreover, $\|u\|_{1,p} \approx \|u\|_p + \|u_1^{\#}\|_p$. Thus, for $p > 1$, $W^{1,p}(\mathbb{R}^n) = C^{1,p}(\mathbb{R}^n)$.

The inequality (4.7) also implies that for $p \geqslant 1$, $\sigma \geqslant 1$, and $u \in W^{1,p}(\mathbb{R}^n)$ we have

$$\fint_B |u - u_B|\, dx \leqslant Cr \left(\fint_{\sigma B} |\nabla u|^p\, dx \right)^{1/p} .$$

Thus, $W^{1,p}(\mathbb{R}^n) \subset P^{1,p} \cap L^p$. On the other hand, it was proved in [56, 19, 28] that $W^{1,p}(\mathbb{R}^n) = P^{1,p} \cap L^p$ for $p \geqslant 1$.

In the case of general metric spaces, we have the following assertion.

Theorem 4.16. *If the measure μ is doubling and $1 \leqslant p < \infty$, then*

$$C^{1,p}(X,d,\mu) = M^{1,p}(X,d,\mu) \subset P^{1,p}(X,d,\mu) \cap L^p(\mu) \subset N^{1,p}(X,d,\mu) .$$

For a proof see [29, Corollary 10.5 and Theorem 9.3], [32, Theorem 3.4 and Lemma 3.6], and [71].

The so-called telescoping argument (infinite iteration of the inequality (4.6) on a decreasing sequence of balls) shows that if $u \in P^{1,p}(X,d,\mu)$, then

$$|u(x) - u(y)| \leqslant Cd(x,y)((\mathcal{M}g^p(x))^{1/p} + (\mathcal{M}g^p(y))^{1/p}) \quad \text{a.e.} \tag{4.8}$$

(see [33]). A version of the same telescoping argument shows also that for $u \in L^1_{\rm loc}$

$$|u(x) - u(y)| \leqslant Cd(x,y)(u_1^{\#}(x) + u_1^{\#}(y)) \quad \text{a.e.}$$

(see [32, Lemma 3.6]). This implies that $C^{1,p} \subset M^{1,p}$ for $p \geqslant 1$. On the other hand, if $u \in M^{1,p}$ and $|u(x) - u(y)| \leqslant d(x,y)(g(x) + g(y))$, then a direct integration with respect to x and y yields

$$\fint_B |u - u_B|\, d\mu \leqslant 4r \fint_B g\, d\mu \leqslant 4r \left(\fint_B g^p\, d\mu \right)^{1/p}.$$

Hence $M^{1,p} \subset P^{1,p} \cap L^p$ and also

$$u_1^{\#} \leqslant 4\mathcal{M}g,$$

which shows that $M^{1,p} \subset C^{1,p}$ for $p > 1$. Thus, $C^{1,p} = M^{1,p}$ for $p > 1$. The case $p = 1$ of this equality is more difficult (see [29, Theorem 9.3] and [71]).

For the proof of the remaining inclusion $P^{1,p} \cap L^p \subset N^{1,p}$ see [29, Corollary 10.5].

If a metric space X has no nonconstant rectifiable curves, then $g = 0$ is an upper gradient of any $u \in L^p$ and hence $N^{1,p}(X, d, \mu) = L^p(\mu)$. On the other hand, the theory of Sobolev spaces $M^{1,p}$, $C^{1,p}$, and $P^{1,p}$ is not trivial in this case. Indeed, a variant of the above telescoping argument leads to the estimate of $|u - u_B|$ by a generalized Riesz potential [33], and hence the fractional integration theorem implies Sobolev embedding theorems. Many results of the classical theory of Sobolev spaces extend to this situation (see, for example, [27, 33, 29]), and we state just one of them.

Theorem 4.17. *Let μ be a doubling measure, and let $s = \log C_d / \log 2$ be the same as in Lemma 4.13. If $u \in L^1_{\mathrm{loc}}(\mu)$, $\sigma \geqslant 1$, and $0 \leqslant g \in L^p(\mu)$, $0 < p < s$ are such that the p-Poincaré inequality*

$$\fint_B |u - u_B|\, d\mu \leqslant r \left(\fint_{\sigma B} g^p\, d\mu \right)^{1/p}$$

holds on every ball B of radius r, then for any $p < q < s$ the Sobolev–Poincaré inequality

$$\left(\fint_B |u - u_B|^{q^*}\, d\mu \right)^{1/q^*} \leqslant Cr \left(\fint_{5\sigma B} g^q\, d\mu \right)^{1/q}$$

holds on every ball B of radius r, where $q^* = sq/(s - q)$ is the Sobolev exponent.

This result implies Sobolev embedding for the spaces $C^{1,p}$, $M^{1,p}$, and $P^{1,p}$, but not for $N^{1,p}$.

Other results for $C^{1,p}$, $M^{1,p}$, and $P^{1,p}$ spaces available in the general case of metric spaces with doubling measure include Sobolev embedding into Hölder continuous functions, Trudinger inequality, compact embedding theorem, embedding on spheres, and extension theorems.

4.7 Spaces supporting the Poincaré inequality

Metric spaces equipped with doubling measures are too general for the theory of $N^{1,p}$ spaces to be interesting. Indeed, if there are no nonconstant rectifiable curves in X, then, as we have already observed, $N^{1,p}(X,d,\mu) = L^p(\mu)$. Thus, we need impose additional conditions on the metric space that will imply, in particular, the existence of many rectifiable curves. Such a condition was discovered by Heinonen and Koskela [46].

We say that (X,d,μ) *supports a p-Poincaré inequality*, $1 \leqslant p < \infty$, if the measure μ is doubling and there exist constants C_P and $\sigma \geqslant 1$ such that for every ball $B \subset X$, every Borel measurable function $u \in L^1(\sigma B)$, and every upper gradient $0 \leqslant g \in L^p(\sigma B)$ of u on σB the following Poincaré type inequality is satisfied:

$$\fint_B |u - u_B|\, d\mu \leqslant C_P r \left(\fint_{\sigma B} g^p\, d\mu \right)^{1/p}. \tag{4.9}$$

Note that this condition immediately implies the existence of rectifiable curves. Indeed, if u is not constant, then $g = 0$ cannot be an upper gradient of u; otherwise, the inequality (4.9) would not be satisfied. More precisely, we have the following assertion (see, for example, [33, Proposition 4.4]).

Theorem 4.18. *If a space X supports a p-Poincaré inequality, then there is a constant $C > 0$ such that any two points $x, y \in X$ can be connected by a curve of length less than or equal to $Cd(x,y)$.*

Clearly, $\mathbb{R}^n$ supports the p-Poincaré inequality for all $1 \leqslant p < \infty$. Another example of spaces supporting Poincaré inequalities is provided by Riemannian manifolds of nonnegative Ricci curvature [8, 72]. There are, however, many examples of spaces supporting Poincaré inequalities which carry some mild geometric structure, but do not resemble Riemannian manifolds [7, 45, 46, 59, 60, 75]. An important class of spaces that support the p-Poincaré inequality is provided by the so-called *Carnot groups* [23, 68, 9] and more general Carnot–Carathéodory spaces [22, 23]. For the sake of simplicity, only the simplest case of the Heisenberg group is described here.

The Heisenberg group $\mathbb{H}_1$ can be identified with $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ equipped with the noncommutative group law $(z_1,t_1)\cdot(z_2,t_2) = (z_1+z_2, t_1+t_2+2\mathrm{Im}\,(z_1\overline{z}_2))$. It is equipped with a non-Riemannian metric $d(x,y) = \|a^{-1} \cdot b\|$, where $\|(z,t)\| = (|z|^4 + t^2)^{1/2}$. This metric is bi-Lipschitz equivalent to another so-called Carnot–Carathéodory metric. The metric d is quite exotic because the Hausdorff dimension of $(\mathbb{H}_1, d)$ is 4, while topological dimension is 3. The applications of the Heisenberg group include several complex variables, subelliptic equations and noncommutative harmonic analysis [78]. More recently, it was a subject of an intense study from the perspective of geometric measure theory [21, 9].

If a space (X, d, μ) supports the p-Poincaré inequality, $u \in N^{1,p}(X, d, \mu)$, and $0 \leqslant g \in L^p(\mu)$ is an upper gradient of u, then the p-Poincaré inequality (4.9) is satisfied and hence the Sobolev embedding (Theorem 4.17) holds. One can actually prove that, in this case, we can take $q = p$, i.e.,

$$\left(\fint_B |u - u_B|^{p^*} \, d\mu \right)^{1/p^*} \leqslant Cr \left(\fint_{5\sigma B} g^p \, d\mu \right)^{1/p}$$

on every ball B of radius r, where $1 \leqslant p < s$ and $p^* = sp/(s-p)$ (see [33]).

A direct application of the Hölder inequality shows that if a space supports a p-Poincaré inequality, then it also supports a q-Poincaré inequality for all $q > p$. On the other hand, we have the following important result of Keith and Zhong [54].

Theorem 4.19. *If a complete metric measure space supports a p-Poincaré inequality for some $p > 1$, then it also supports a q-Poincaré inequality for some $1 \leqslant q < p$.*

This important result implies that, in the case of spaces supporting the p-Poincaré inequality, other approaches to Sobolev spaces described in the previous section are equivalent.

Theorem 4.20. *If the space supports the p-Poincaré inequality, $1 < p < \infty$, then $C^{1,p}(X, d, \mu) = M^{1,p}(X, d, \mu) = P^{1,p}(X, d, \mu) \cap L^p(\mu) = N^{1,p}(X, d, \mu)$.*

Indeed, prior to the work of Keith and Zhong it was known that the spaces are equal provided that the space supports the q-Poincaré inequality for some $1 \leqslant q < p$ (see, for example, [29, Theorem 11.3]).

Spaces supporting Poincaré inequalities play a fundamental role in the modern theory of quasiconformal mappings [46, 47], geometric rigidity problems [7], nonlinear subelliptic equations (see, for example, [11, 22, 20, 33, 34]), and nonlinear potential theory [1, 6].

Although the known examples show that spaces supporting a Poincaré inequality can be very exotic, surprisingly, one can prove that such spaces are always equipped with a weak differentiable structure [13, 53].

5 Sobolev Mappings between Metric Spaces

Throughout this section, we assume that (X, d, μ) is a metric measure space equipped with a doubling measure. Let Y be another metric space. The construction of the space of Sobolev mappings between metric spaces $N^{1,p}(X, Y)$ is similar to that in Sect. 3 with the difference that the classical Sobolev space is replaced by the Sobolev space $N^{1,p}$. The space $N^{1,p}(X, Y)$ was introduced in [47].

Let V be a Banach space. Following [47], we say that $F \in \widetilde{N}^{1,p}(X,V)$ if $F \in L^p(X,V)$ (in the Bochner sense) and there is a Borel measurable function $0 \leqslant g \in L^p(\mu)$ such that

$$\|F(\gamma(a)) - F(\gamma(b))\| \leqslant \int_\gamma g$$

for every rectifiable curve $\gamma : [a,b] \to X$. We call g an *upper gradient* of F. We also define

$$\|F\|_{1,p} = \|F\|_p + \inf_g \|g\|_p,$$

where the infimum is taken over all upper gradients of F. Now we define $N^{1,p}(X,V) = \widetilde{N}^{1,p}(X,V)/\sim$, where $F_1 \sim F_2$ when $\|F_1 - F_2\|_{1,p} = 0$.

As in the case of $N^{1,p}(X,d,\mu)$ spaces, the p-upper gradient can be replaced by p-weak upper gradient in the above definition. The following two results were proved in [47] (see also [31] for Theorem 5.2).

Theorem 5.1. *$N^{1,p}(X,V)$ is a Banach space.*

Theorem 5.2. *Suppose that the space (X,d,μ) supports the p-Poincaré inequality for some $1 \leqslant p < \infty$. Then for every Banach space V the pair (X,V) supports the p-Poincaré inequality in the following sense: there is a constant $C > 0$ such that for every ball $B \subset X$, for every $F \in L^1(6\sigma B, V)$, and for every $0 \leqslant g \in L^p(6\sigma B)$ being a p-weak upper gradient of F on $6\sigma B$ the following inequality is satisfied:*

$$⨍_B \|F - F_B\| \, d\mu \leqslant C(\operatorname{diam} B) \left(⨍_{6\sigma B} g^p \, d\mu \right)^{1/p}. \tag{5.1}$$

The Poincaré inequality (5.1) and the standard telescoping argument implies the following pointwise inequality: if $F \in N^{1,p}(X,V)$ and $0 \leqslant g \in L^p(\mu)$ is a p-weak upper gradient of F, then

$$\|F(x) - F(y)\| \leqslant Cd(x,y)((\mathcal{M}g^p(x))^{1/p} + (\mathcal{M}g^p(y))^{1/p})$$

almost everywhere, where, on the right-hand side, we have the maximal function, just like in the case of the equality (4.8).

In particular, F restricted to the set $E_t = \{x : \mathcal{M}g^p < t^p\}$ is Lipschitz continuous with the Lipschitz constant Ct. Using the Lipschitz extension of $F|_{E_t}$ to the entire space X (McShane extension), one can prove [31] the following assertion.

Theorem 5.3. *Suppose that the space (X,d,μ) supports the p-Poincaré inequality for some $1 \leqslant p < \infty$ and V is a Banach space. If $F \in N^{1,p}(X,V)$,* *then for every* $\varepsilon > 0$ *there is a Lipschitz mapping* $G \in \mathrm{Lip}\,(X, V)$ *such that* $\mu\{x : F(x) \neq G(x)\} < \varepsilon$ *and* $\|F - G\|_{1,p} < \varepsilon$*.*

As we have seen in the previous section, the Poincaré inequality plays a crucial role in the development of the theory of Sobolev spaces on metric spaces. Since such an inequality is also valid for $N^{1,p}(X, V)$ spaces, Theorem 5.2, many results true for $N^{1,p}(X, d, \mu)$ like, for example, Sobolev embedding theorems can be generalized to $N^{1,p}(X, V)$ spaces (see [47]).

Now, if Y is a metric space isometrically embedded into a Banach space V, $Y \subset V$, we define

$$N^{1,p}(X, Y) = \{F \in N^{1,p}(X, V) : F(X) \subset Y\}\,.$$

Since $N^{1,p}(X, V)$ is a Banach space, $N^{1,p}(X, Y)$ is equipped with a norm metric.

If X is an open set in $\mathbb{R}^n$ or X is a compact manifold, then the space $N^{1,p}(X, d, \mu)$ is equivalent with the classical Sobolev space (see Theorem 4.9). Hence, in this case, the definition of $N^{1,p}(\Omega, Y)$ (or $N^{1,p}(M, Y)$) is equivalent with that of $R^{1,p}(\Omega, Y)$ (or $R^{1,p}(M, Y)$) described in Sect. 3 (see [47]).

If $F \in N^{1,p}(X, Y)$, then, according to Theorem 5.3, F can be approximated by Lipschitz mappings $\mathrm{Lip}\,(X, V)$ and the question is: Under what conditions F can be approximated by $\mathrm{Lip}\,(X, Y)$ mappings?

This is a question about extension of the theory described in Sect. 2 to the case of Sobolev mappings between metric spaces and it was formulated explicitly by Heinonen, Koskela, Shanmugalingam, and Tyson [47, Remark 6.9].

An answer to this question cannot be easy because, as soon as we leave the setting of manifolds, we have many unpleasant counterexamples like those in Sect. 3. A particularly dangerous situation is created by the lack of stability with respect to bi-Lipschitz deformations of the target (Theorem 3.10). Indeed, in most situations, there is no canonical way to choose a metric on Y and we are free to choose any metric in the class of bi-Lipschitz equivalent metrics.

An example of spaces supporting the p-Poincaré inequality is provided by the Heisenberg group and, more generally, Carnot groups and Carnot–Carathéodory spaces. In this setting, Gromov [23, Sect. 2.5.E] stated as an open problem the extension of the results from Sect. 2 to the case of mappings from Carnot–Carathéodory spaces to Riemannian manifolds. Thus, the question of Heinonen, Koskela, Shanmugalingam, and Tyson can be regarded and a more general form of Gromov's problem.

The following result was proved in [31] (see Theorem 3.9 above).

Theorem 5.4. *Suppose that* (X, d, μ) *is a doubling metric measure space of finite measure* $\mu(X) < \infty$ *and* Y_1, Y_2 *are two bi-Lipschitz homeomorphic metric spaces of finite diameter isometrically embedded into Banach spaces* V_1 *and* V_2 *respectively. Suppose that Lipschitz mappings* $\mathrm{Lip}\,(X, Y_1)$ *are dense in* $N^{1,p}(X, Y_1)$, $1 \leqslant p < \infty$, *in the following strong sense*: *for any* $f \in$

$N^{1,p}(X, Y_1)$ and $\varepsilon > 0$ there is $f_\varepsilon \in \mathrm{Lip}\,(X, Y_1)$ such that $\mu(\{x : f(x) \neq f_\varepsilon(x)\}) < \varepsilon$ and $\|f - f_\varepsilon\|_{1,p} < \varepsilon$. Then the Lipschitz mappings $\mathrm{Lip}\,(X, Y_2)$ *are dense in $N^{1,p}(X, Y_2)$.*

This result shows that, in the case in which we can prove strong density, there is no problem with the bi-Lipschitz invariance of the density.

It turns out that also White's theorem (Theorem 2.5) and the density result of Schoen and Uhlenbeck (Theorems 2.1 and 3.7) can be generalized to the setting of mappings between metric spaces. Theorem 5.4 plays a crucial role in the proof.

Theorem 5.5. *Let (X, d, μ) be a metric measure space of finite measure $\mu(X) < \infty$ supporting the p-Poincaré inequality. If $p \geqslant s = \log C_d / \log 2$ and Y is a compact metric doubling space which is bi-Lipschitz homeomorphic to a Lipschitz neighborhood retract of a Banach space, then for every isometric embedding of Y into a Banach space Lipschitz mappings* $\mathrm{Lip}\,(X, Y)$ *are dense in $N^{1,p}(X, Y)$. Moreover, for every $f \in N^{1,p}(X, Y)$ there is $\varepsilon > 0$ such that if $f_1, f_2 \in \mathrm{Lip}\,(X, Y)$ satisfy $\|f - f_i\|_{1,p} < \varepsilon$, $i = 1, 2$, then the mappings f_1 and f_2 are homotopic.*

5.1 Lipschitz polyhedra

By a *simplicial complex* we mean a finite collection K of simplexes in some Euclidean space $\mathbb{R}^\nu$ such that

1) if $\sigma \in K$ and τ is a face of σ, then $\tau \in K$;

2) if $\sigma, \tau \in K$, then either $\sigma \cap \tau = \varnothing$ or $\sigma \cap \tau$ is a common face of σ and τ.

The set $|K| = \bigcup_{\sigma \in K} \sigma$ is called a *rectilinear polyhedron*. By a *Lipschitz polyhedron* we mean any metric space which is bi-Lipschitz homeomorphic to a rectilinear polyhedron. The main result of [31] reads as follows.

Theorem 5.6. *Let Y be a Lipschitz polyhedron, and let $1 \leqslant p < \infty$. Then the class of Lipschitz mappings* $\mathrm{Lip}\,(X, Y)$ *is dense in $N^{1,p}(X, Y)$ for every metric measure space X of finite measure that supports the p-Poincaré inequality if and only if $\pi_1(Y) = \pi_2(Y) = \ldots = \pi_{[p]}(Y) = 0$.*

Observe that the density of Lipschitz mappings does not depend on the particular choice of the metric in Y in the class of bi-Lipschitz equivalent metrics, only on the topology of Y. This is because, in the proof of Theorem 5.6, one shows the strong approximation property described in Theorem 5.4. Theorem 5.6 can be regarded as a partial answer to the problems of Heinonen, Koskela, Shanmugalingam, and Tyson and also to the problem of Gromov.

Acknowledgement. The work was supported by NSF (grant DMS-0500966).

References

1. Aikawa, H., Shanmugalingam, N.: Hölder estimates of p-harmonic extension operators. J. Differ. Equations **220**, 18-45 (2006)
2. Ambrosio, L.: Metric space valued functions of bounded variation. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **17**, 439-478 (1990)
3. Bethuel, F.: The approximation problem for Sobolev maps between two manifolds. Acta Math. **167**, 153-206 (1991)
4. Bethuel, F.: On the singular set of stationary harmonic maps. Manuscr. Math. **78**, 417-443 (1993)
5. Bethuel, F., Zheng, X.M.: Density of smooth functions between two manifolds in Sobolev spaces. J. Funct. Anal. **80**, 60-75 (1988)
6. Björn, A., Björn, J., Shanmugalingam, N.: The Perron method for p-harmonic functions in metric spaces. J. Differ. Equations **195**, 398-429 (2003)
7. Bourdon, M., Pajot, H.: Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings. Proc. Am. Math. Soc. **127**, 2315-2324 (1999)
8. Buser, P.: A note on the isoperimetric constant. Ann. Sci. École Norm. Sup. (4) **15**, 213-230 (1982)
9. Capogna, L., Danielli, D., Pauls, S.D., Tyson, J.T.: An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem. Birkhäuser, Basel (2007)
10. Calderón, A.P.: Estimates for singular integral operators in terms of maximal functions. Stud. Math. **44** , 563-582 (1972)
11. Capogna, L., Garofalo, N.: Regularity of minimizers of the calculus of variations in Carnot groups via hypoellipticity of systems of Hörmander type. J. Eur. Math. Soc. (JEMS) **5**, 1-40 (2003)
12. Capogna, L., Lin, F.-H.: Legendrian energy minimizers. I. Heisenberg group target. Calc. Var. Partial Differ. Equ. **12** , 145-171 (2001)
13. Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal. **9**, 428-517 (1999)
14. Coifman, R., Weiss, G.: Analyse Harmonique sur Certains Espaces Homogenes. Lect. Notes Math. **242**, Springer (1971)
15. Diestel, J., Uhl, J.J., Jr.: Vector Measures. Am. Math. Soc., Providence, RI (1977)
16. Duzaar, F., Mingione, G.: The p-harmonic approximation and the regularity of p-harmonic maps. Calc. Var. Partial Differ. Equ. **20**, 235-256 (2004)
17. Eells, J., Fuglede, B.: Harmonic Maps between Riemannian Polyhedra. Cambridge Univ. Press, Cambridge (2001)
18. Eells, J., Lemaire, L.: A report on harmonic maps. Bull. London Math. Soc. **10**, 1-68 (1978)
19. Franchi, B.; Hajłasz, P.; Koskela, P.: Definitions of Sobolev classes on metric spaces. Ann. Inst. Fourier (Grenoble) **49**, 1903-1924 (1999)
20. Franchi, B., Gutiérrez, C.E., Wheeden, R.L.: Weighted Sobolev–Poincaré inequalities for Grushin type operators. Commun. Partial Differ. Equ. **19**, 523-604 (1994)
21. Franchi, B., Serapioni, R., Serra Cassano, F.: Rectifiability and perimeter in the Heisenberg group. Math. Ann. **321**, 479-531 (2001)
22. Garofalo, N., Nhieu, D.-M.: Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. Commun. Pure Appl. Math. **49**, 1081-1144 (1996)

23. Gromov, M.: Carnot-Carathéodory spaces seen from within. In: Sub-Riemannian Geometry, pp. 79-323, Progr. Math. **144**, Birkhäuser, Basel (1996)
24. Gromov, M., Schoen, R.: Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. Inst. Hautes Études Sci. Publ. Math. **76**, 165-246 (1992)
25. Hajłasz, P.: Equivalent statement of the Poincaré conjecture, Ann. Mat. Pura Appl. **167**, 25-31 (1994)
26. Hajłasz, P.: Approximation of Sobolev mappings. Nonlinear Anal., Theory Methods Appl. **22**, 1579-1591 (1994)
27. Hajłasz, P.: Sobolev spaces on an arbitrary metric space. Potential Anal. **5**, 403-415 (1996)
28. Hajłasz, P.: A new characterization of the Sobolev space. Stud. Math. **159**, 263-275 (2003)
29. Hajłasz, P.: Sobolev spaces on metric measure spaces. In: Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces (Paris, 2002), pp. 173-218, Contemp. Math. **338**, Am. Math. Soc., Providence, RI (2003)
30. Hajłasz, P.: Sobolev mappings: Lipschitz density is not a bi-Lipschitz invariant of the target. Geom. Funct. Anal. **17**, 435-467 (2007)
31. Hajłasz, P. Density of Lipschitz mappings in the class of Sobolev mappings between metric spaces. Math. Ann. [To appear]
32. Hajłasz, P., Kinnunen, J.: Hölder quasicontinuity of Sobolev functions on metric spaces. Rev. Mat. Iberoam. **14**, 601-622 (1998)
33. Hajłasz, P., Koskela, P.: Sobolev met Poincaré. Mem. Am. Math. Soc. no. 688 (2000)
34. Hajłasz, P., Strzelecki, P.: Subelliptic p-harmonic maps into spheres and the ghost of Hardy spaces, Math. Ann. **312**, 341-362 (1998)
35. Hajłasz, P., Tyson, J.: Sobolev Peano cubes. Michigan Math. J. [To appear]
36. Hang, F.: On the weak limits of smooth maps for the Dirichlet energy between manifolds. Commun. Anal. Geom. **13**, 929-938 (2005)
37. Hang, F.: Density problems for $W^{1,1}(M,N)$. Commun. Pure Appl. Math. **55**, 937-947 (2002)
38. Hang, F., Lin, F.: Topology of Sobolev mappings. Math. Res. Lett. **8**, 321-330 (2001)
39. Hang, F., Lin, F.: Topology of Sobolev mappings II. Acta Math. **191**, 55-107 (2003)
40. Hang, F., Lin, F.: Topology of Sobolev mappings. III. Commun. Pure Appl. Math. **56**, 1383-1415 (2003)
41. Hardt, R.: Singularities of harmonic maps. Bull. Am. Math. Soc. (N.S.) **34**, 15-34 (1997)
42. Hardt, R., Lin, F.-H.: Mappings minimizing the L^p norm of the gradient. Commun. Pure Appl. Math. **40**, no. 5, 555-588 (1987)
43. Heinonen, J.: Lectures on Analysis on Metric Spaces. Springer-Verlag, New York (2001)
44. Heinonen, J.: Nonsmooth calculus. Bull. Am. Math. Soc. (N.S.) **44**, 163-232 (2007)
45. Hanson, B., Heinonen, J. An n-dimensional space that admits a Poincaré inequality but has no manifold points. Proc. Am. Math. Soc. **128**, 3379-3390 (2000)
46. Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. **181**, 1-61 (1998)
47. Heinonen, J., Koskela, P., Shanmugalingam, N., Tyson, J.T.: Sobolev classes of Banach space-valued functions and quasiconformal mappings. J. Anal. Math. **85**, 87-139 (2001)
48. Hélein, F.: Harmonic Maps, Conservation Laws and Moving Frames. Cambridge Univ. Press, Cambridge (2002)
49. Jost, J.: Equilibrium maps between metric spaces. Calc. Var. Partial Differ. Equ. **2**, 173-204 (1994)

50. Jost, J.: Generalized Dirichlet forms and harmonic maps. Calc. Var. Partial Differ. Equ. **5**, 1-19 (1997)
51. Jost, J.: Nonlinear Dirichlet forms. In: New Directions in Dirichlet Forms. pp. 1-47, AMS/IP Stud. Adv. Math. **8**, Am. Math. Soc., Providence, RI (1998)
52. Jost, J., Zuo, K.: Harmonic maps into Bruhat-Tits buildings and factorizations of p-adically unbounded representations of π_1 of algebraic varieties. I. J. Algebraic Geom. **9**, 1-42 (2000)
53. Keith, S.: A differentiable structure for metric measure spaces. Adv. Math. **183**, 271-315 (2004)
54. Keith, S., Zhong, X.: The Poincaré inequality is an open ended condition. Ann. Math. **167**, 575-599 (2008)
55. Korevaar, N.J., Schoen, R.M.: Sobolev spaces and harmonic maps for metric space targets. Commun. Anal. Geom. **1**, 561-659 (1993)
56. Koskela, P., MacManus, P.: Quasiconformal mappings and Sobolev spaces. Stud. Math. **131**, 1-17 (1998)
57. Koskela P., Saksman, E.: Pointwise characterizations of Hardy–Sobolev functions. Math. Res. Lett. [To appear]
58. Kronz, M.: Some function spaces on spaces of homogeneous type. Manuscr. Math. **106**, 219-248 (2001)
59. Laakso, T.J.: Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. Geom. Funct. Anal. **10**, 111-123 (2000)
60. Laakso, T.J.: Erratum to: "Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality", GAFA 10 (2000). Geom. Funct. Anal. **12**, 650 (2002)
61. Lindenstrauss, J., Preiss, D.: On Fréchet differentiability of Lipschitz maps between Banach spaces. Ann. Math. (2) **157**, 257-288 (2003)
62. Luukkainen, J., Saksman, E.: Every complete doubling metric space carries a doubling measure. Proc. Am. Math. Soc. **126**, 531-534 (1998)
63. Luukkainen, J., Väisälä, J.: Elements of Lipschitz topology. Ann. Acad. Sci. Fenn. Ser. A I Math. **3**, 85-122 (1977)
64. Marcus, M., Mizel, V.J.: Every superposition operator mapping one Sobolev space into another is continuous. J. Funct. Anal. **33**, 217-229 (1979)
65. Mou, L., Yang, P.: Regularity for n-harmonic maps. J. Geom. Anal. **6**, 91-112 (1996)
66. Pakzad, M.R.: Weak density of smooth maps in $W^{1,1}(M,N)$ for non-Abelian $\pi_1(N)$. Ann. Global Anal. Geom. **23**, 1-12 (2003)
67. Pakzad, M.R., Rivière, T.: Weak density of smooth maps for the Dirichlet energy between manifolds. Geom. Funct. Anal. **13**, 223-257 (2003)
68. Pansu, P.: Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. Ann. Math. **129**, 1-60 (1989)
69. Rivière, T.: Conservation laws for conformally invariant variational problems. Invent. Math. **168**, 1-22 (2007)
70. Reshetnyak, Yu. G.: Sobolev classes of functions with values in a metric space (Russian). Sib. Mat. Zh. **38**, 657-675 (1997); English transl. Sib. Math. J. **38**, 567-583 (1997)
71. Sadhu, S.: A characterization of Sobolev spaces on metric measure spaces. [In preparation]
72. Saloff-Coste, L.: Aspects of Sobolev Type Inequalities. Cambridge Univ. Press, Cambridge (2002)
73. Schoen, R., Uhlenbeck, K.: The Dirichlet problem for harmonic maps. J. Differ. Geom. **18** 253-268 (1983)
74. Schoen, R., Uhlenbeck, K.: Approximation theorems for Sobolev mappings [Unpublished manuscript]

75. Semmes, S.: Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities. Sel. Math., New Ser. **2** 155-295 (1996)
76. Serbinowski, T.: Boundary regularity of harmonic maps to nonpositively curved metric spaces. Commun. Anal. Geom. **2**, 139-153 (1994)
77. Shanmugalingam, N.: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. Rev. Mat. Iberoamericana **16**, 243-279 (2000)
78. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Univ. Press, Princeton, NJ (1993)
79. Strzelecki, P.: Regularity of generalized sphere valued p-harmonic maps with small mean oscillations. Manuscr. Math. **112**, 473-487 (2003)
80. Toro, T., Wang, C.: Compactness properties of weakly p-harmonic maps into homogeneous spaces. Indiana Univ. Math. J. **44**, 87-113 (1995)
81. Troyanov, M.: Approximately Lipschitz mappings and Sobolev mappings between metric spaces. In: Proceedings on Analysis and Geometry. Novosibirsk, 1999. pp. 585-594. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.
82. Volberg, A.L., Konyagin, S.V.: On measures with the doubling condition (Russian). Izv. Acad. Nauk SSSR **51**, 666-675 (1987); English transl.: Math. USSR Izv. **30**, 629-638 (1988)
83. Wang, C.: Subelliptic harmonic maps from Carnot groups. Calc. Var. Partial Differ. Equ. **18**, 95-115 (2003)
84. Wang, C.: A compactness theorem of n-harmonic maps. Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, 509-519 (2005)
85. White B.: Infima of energy functionals in homotopy classes of mappings. J. Differ. Geom. **23**, 127-142 (1986)