

SOBOLEV MAPPINGS BETWEEN MANIFOLDS AND METRIC SPACES

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For $\Omega \subset \mathbb{R}^n$, and $1 \leq p < \infty$, $W^{1,p}(\Omega)$ denotes the usual Sobolev space of functions for which $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p < \infty$. This definition can easily be extended to the case of Riemannian manifolds $W^{1,p}(M)$. Let now M and N be compact Riemannian manifolds. We can always assume that N is isometrically embedded in the Euclidean space \mathbb{R}^ν (Nash's theorem). We also assume that the manifold N has no boundary, while M may have boundary. This allows one to define the class of Sobolev mappings between the two manifolds as follows

$$W^{1,p}(M, N) = \{u \in W^{1,p}(M, \mathbb{R}^\nu) \mid u(x) \in N \text{ a.e.}\}$$

$W^{1,p}(M, N)$ is equipped with the metric inherited from the norm $\varrho(u, v) = \|u - v\|_{1,p}$. The space $W^{1,p}(M, N)$ provides a natural setting for geometric variational problems like, e.g. *weakly p -harmonic mappings* (called *weakly harmonic mappings* when $p = 2$). Weakly p -harmonic mappings are stationary points of the functional

$$I(u) = \int_M |\nabla u|^p, \quad \text{for } u \in W^{1,p}(M, N).$$

Because of the constrain in the image (manifold N) one has to clarify how the variation of this functional is defined. Let $\mathcal{U} \subset \mathbb{R}^\nu$ be a tubular neighborhood of N and $\pi : \mathcal{U} \rightarrow N$ be the smooth nearest point projection. For $\varphi \in C_0^\infty(M, \mathbb{R}^\nu)$, and $u \in W^{1,p}(M, N)$ the mapping $u + t\varphi$ takes on values into \mathcal{U} , provided $|t|$ is sufficiently small. Then we say that u is weakly p -harmonic if

$$\frac{d}{dt} \Big|_{t=0} I(\pi(u + t\varphi)) = 0 \quad \text{for all } \varphi \in C_0^\infty(M, \mathbb{R}^\nu).$$

The condition that the mappings take values into the manifold N is a constrain that makes the corresponding Euler-Lagrange system

$$(0.1) \quad -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-2} A(u)(\nabla u, \nabla u),$$

very nonlinear and difficult to handle. Here A is a second fundamental form of the embedding of N into the Euclidean space, see e.g. [4], [16], [34], [41], [42], [48], [65], [69], [79], [80], [83], [84]. There is a huge and growing literature on the subject and it is impossible to list here all relevant papers, but the

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reader easily find other papers following the references in the papers cited above.

Our main focus in this paper will be the theory of Sobolev mappings between manifolds, and later, the theory of Sobolev mappings between metric spaces, rather than applications of this theory to variational problems, and the above example was just to illustrate one of many areas in which the theory applies.

In connection with the theory of p -harmonic mappings Eells and Lemaire, [18], raised a question about density of smooth mappings $C^\infty(M, N)$ in $W^{1,p}(M, N)$. If $p \geq n = \dim M$, then smooth mappings are dense in $W^{1,p}(M, N)$, [73], [74], but if $p < n$, the answer depends on the topology of manifolds M and N . Recently Hang and Lin, [39], found a necessary and sufficient condition for the density in terms of algebraic topology. Their result is a correction of an earlier result of Bethuel [3] and a generalization of a result of Hajlasz [26]. To emphasize the connection of the problem with algebraic topology let us mention that it is possible to reformulate the Poincaré Conjecture (now a theorem) in terms of approximability of Sobolev mappings, [25]. The theory of Sobolev mappings between manifolds has been extended to the case of Sobolev mappings with values into metric spaces. The first papers on this subject include the work of Ambrosio, [2], on limits of classical variational problems and the work of Gromov and Schoen, [24], on Sobolev mappings into the Bruhat-Tits buildings, with applications to rigidity questions for discrete groups. Later the theory of Sobolev mappings with values into metric spaces was developed in a more elaborated form by Korevaar and Schoen, [55], in their approach to the theory of harmonic mappings into Alexandrov spaces of non-positive curvature. Other papers on Sobolev mappings from a manifold into a metric space include [12], [17], [49], [50], [51], [52], [70], [76]. Finally analysis on metric spaces, the theory of Carnot–Carathéodory spaces and the theory of quasiconformal mappings between metric spaces led to the theory of Sobolev mappings between metric spaces, [46], [47], [58], [81], among which the theory of Newtonian-Sobolev mappings $N^{1,p}(X, Y)$ is particularly important.

In Section 1 we discuss fundamental results concerning the density of smooth mappings in $W^{1,p}(M, N)$. Section 2 is devoted to a construction of the class of Sobolev mappings from a manifold into a metric space. We also show there that several natural questions to the density problem have negative answers when we consider mappings from a manifold into a metric space. In Section 3 we explain the construction and basic properties of Sobolev spaces on metric-measure spaces and in final Section 4 we discuss recent development of the theory of Sobolev mappings between metric spaces, including results about approximation of mappings.

Notation in the paper is fairly standard. We assume that all manifolds are compact (with or without boundary), smooth and connected. We always

assume that such a manifold is equipped with a Riemannian metric, but since all such metrics are equivalent it is not important with which metric we work. By a closed manifold we mean a smooth compact manifold without boundary. The integral average of a function u over a set E will be denoted by

$$u_E = \int_E u d\mu = \mu(E)^{-1} \int_E u d\mu.$$

Balls will be denoted by B and for $\sigma > 0$, σB denotes the ball concentric with B whose radius is σ times that of B . The symbol C will stand for a general constant whose actual value may change within a single string of estimates. We write $A \approx B$ if there is a constant $C \geq 1$ such that $C^{-1}A \leq B \leq CA$.

1. SOBOLEV MAPPINGS BETWEEN MANIFOLDS

It is easy and well known that smooth functions are dense in the Sobolev space $W^{1,p}(M)$. Thus if N is isometrically embedded into \mathbb{R}^ν it follows that every $W^{1,p}(M, N)$ mapping can be approximated by $C^\infty(M, \mathbb{R}^\nu)$ mappings and the question is whether we can approximate $W^{1,p}(M, N)$ by $C^\infty(M, N)$ mappings. It was answered in the affirmative by Schoen and Uhlenbeck [73], [74] in the case in which $p \geq n = \dim M$.

Theorem 1.1. *If $p \geq n = \dim M$, then the class of smooth mappings $C^\infty(M, N)$ is dense in the Sobolev space $W^{1,p}(M, N)$.*

*Proof.*¹ Assume that N is isometrically embedded in some Euclidean space \mathbb{R}^ν . If $p > n$, then the result is very easy. Indeed, let $u_k \in C^\infty(M, \mathbb{R}^\nu)$ be a sequence of smooth mappings that converge to u in the norm of $W^{1,p}$. Since $p > n$ the Sobolev embedding theorem implies that u_k converges uniformly to u . Hence for $k \geq k_0$ values of the mappings u_k belong to a tubular neighborhood $\mathcal{U} \subset \mathbb{R}^\nu$ of N from which there is a smooth nearest point projection $\pi : \mathcal{U} \rightarrow N$. Now $\pi \circ u_k \in C^\infty(M, N)$ and $\pi \circ u_k \rightarrow \pi \circ u = u$ in the $W^{1,p}$ norm. If $p = n$, then we do not have uniform convergence, but one still can prove that the values of the approximating sequence u_k whose construction is based locally on the convolution approximation belong to the tubular neighborhood of N for all sufficiently large k . This follows from the Poincaré inequality. To see this it suffices to consider the localized problem where the mappings are defined on an Euclidean ball. Let $u \in W^{1,n}(B^n(0, 1), N)$ and let \bar{u} be the extension of u to a neighborhood of the ball (by reflection). We define $u_\varepsilon = \bar{u} * \varphi_\varepsilon$, where φ_ε is a standard mollifying

¹See also Theorem 2.7 and Theorem 4.5.

kernel. Poincaré inequality yields

$$(1.1) \quad \left(\int_{B^n(x,\varepsilon)} |\bar{u}(y) - u_\varepsilon(x)|^n dy \right)^{1/n} \leq Cr \left(\int_{B^n(x,\varepsilon)} |\nabla \bar{u}|^n \right)^{1/n} \\ = C' \left(\int_{B^n(x,\varepsilon)} |\nabla \bar{u}|^n \right)^{1/n}.$$

The right hand side (as a function of x) converges to 0 as $\varepsilon \rightarrow 0$ uniformly on $B^n(0, 1)$. Since

$$\text{dist}(u_\varepsilon(x), N) \leq |\bar{u}(y) - u_\varepsilon(x)|$$

for all y , we conclude from (1.1) that

$$\text{dist}(u_\varepsilon(x), N) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

uniformly on $B^n(0, 1)$. Hence for $\varepsilon < \varepsilon_0$ values of the smooth mappings u_ε belong to \mathcal{U} and thus $\pi \circ u_\varepsilon \rightarrow \pi \circ u = u$ as $\varepsilon \rightarrow 0$. \square

Arguments used in the above proof lead to the following result.

Proposition 1.2. *If $u \in W^{1,p}(M, N)$ can be approximated by continuous Sobolev mappings $C^0 \cap W^{1,p}(M, N)$, then it can be approximated by smooth $C^\infty(M, N)$ mappings.*

Proof. Indeed, if $v \in C^0 \cap W^{1,p}(M, N)$, then $v_\varepsilon \rightrightarrows v$ uniformly and hence $\pi \circ v_\varepsilon \rightarrow \pi \circ v = v$ in $W^{1,p}$. \square

A basic tool in the study of Sobolev mappings between manifolds is a variant of the Fubini theorem for Sobolev functions. Let us illustrate it in a simplest setting. Suppose that $u, u_i \in W^{1,p}([0, 1]^n)$, $\|u - u_i\|_{1,p} \rightarrow 0$ as $i \rightarrow \infty$. Denote points in the cube by (t, x) , where $t \in [0, 1]$, $x \in [0, 1]^{n-1}$. Then

$$\int_{[0,1]^n} |u - u_i|^p + |\nabla u - \nabla u_i|^p \\ = \int_0^1 \left(\int_{[0,1]^{n-1}} |u - u_i|^p + |\nabla u - \nabla u_i|^p dx \right) dt = \int_0^1 F_i(t) dt \xrightarrow{i \rightarrow \infty} 0.$$

Hence $F_i \rightarrow 0$ in $L^1(0, 1)$ and so there is a subsequence u_{i_j} such that $F_{i_j}(t) \rightarrow 0$ for a.e. $t \in (0, 1)$. That means, for a.e. $t \in [0, 1]$, $u(t, \cdot), u_{i_j}(t, \cdot) \in W^{1,p}([0, 1]^{n-1})$ and $u_{i_j}(t, \cdot) \rightarrow u(t, \cdot)$ in $W^{1,p}([0, 1]^{n-1})$. Clearly the same argument applies to lower dimensional slices of the cube.

As it has already been pointed out, if $p < n = \dim M$, then density of smooth mappings does not always hold. The first example of this type was provided by Schoen and Uhlenbeck and it is actually quite simple. A direct computation shows that the radial projection

$$u_0(x) = x/|x| : B^n(0, 1) \setminus \{0\} \rightarrow S^{n-1}$$

belongs to the Sobolev space $W^{1,p}(B^n, S^{n-1})$ for all $1 \leq p < n$. Shoen and Uhlenbeck [73], [74] proved

Theorem 1.3. *If $n - 1 \leq p < n$, then the mapping u_0 cannot be approximated by smooth mappings $C^\infty(B^n, S^{n-1})$ in the norm of $W^{1,p}$.*

Proof. Suppose that there is a sequence $u_k \in C^\infty(B^n, S^{n-1})$ such that $\|u_k - u_0\|_{1,p} \rightarrow 0$ as $k \rightarrow \infty$ for some $n - 1 \leq p < n$. Then it follows from the Fubini theorem that there is a subsequence (still denoted by u_k) such that for almost every $0 < r < 1$

$$u_k|_{S^{n-1}(0,r)} \rightarrow u_0|_{S^{n-1}(0,r)}$$

in the norm of $W^{1,p}(S^{n-1}(0,r))$. If $n - 1 < p < n$, then the Sobolev embedding theorem into Hölder continuous functions implies that u_k restricted to such spheres converges uniformly to u_0

$$u_k|_{S^{n-1}(0,r)} \rightrightarrows u_0|_{S^{n-1}(0,r)}$$

which is impossible, because the Brouwer degree² of $u_k|_{S^{n-1}(0,r)}$ is 0 and the degree of $u_0|_{S^{n-1}(0,r)}$ is 1. The case $p = n - 1$ needs a different, but related argument. The degree of a mapping $v : M \rightarrow N$ between two oriented $(n - 1)$ -dimensional compact manifolds without boundary can be defined by an integral formula $\deg v = \int_M \det Dv / \text{vol } N$ and it follows from the Hölder inequality that degree is continuous in the $W^{1,n-1}$ norm. This implies that if $u_k \rightarrow u_0$ in $W^{1,n-1}(S^{n-1}(0,r))$, then the degree of $u_k|_{S^{n-1}(0,r)}$ which is 0 converges to degree of $u_0|_{S^{n-1}(0,r)}$ is 1. Again we obtain a contradiction. \square

It turns out, however, that for $1 \leq p < n - 1$ smooth maps are dense in $W^{1,p}(B^n, S^{n-1})$. Indeed, the following result was proved by Bethuel and Zheng [5].

Theorem 1.4. *For $1 \leq p < k$ smooth mappings $C^\infty(M, S^k)$ are dense in $W^{1,p}(M, S^k)$.*

Proof. Let $u \in W^{1,p}(M, S^k)$. It is easy to see that for every $x \in S^k$ and $\delta > 0$ there is a Lipschitz retraction $\pi_{x,\delta} : S^k \rightarrow S^k \setminus B(x, \delta)$, i.e. $\pi_{x,\delta} \circ \pi_{x,\delta} = \pi_{x,\delta}$, with the Lipschitz constant bounded by $C\delta^{-1}$. Now we consider the mapping $u_{x,\delta} = \pi_{x,\delta} \circ u$. Since $u_{x,\delta}$ maps M into the set $S^k \setminus B(x, \delta)$ which is diffeomorphic with a closed k dimensional ball it is easy to see that $u_{x,\delta}$ can be approximated by smooth maps from M to $S^k \setminus B(x, \delta) \subset S^k$. Thus it remains to prove that for every $\varepsilon > 0$ there is $\delta > 0$ and $x \in S^k$ such that $\|u - u_{x,\delta}\|_{1,p} < \varepsilon$.

There are $C\delta^{-k}$ disjoint balls of radius δ on S^k . Denote such a family of balls by $B(x_i, \delta)$, $i = 1, 2, \dots, N_\delta$, where $N_\delta \approx \delta^{-k}$. Note that the mapping

²The degree is 0 because u_k has continuous (actually smooth) extension to the entire ball.

$u_{x_i, \delta}$ differs from u on the set $u^{-1}(B(x_i, \delta))$ and this is a family of $N_\delta \approx \delta^{-k}$ disjoint subset of M . Therefore there is i such that

$$\int_{u^{-1}(B(x_i, \delta))} |u|^p + |\nabla u|^p \leq C\delta^k \|u\|_{1,p}^p.$$

Using the fact that the Lipschitz constant of $\pi_{x_i, \delta}$ is bounded by $C\delta^{-1}$ it is easy to see that

$$\int_{u^{-1}(B(x_i, \delta))} |\nabla u_{x_i, \delta}|^p \leq C\delta^{-p} \int_{u^{-1}(B(x_i, \delta))} |\nabla u|^p \leq C\delta^{k-p} \|u\|_{1,p}^p$$

Since $u = u_{x_i, \delta}$ on the complement of the set $u^{-1}(B(x_i, \delta))$ we have

$$\begin{aligned} \|\nabla u - \nabla u_{x_i, \delta}\|_p &= \left(\int_{u^{-1}(B(x_i, \delta))} |\nabla u - \nabla u_{x_i, \delta}|^p \right)^{1/p} \\ &\leq \left(\int_{u^{-1}(B(x_i, \delta))} |\nabla u|^p \right)^{1/p} + \left(\int_{u^{-1}(B(x_i, \delta))} |\nabla u_{x_i, \delta}|^p \right)^{1/p} \\ &\leq C(\delta^{k/p} + \delta^{(k-p)/p}) \|u\|_{1,p}. \end{aligned}$$

Since $k - p > 0$, this implies that given $\varepsilon > 0$ there is $\delta > 0$ and $x \in S^k$ such that $\|\nabla u - \nabla u_{x, \delta}\|_{1,p} < \varepsilon$. It remains to observe that the mappings u and $u_{x, \delta}$ are also close in the L^p norm. Indeed, they are both uniformly bounded (as mappings into the unit sphere) and they coincide outside a set of very small measure. \square

The above two results show that the answer to the problem of density of smooth mappings in the Sobolev space $W^{1,p}(M, N)$ depends of the topology of the manifold N and perhaps also on the topology of the manifold M . We will find now necessary conditions for the density of $C^\infty(M, N)$ in $W^{1,p}(M, N)$.

The density result, Theorem 1.1, implies that if M and N are two smooth oriented compact manifolds without boundary, both of dimension n , then we can define degree of mappings in the class $W^{1,n}(M, N)$. Indeed, if $u \in C^\infty(M, N)$, then the degree is defined in terms of the integral of the Jacobian and then it can be extended to the entire space $W^{1,n}(M, N)$ by the density of smooth mappings. Thus

$$\text{deg} : W^{1,n}(M, N) \rightarrow \mathbf{Z}$$

is a continuous function and it coincides with the classical degree on the subclass of smooth mappings. It turns out, however, that not only degree, but also homotopy classes can be defined. This follows from the following result of White [85].

Theorem 1.5. *Let M and N be closed manifolds and let $n = \dim M$. Then for every $f \in W^{1,n}(M, N)$, there is $\varepsilon > 0$ such that any two smooth mappings $g_1, g_2 : M \rightarrow N$ satisfying $\|f - g_i\|_{1,n} < \varepsilon$, for $i = 1, 2$, are homotopic.*

Note that Theorem 1.5 is also a special case of Theorem 1.8 and Theorem 4.5 below.

We will use the above result to find the first necessary condition for the density of smooth mappings in the Sobolev space. The following result is due to Bethuel and Zheng [5] and Bethuel [3]. A simplified proof provided below is taken from [25]. Let $[p]$ denotes the largest integer less than or equal to p . In the following theorem π_k stands for the homotopy group.

Theorem 1.6. *If $\pi_{[p]}(N) \neq 0$ and $1 \leq p < n = \dim M$, then the smooth mappings $C^\infty(M, N)$ are not dense in $W^{1,p}(M, N)$.*

Proof. It is easy to construct a smooth mapping $f : B^{[p]+1} \rightarrow S^{[p]}$ with two singular points such that f restricted to small spheres centered at the singularities have degree $+1$ and -1 respectively and f maps a neighborhood of the boundary of the ball $B^{[p]+1}$ into a point. We can model the singularities on the radial projection mapping as in Theorem 1.3 so the mapping f belongs to $W^{1,p}$. Let now $g : B^{[p]+1} \times S^{n-[p]-1} \rightarrow S^{[p]}$ be defined by $g(b, s) = f(b)$. We can embed the torus $B^{[p]+1} \times S^{n-[p]-1}$ into the manifold M and extend the mapping on the completion of this torus as a mapping into a point. Clearly $g \in W^{1,p}(M, S^{[p]})$. Now let $\varphi : S^{[p]} \rightarrow N$ be a smooth representative of a nontrivial homotopy class. We will prove that the mapping $\varphi \circ g \in W^{1,p}(M, N)$ cannot be approximated by smooth mappings from $C^\infty(M, N)$. By contrary suppose that $u_k \in C^\infty(M, N)$ converges to $\varphi \circ f$ in the norm of $W^{1,p}$. In particular $u_k \rightarrow \varphi \circ g$ in $W^{1,p}(B^{[p]+1} \times S^{n-[p]-1}, N)$. It follows from the Fubini theorem there is a subsequence of u_k (still denoted by u_k) such that for almost every $s \in S^{n-[p]-1}$, u_k restricted to the slice $B^{[p]+1} \times \{s\}$ converges to the corresponding restriction of $\varphi \circ g$ in the Sobolev norm. Take such a slice and denote it simply by $B^{[p]+1}$. Again by the Fubini theorem u_k restricted to almost every sphere centered at the $+1$ singularity of f converges to the corresponding restriction of $\varphi \circ g$ in the Sobolev norm. Denote such a sphere by $S^{[p]}$. Hence $u_k|_{S^{[p]}} \rightarrow \varphi \circ g|_{S^{[p]}}$ in the space $W^{1,p}(S^{[p]}, N)$. Now the mapping $u_k|_{S^{[p]}} : S^{[p]} \rightarrow N$ is contractible (because it has a smooth extension to the ball), while $\varphi \circ g|_{S^{[p]}} : S^{[p]} \rightarrow N$ is a smooth representative of a nontrivial homotopy class $\pi_{[p]}(N)$, so $u_k|_{S^{[p]}}$ cannot be homotopic to $\varphi \circ g|_{S^{[p]}}$ which contradicts the theorem of White (Theorem 1.5). \square

It turns out that in some cases the condition $\pi_{[p]}(N) = 0$ is also sufficient for the density of smooth mappings. The following statement is due to Bethuel [3].

Theorem 1.7. *If $1 \leq p < n$, then smooth mappings $C^\infty(B^n, N)$ are dense in $W^{1,p}(B^n, N)$ if and only if $\pi_{[p]}(N) = 0$.*

Actually Bethuel [3] claimed a stronger result that $\pi_{[p]}(N) = 0$ is a necessary and sufficient condition for the density of $C^\infty(M, N)$ mappings in $W^{1,p}(M, N)$ for any compact manifold M of dimension $\dim M = n > p$. This, however, turned out to be false: Hang and Lin, [38], provided a counterexample to Bethuel's claim by demonstrating that despite the equality $\pi_3(\mathbb{C}\mathbb{P}^2) = 0$, $C^\infty(\mathbb{C}\mathbb{P}^3, \mathbb{C}\mathbb{P}^2)$ is *not* dense in $W^{1,3}(\mathbb{C}\mathbb{P}^3, \mathbb{C}\mathbb{P}^2)$. Bethuel's claim made people believe that the problem of density of smooth mappings in the Sobolev space has a local nature. However the example of Hang and Lin and Theorem 1.7 shows that there might be global obstacles. Indeed, the mapping constructed by Hang and Lin cannot be approximated by smooth mappings $C^\infty(\mathbb{C}\mathbb{P}^3, \mathbb{C}\mathbb{P}^2)$, however, since $\pi_3(\mathbb{C}\mathbb{P}^2) = 0$, Theorem 1.7 shows that this mapping can be smoothly approximated in a neighborhood of any point in $\mathbb{C}\mathbb{P}^3$.

Therefore when searching for a necessary and sufficient condition for the density of smooth mappings one has to take into account the topology of both manifolds M and N , or rather the interplay between the topology of M and the topology of N . Now we will find such a necessary condition for the density of smooth mappings. Before we start we need to say a few words about the behavior of Sobolev mappings on k -dimensional skeletons of generic smooth triangulations.

Let the manifold M be equipped with a smooth triangulation M^k , $k = 0, 1, 2, \dots, n = \dim M$. Since the skeletons of the triangulation are piecewise smooth it is not difficult to define the Sobolev space on skeletons $W^{1,p}(M^k)$. There is no problem with the definition of Sobolev functions in the interiors of the simplexes, but one needs to clarify how the Sobolev functions meet at the boundaries, so that the function belongs to the Sobolev space not only in each of the simplexes, but on the whole skeleton M^k . One possibility is to define the Sobolev norm $\|u\|_{1,p}$ for functions u that are Lipschitz continuous on M^k and then define $W^{1,p}$ by completion. Suppose now that $u \in W^{1,p}(M)$. If $v \in W^{1,p}([0, 1]^n)$, then, in general, it is not true that the function v restricted to *each* slice $\{t\} \times [0, 1]^{n-1}$ belongs to the Sobolev space $W^{1,p}([0, 1]^{n-1})$, but it is true for almost all $t \in [0, 1]$. For the same reason u restricted to M^k does not necessarily belong to the Sobolev space $W^{1,p}(M^k)$. This problem can, however, be handled. Indeed, faces of the k dimensional skeleton M^k can be translated in the remaining directions which form an $n - k$ dimensional space. Hence, roughly speaking with each skeleton M^k we can associate an $n - k$ dimensional family of skeletons.³ Now u restricted to almost every skeleton in this family belongs to the Sobolev space $W^{1,p}$ on that skeleton, by the Fubini theorem. We briefly summarize this construction by saying that if $u \in W^{1,p}(M)$, then u restricted to a generic k

³This is not entirely obvious because we translate different faces in different directions and we have to make sure that after all, the faces glue together, so that we still have a k -dimensional skeleton. This, however, can be done and there are no unexpected surprises.

dimensional skeleton M^k belongs to the Sobolev space $W^{1,p}(M^k)$. Moreover, if $u, u_i \in W^{1,p}(M)$, $\|u - u_i\|_{1,p} \rightarrow 0$, then there is a subsequence u_{i_j} such that $u_{i_j} \rightarrow u$ in $W^{1,p}(M^k)$ on generic k -dimensional skeletons. This follows from the Fubini theorem argument explained above.

We say that two continuous mappings $f, g : M \rightarrow N$ are k -homotopic, $0 \leq k \leq n = \dim M$, if the restrictions of both mappings to the k -dimensional skeleton of a triangulation of M are homotopic. Using elementary topology one can prove that the above definition does not depend on the choice of a triangulation of M (cf. [39, Lemma 2.1]). Theorem 1.5 is a special case of a more general result of White [85].

Theorem 1.8. *Let M and N be closed manifolds and let $n = \dim M$. Then for every $f \in W^{1,p}(M, N)$, $1 \leq p \leq n$ there is $\varepsilon > 0$ such that any two Lipschitz mappings $g_1, g_2 : M \rightarrow N$ satisfying $\|f - g_i\|_{1,p} < \varepsilon$, $i = 1, 2$ are $[p]$ -homotopic.*

Another result that we will frequently use is the homotopy extension theorem. We will state it only in a special case.

Theorem 1.9. *Let M be a smooth compact manifold equipped with a smooth triangulation M^k , $k = 0, 1, 2, \dots, n = \dim M$. Then for any topological space X every continuous mapping*

$$H : (M \times \{0\}) \cup (M^k \times [0, 1]) \rightarrow X$$

has a continuous extension to $\tilde{H} : M \times [0, 1] \rightarrow X$.

In particular the theorem implies that if $f : M \rightarrow N$ is continuous and $g : M^k \rightarrow N$ is homotopic to $f|_{M^k}$, then g admits a continuous extension to $\tilde{g} : M \rightarrow N$. We will apply this observation below.

In the proof of necessity of the condition $\pi_{[p]}(N) = 0$ we constructed a map with the $(n - [p] - 1)$ -dimensional singularity. The condition we will present now will actually imply $\pi_{[p]}(N) = 0$ and, not surprisingly, our argument will also involve a construction of a map with the $(n - [p] - 1)$ -dimensional singularity.

Let $1 \leq p < n = \dim M$ and suppose that smooth mappings $C^\infty(M, N)$ are dense in $W^{1,p}(M, N)$. Assume that M is endowed with a smooth triangulation. Let $h : M^{[p]} \rightarrow N$ be a Lipschitz mapping. Observe that if $f \in W^{1,p}(S^k, N)$, then the integration in spherical coordinates easily implies that the mapping $\bar{f}(x) = f(x/|x|)$ belongs to $W^{1,p}(B^{k+1}, N)$, provided $p < k + 1$. Clearly, the ball B^{k+1} can be replaced by a $(k + 1)$ -dimensional simplex and S^k by its boundary. For this reason the mapping $h : M^{[p]} \rightarrow N$ can be extended to a mapping in $W^{1,p}(M^{[p]+1}, N)$. The extension will have singularity consisting of one point in each $([p] + 1)$ -dimensional simplex in

$M^{[p]+1}$. Next we can extend the mapping to $W^{1,p}(M^{[p]+2}, N)$. Now the singularity will be one dimensional. We can continue this process by extending the mapping to higher dimensional skeletons. Eventually we will obtain a mapping $\bar{h} \in W^{1,p}(M, N)$ with the $(n - [p] - 1)$ -dimensional singularity located on a dual skeleton to $M^{[p]}$.

Let $u_i \in C^\infty(M, N)$ be such that $\|\bar{h} - u_i\|_{1,p} \rightarrow 0$ as $i \rightarrow \infty$. It follows from the Fubini theorem that there is a subsequence u_{i_j} such that $u_{i_j} \rightarrow \bar{h}$ in $W^{1,p}$ on generic $[p]$ -dimensional skeletons, so

$$u_{i_j} \rightarrow \bar{h} \quad \text{in } W^{1,p}(\tilde{M}^{[p]}, N),$$

where $\tilde{M}^{[p]}$ is a ‘‘tilt’’ of $M^{[p]}$. Since \bar{h} and u_{i_j} are Lipschitz it follows from Theorem 1.8 that u_{i_j} is homotopic to \bar{h} on $\tilde{M}^{[p]}$ for all $j \geq j_0$. Now it follows from the homotopy extension theorem (Theorem 1.9) that the mapping $\bar{h}|_{\tilde{M}^{[p]}}$ admits an extension to a continuous mapping $\bar{h} : M \rightarrow N$, hence also $h : M^{[p]} \rightarrow N$ can be extended to a continuous mapping $h' : M \rightarrow N$.

We proved that every Lipschitz mapping $h : M^{[p]} \rightarrow N$ admits a continuous extension $h' : M \rightarrow N$. Since every continuous mapping $f : M^{[p]} \rightarrow N$ is homotopic to a Lipschitz mapping, another application of the homotopy extension theorem implies that also f has continuous extension. We proved

Proposition 1.10. *If $1 \leq p < n = \dim M$ and $C^\infty(M, N)$ is dense in $W^{1,p}(M, N)$, then every continuous mapping $f : M^{[p]} \rightarrow N$ can be extended to a continuous mapping $f' : M \rightarrow N$.*

The following result provides a characterization of the property described in the above proposition.

We say that M has $(k-1)$ -extension property with respect to N if for every continuous mapping $f \in C(M^k, N)$, $f|_{M^{k-1}}$ has a continuous extension to $\tilde{f} \in C(M, N)$.

Proposition 1.11. *If $1 \leq k < n = \dim M$, then every continuous mapping $f : M^k \rightarrow N$ can be extended to a continuous mapping $f' : M \rightarrow N$ if and only if $\pi_k(N) = 0$ and M has the $(k-1)$ -extension property with respect to N .*

Proof. Suppose that every continuous mapping $f : M^k \rightarrow N$ has a continuous extension to $f' : M \rightarrow N$. Then obviously M has the $(k-1)$ -extension property with respect to N . We need to prove that $\pi_k(N) = 0$. Suppose $\pi_k(N) \neq 0$. Let Δ be a $(k+1)$ -dimensional simplex in M^{k+1} and let $\partial\Delta$ be its boundary. It is easy to see that there is a continuous retraction $\pi : M^k \rightarrow \partial\Delta$. Let $\varphi : \partial\Delta \rightarrow N$ be a representative of a nontrivial element in the homotopy group $\pi_k(N)$. Then φ cannot be extended to Δ , hence also $f = \varphi \circ \pi : M^k \rightarrow N$ has no continuous extension to M . Contradiction.

Now suppose that $\pi_k(N) = 0$ and M has the $(k-1)$ -extension property with respect to N . Let $f : M^k \rightarrow N$ be continuous. We need to show that f can be continuously extended to M . Let $\tilde{f} : M \rightarrow N$ be a continuous extension of $f|_{M^{k-1}}$.

The set $M^k \times [0, 1]$ is the union of $(k+1)$ -dimensional cells $\Delta \times [0, 1]$, where Δ is a k -dimensional simplex in M^k . Denote the points in $\Delta \times [0, 1]$ by (x, t) and define the mapping on the boundary on each cell as follows

$$\begin{aligned} H(x, 0) &= \tilde{f}(x) & \text{for } x \in \Delta, \\ H(x, 1) &= f(x) & \text{for } x \in \Delta, \\ H(x, t) &= \tilde{f}(x) = f(x) & \text{for } x \in \partial\Delta. \end{aligned}$$

Because $\pi_k(N) = 0$, H can be continuously extended to the interior of each cell. Denote the extension by $H : M^k \times [0, 1] \rightarrow N$. Now it follows from the homotopy extension theorem that $f : M^k \rightarrow N$ admits a continuous extension $f' : M \rightarrow N$. \square

Thus if $f \in W^{1,p}(M, N)$, $1 \leq p < n = \dim M$, can be approximated by smooth mappings, then $\pi_{[p]}(N) = 0$ and for every continuous mapping $g : M^{[p]} \rightarrow N$, $g|_{M^{[p]-1}}$ has a continuous extension to M .

Actually this property was used by Hang and Lin [38] to demonstrate that $C^\infty(\mathbb{C}\mathbb{P}^3, \mathbb{C}\mathbb{P}^2)$ mappings are not dense in $W^{1,3}(\mathbb{C}\mathbb{P}^3, \mathbb{C}\mathbb{P}^2)$ (despite the fact that $\pi_3(\mathbb{C}\mathbb{P}^2) = 0$).

Since the extension property is of a topological nature, it is easier to work with the natural CW structure of $\mathbb{C}\mathbb{P}^n$ rather than with the triangulation and the extension property can be equivalently formulated for CW structures.

It is well known that $\mathbb{C}\mathbb{P}^n$ has a natural CW structure $\mathbb{C}\mathbb{P}^0 \subset \mathbb{C}\mathbb{P}^1 \subset \dots \subset \mathbb{C}\mathbb{P}^n$. If $M = \mathbb{C}\mathbb{P}^3$, then $M^2 = M^3 = \mathbb{C}\mathbb{P}^1$. Now it follows from elementary algebraic topology that the identity mapping

$$i : M^3 = \mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^2$$

cannot be continuously extended to $\tilde{i} : \mathbb{C}\mathbb{P}^3 \rightarrow \mathbb{C}\mathbb{P}^2$ and since $M^2 = M^3$ we also have that $i|_{M^2}$ has no continuous extension. Thus $C^\infty(\mathbb{C}\mathbb{P}^3, \mathbb{C}\mathbb{P}^2)$ mappings are not dense in $W^{1,p}(\mathbb{C}\mathbb{P}^3, \mathbb{C}\mathbb{P}^2)$. See [38, pp. 327-328] for more details.

It turns out that the above necessary condition for the density is also sufficient. Namely the following result was proved by Hang and Lin, [39].

Theorem 1.12. *Assume that M and N are compact smooth Riemannian manifolds without boundary. If $1 \leq p < \dim M$, then smooth mappings $C^\infty(M, N)$ are dense in $W^{1,p}(M, N)$ if and only if $\pi_{[p]}(N) = 0$ and M has the $([p]-1)$ -extension property with respect to N .*

The following two corollaries easily follow from the theorem, see [39].

Corollary 1.13. *If $1 \leq p < n = \dim M$, k is an integer such that $0 \leq k \leq [p] - 1$ and $\pi_i(M) = 0$ for $1 \leq i \leq k$, $\pi_i(N) = 0$ for $k + 1 \leq i \leq [p]$. Then $C^\infty(M, N)$ is dense in $W^{1,p}(M, N)$.*

Corollary 1.14. *If $1 \leq p < n = \dim M$, $\pi_i(N) = 0$ for $[p] \leq i \leq n - 1$, then $C^\infty(M, N)$ is dense in $W^{1,p}(M, N)$.*

In particular Corollary 1.13 with $k = 0$ gives the following result that has previously been proved in [26].

Corollary 1.15. *If $1 \leq p < n = \dim M$ and $\pi_1(N) = \pi_2(N) = \dots = \pi_{[p]}(N) = 0$, then $C^\infty(M, N)$ is dense in $W^{1,p}(M, N)$.*

The reason why we stated this corollary in addition to Corollary 1.13 is that in the case of Sobolev mappings from metric spaces supporting Poincaré inequalities into Lipschitz polyhedra the homotopy condition from Corollary 1.15 turns out to be necessary and sufficient for the density, see Theorem 4.6.

Another interesting question regarding density of smooth mappings is the question about the density in the sequential weak topology. We will not discuss this topic here and we refer the reader to [26], [36], [37], [39], [40], [66], [67].

2. SOBOLEV MAPPINGS INTO METRIC SPACES

There have been several approaches to the definition of the class of Sobolev mappings from a manifold, or just an open set in \mathbb{R}^n , into a metric space, see e.g. [2], [24], [49], [55], [70]. The approach presented here is taken from [35] and it is an elaboration of ideas of Ambrosio [2] and Reshetnyak [70]. One of the benefits of the construction presented here is that the Sobolev space of mappings into a metric space is equipped in a natural way with a metric, so one can ask whether the class of Lipschitz mappings is dense. In the case of mappings into metric spaces it does not make sense to talk about smooth mappings so we need to consider Lipschitz mappings instead.

Since every metric space X admits an isometric embedding into a Banach space⁴ V , the idea is to define the Sobolev space of functions with values into a Banach space V and then define the Sobolev space of mappings with values into X as

$$W^{1,p}(M, X) = \{f \in W^{1,p}(M, V) \mid f(M) \subset X\}.$$

⁴Every metric space admits an isometric embedding into the Banach space $V = \ell^\infty(X)$ of bounded functions on X . If in addition X is separable, then X admits an isometric embedding into ℓ^∞ .

Since $W^{1,p}(M, V)$ is a Banach space, this approach equips $W^{1,p}(M, X)$ with a natural metric inherited from the norm of $W^{1,p}(M, V)$, just like in the case of Sobolev mappings between manifolds. With this metric at hand we can ask under what conditions the class of Lipschitz mappings $\text{Lip}(M, X)$ is dense in $W^{1,p}(M, X)$.

On the other hand the approach described above depends on the isometric embedding of X into V , so it is useful to find another, equivalent and intrinsic approach independent of the embedding. In this section we will describe both such approaches. In our approach we will follow [35], where the reader can find detailed proofs of results stated here.

For the sake of simplicity we will consider Sobolev functions defined on a domain in \mathbb{R}^n rather than on a manifold, but all the statements can easily be generalized to the case of Sobolev functions defined on manifolds.

Before we define the Sobolev space of functions with values into a Banach space we need briefly recall the notion of the Bochner integral, see [15].

Let V be a Banach space, $E \subset \mathbb{R}^n$ a measurable set, and $1 \leq p \leq \infty$. We say that $f \in L^p(E, V)$ if

- (1) f is *essentially separable valued*, i.e. $f(E \setminus Z)$ is a separable subset of V for some set Z of Lebesgue measure zero;
- (2) f is *weakly measurable*, i.e. for every $v^* \in V^*$, $\langle v^*, f \rangle$ is measurable;
- (3) $\|f\| \in L^p(E)$.

If $f = \sum_{i=1}^k a_i \chi_{E_i} : E \rightarrow V$ is a simple function, then the Bochner integral is defined as

$$\int_E f(x) dx = \sum_{i=1}^k a_i |E_i|$$

and for $f \in L^1(E, V)$ the Bochner integral is defined as the limit of integrals of simple functions that converge to f a.e. The following two properties of the Bochner integral are well known

$$\left\| \int_E f(x) dx \right\| \leq \int_E \|f(x)\| dx,$$

and

$$(2.1) \quad \left\langle v^*, \int_E f(x) dx \right\rangle = \int_E \langle v^*, f(x) \rangle dx \quad \text{for all } v^* \in V^*.$$

In the theory of the Bochner integral a measurable set $E \subset \mathbb{R}^n$ can be replaced by a more general measure space. We will need such a more general setting later in Section 4.

Let now $\Omega \subset \mathbb{R}^n$ be an open set and V a Banach space. It is natural to define the Sobolev space $W^{1,p}(\Omega, V)$ using the notion of weak derivative,

just like in the case of real valued functions. We say that $f \in W^{1,p}(\Omega, V)$ if $f \in L^p(\Omega, V)$ and for $i = 1, 2, \dots, n$ there are functions $f_i \in L^p(\Omega, V)$ such that

$$\int_{\Omega} \frac{\partial \varphi}{\partial x_i}(x) f(x) dx = - \int_{\Omega} \varphi(x) f_i(x) dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

We denote $f_i = \partial f / \partial x_i$ and call these functions *weak partial derivatives*. We also write $\nabla f = (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$ and

$$|\nabla f| = \left(\sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|^2 \right)^{1/2}.$$

The space $W^{1,p}(\Omega, V)$ is equipped with the norm

$$\|f\|_{1,p} = \left(\int_{\Omega} \|f\|^p \right)^{1/p} + \left(\int_{\Omega} |\nabla f|^p \right)^{1/p}.$$

It is an easy exercise to show that $W^{1,p}(\Omega, V)$ is a Banach space.

The problem with this definition is that it is not clear what conditions are needed to guarantee that Lipschitz functions belong to the Sobolev space. Indeed, a Lipschitz function $f : [0, 1] \rightarrow V$ need not be differentiable in the Fréchet sense at any point, unless V has the Radon-Nikodym property, cf. [61, p. 259]. Since we want to work with Sobolev mappings from the geometric point of view it is a very unpleasant situation.

There is another, more geometric, definition of the Sobolev space of functions with values in Banach spaces which we describe now. The definition below is motivated by the work of Ambrosio [2] and Reshetnyak [70].

Let $\Omega \subset \mathbb{R}^n$ be an open set, V a Banach space and $1 \leq p < \infty$. The space $R^{1,p}(\Omega, V)$ is the class of all functions $f \in L^p(\Omega, V)$ such that:

- (1) for every $v^* \in V^*$, $\|v^*\| \leq 1$ we have $\langle v^*, f \rangle \in W^{1,p}(\Omega)$;
- (2) there is a nonnegative function $g \in L^p(\Omega)$ such that

$$(2.2) \quad |\nabla \langle v^*, f \rangle| \leq g \quad \text{a.e.}$$

for every $v^* \in V^*$ with $\|v^*\| \leq 1$.

Using arguments similar to those in the proof of the completeness of L^p one can easily show that $R^{1,p}(\Omega, V)$ is a Banach space with respect to the norm

$$\|f\|_{R^{1,p}} = \|f\|_p + \inf \|g\|_p$$

where the infimum is over the class of all functions g satisfying the inequality (2.2). Using the definitions and the property (2.1) one can easily prove the following result, see [35].

Proposition 2.1. *If $\Omega \subset \mathbb{R}^n$ is open and V is a Banach space, then $W^{1,p}(\Omega, V) \subset R^{1,p}(\Omega, V)$ and $\|f\|_{R^{1,p}} \leq \|f\|_{1,p}$ for all $f \in W^{1,p}(\Omega, V)$.*

However we can prove the opposite inclusion only under additional assumptions about the space V , see [35].

Theorem 2.2. *If $\Omega \subset \mathbb{R}^n$ is open, $V = Y^*$ is dual to a separable Banach space Y , and $1 \leq p < \infty$, then $W^{1,p}(\Omega, V) = R^{1,p}(\Omega, V)$ and $\|f\|_{R^{1,p}} \leq \|f\|_{1,p} \leq \sqrt{n}\|f\|_{R^{1,p}}$.*

Idea of the proof. One only needs to prove the inclusion $R^{1,p} \subset W^{1,p}$ along with the estimate for the norm. Actually the proof of this inclusion is quite long and it consists of several steps. In the sketch provided below many delicate steps are omitted.

By the canonical embedding $Y \subset Y^{**} = V^*$ elements of the Banach space Y can be interpreted as functionals on V . Observe that if $u : [0, 1] \rightarrow V$ is absolutely continuous, then for every $v^* \in Y$ the function $\langle v^*, u \rangle$ is absolutely continuous, so it is differentiable a.e. and satisfies the integration by parts formula. Since the space Y is separable we have a.e. differentiability of $\langle v^*, u \rangle$ and the integration by parts for all v^* from a countable and dense subset of Y . This implies that the function $u : [0, 1] \rightarrow V$ is differentiable in a certain weak sense known as the w^* -differentiability. Moreover the w^* -derivative $u' : [0, 1] \rightarrow V$ satisfies the integration by parts formula

$$\int_0^1 \varphi'(t)u(t) dt = - \int_0^1 \varphi(t)u'(t) dt.$$

Using this fact and the Fubini theorem one can prove that a function $f \in L^p(\Omega, V)$ that is absolutely continuous on almost all lines parallel to coordinate axes and such that the w^* -partial derivatives of f satisfy $\|\partial f / \partial x_i\| \leq g$ a.e. for some $g \in L^p(\Omega)$ belongs to the Sobolev space $W^{1,p}(\Omega, V)$, $\|f\|_{1,p} \leq \|f\|_p + \sqrt{n}\|g\|_p$. This fact is similar to the well-known characterization of the Sobolev space $W^{1,p}(\Omega)$ by absolute continuity on lines.

In the last step one proves that if $f \in R^{1,p}(\Omega, V)$, then f is absolutely continuous on almost all lines parallel to the coordinate axes and the w^* -partial derivatives satisfy $\|\partial f / \partial x_i\| \leq g$, where the function $g \in L^p(\Omega)$ satisfies (2.2).

The above facts put together easily imply the result. □

One can prove the following more geometric characterization of the space $R^{1,p}(\Omega, V)$ which is very useful, see [35].

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be open, V a Banach space and $1 \leq p < \infty$. Then $f \in R^{1,p}(\Omega, V)$ if and only if $f \in L^p(\Omega, V)$ and there is a nonnegative function $g \in L^p(\Omega)$ such that for every Lipschitz continuous function $\varphi : V \rightarrow \mathbb{R}$, $\varphi \circ f \in W^{1,p}(\Omega)$ and $|\nabla(\varphi \circ f)| \leq \text{Lip}(\varphi)g$ a.e.*

Idea of the proof. One implication is obvious. Indeed, if a function f satisfies the condition described in the above theorem, then it belongs to the space $R^{1,p}(\Omega, V)$, because for $v^* \in V^*$, $\|v^*\| \leq 1$, $\varphi(v) = \langle v^*, v \rangle$ is 1-Lipschitz continuous and hence $\langle v^*, f \rangle \in W^{1,p}(\Omega)$ with $|\nabla \langle v^*, f \rangle| \leq g$ a.e.

In the other implication we use the fact that $R^{1,p}(\Omega, V)$ functions are absolutely continuous on almost all lines parallel to coordinate axes. This implies that if $\varphi : V \rightarrow \mathbb{R}$ is Lipschitz continuous, then also $\varphi \circ f$ is absolutely continuous on almost all lines and hence $\varphi \circ f \in W^{1,p}(\Omega)$ by the characterization of $W^{1,p}(\Omega)$ in terms of absolute continuity on lines. \square

Now we are ready to define the Sobolev space of mappings with values into an arbitrary metric space. Let $\Omega \subset \mathbb{R}^n$ be open and X a metric space. We can assume that X is isometrically embedded into a Banach space V . We have now two natural definitions

$$W^{1,p}(\Omega, X) = \{f \in W^{1,p}(\Omega, V) \mid f(\Omega) \subset X\}$$

and

$$R^{1,p}(\Omega, X) = \{f \in R^{1,p}(\Omega, V) \mid f(\Omega) \subset X\}$$

Both spaces $W^{1,p}(\Omega, X)$ and $R^{1,p}(\Omega, X)$ are endowed with the norm metric.

Since every Lipschitz function $\varphi : X \rightarrow \mathbb{R}$ can be extended to a Lipschitz function $\tilde{\varphi} : V \rightarrow \mathbb{R}$ with the same Lipschitz constant (McShane extension) we easily see that *if X is compact and Ω is bounded, then $f \in R^{1,p}(\Omega, X)$ if and only if there is a nonnegative function $g \in L^p(\Omega)$ such that for every Lipschitz continuous function $\varphi : X \rightarrow \mathbb{R}$ we have $\varphi \circ f \in W^{1,p}(\Omega)$ and $|\nabla(\varphi \circ f)| \leq \text{Lip}(\varphi)g$ a.e.*

We assume here compactness of X and boundedness of Ω to avoid problems with the L^p integrability of f .

Observe that the last characterization of the space $R^{1,p}(\Omega, X)$ is independent of the isometric embedding of X into a Banach space.

As a direct application of Theorem 2.2 we have

Theorem 2.4. *If $\Omega \subset \mathbb{R}^n$ is open, $V = Y^*$ is dual to a separable Banach space Y , $1 \leq p < \infty$, and $X \subset V$, then $W^{1,p}(\Omega, X) = R^{1,p}(\Omega, X)$.*

The most interesting case is when the space X is separable. In this case X admits an isometric embedding to $V = \ell^\infty$, which is dual to a separable Banach space, $\ell^\infty = (\ell^1)^*$ and hence Theorem 2.4 applies.

With a minor effort one can extend the above arguments to the case of Sobolev spaces defined on a manifold, which leads to the spaces $W^{1,p}(M, X)$ and $R^{1,p}(M, X)$.

The following theorem is the main result in [35].

Suppose that any two points $x, y \in X$ can be connected by a curve of finite length. Then $d_\ell(x, y)$ defined as the infimum of lengths of curves connecting x to y is a metric. We call it *length metric*. Since $d_\ell(x, y) \geq d(x, y)$, it easily follows that if X is compact with respect to d_ℓ , then X is compact with respect to d .

Theorem 2.5. *Let X be a metric space, compact with respect to the length metric. If $n \geq 2$, then there is a continuous Sobolev mapping $f \in C^0 \cap W^{1,n}([0, 1]^n, X)$ such that $f([0, 1]^n) = X$.*

2.1. Density. Once the space of Sobolev mappings with values into metric spaces has been defined, we can ask under what conditions Lipschitz mappings $\text{Lip}(M, X)$ are dense in $W^{1,p}(M, X)$ or in $R^{1,p}(M, X)$. In this section we will follow [30] and provide several counterexamples to natural questions and very few positive results. For the simplicity sake we assume that the metric space X is compact and it admits an isometric embedding into the Euclidean space. Thus $X \subset \mathbb{R}^\nu$ and we simply define

$$W^{1,p}(M, X) = \{f \in W^{1,p}(M, \mathbb{R}^\nu) \mid f(M) \subset X\}.$$

If M and N are smooth compact manifolds, $\dim M = n$, then, as we know (Theorem 1.1), smooth mappings are dense in $W^{1,n}(M, N)$. The key property of N used in the proof was the existence of a smooth nearest point projection from a tubular neighborhood of N . The proof employed the fact that the composition with the smooth nearest point projection is continuous in the Sobolev norm. It turns out that the composition with a Lipschitz mapping need not be continuous in the Sobolev norm, [30].

Theorem 2.6. *There is a Lipschitz function $\varphi \in \text{Lip}(\mathbb{R}^2)$ with compact support such that the operator $\Phi : W^{1,p}([0, 1], \mathbb{R}^2) \rightarrow W^{1,p}([0, 1])$ defined as composition $\Phi(f) = \varphi \circ f$ is not continuous for any $1 \leq p < \infty$.*

The proof of continuity of the composition with a *smooth* function φ is based on the chain rule and the continuity of the derivative $\nabla\varphi$. If φ is just Lipschitz continuous, then $\nabla\varphi$ is only measurable, so the proof does not work and the existence of the example as in the theorem above is not surprising after all, see, however [64].

Although the composition with a Lipschitz mapping is not continuous in the Sobolev norm, we can still prove that Theorem 1.1 is true if we replace N by a compact Lipschitz neighborhood retract.

We say that a closed set $X \subset \mathbb{R}^\nu$ is a *Lipschitz neighborhood retract* if there is an open neighborhood $\mathcal{U} \subset \mathbb{R}^\nu$ of X , $X \subset \mathcal{U}$, and a Lipschitz retraction $\pi : \mathcal{U} \rightarrow X$, $\pi \circ \pi = \pi$.

The following result was proved in [30].

Theorem 2.7. *Let $X \subset \mathbb{R}^\nu$ be a compact Lipschitz neighborhood retract. Then for every smooth compact n -dimensional manifold M , Lipschitz mappings $\text{Lip}(M, X)$ are dense in $W^{1,p}(M, X)$ for $p \geq n$.*

Sketch of the proof. If $f \in W^{1,p}(M, X)$ and $f_i \in C^\infty(M, \mathbb{R}^\nu)$, is a smooth approximation based on the mollification, then $\|f_i - f\|_{1,p} \rightarrow 0$ and for all sufficiently large i , the values of f_i belong to \mathcal{U} (Sobolev embedding for $p > n$ and Poincaré inequality for $p = n$), but there is no reason to claim that $\pi \circ f_i \rightarrow \pi \circ f = f$. To overcome this problem one needs to construct another approximation $f_t \in \text{Lip}(M, \mathbb{R}^\nu)$ such that (1) Lipschitz constant of f_t is bounded by Ct ; (2) $t^p |\{f \neq f_t\}| \rightarrow 0$ as $t \rightarrow \infty$; (3) $\sup_{x \in M} \text{dist}(f_t(x), X) \rightarrow 0$ as $t \rightarrow \infty$. The construction of such an approximation is not easy, but once we have it a routine calculation shows that $\|f - \pi \circ f_t\|_{1,p} \rightarrow 0$ as $t \rightarrow \infty$. Indeed,

$$\begin{aligned} & \left(\int_M |\nabla f - \nabla(\pi \circ f_t)|^p \right)^{1/p} \\ & \leq \left(\int_{\{f \neq f_t\}} |\nabla f|^p \right)^{1/p} + \left(\int_{\{f \neq f_t\}} |\nabla(\pi \circ f_t)|^p \right)^{1/p} \\ & \leq \left(\int_{\{f \neq f_t\}} |\nabla f|^p \right)^{1/p} + Ct |\{f \neq f_t\}|^{1/p} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

The proof is complete. \square

The class of Lipschitz neighborhood retracts contains Lipschitz submanifolds of \mathbb{R}^ν , [63, Theorem 5.13].

In the following example X is replaced by an n -dimensional submanifold of the Euclidean space such that it is smooth except for a just one point, and we no longer have density of Lipschitz mappings, [30].

Theorem 2.8. *Let $M \subset \mathbb{R}^\nu$ be a closed n -dimensional manifold. Then there is a homeomorphism $\Phi \in C^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu)$ which is a diffeomorphism in $\mathbb{R}^\nu \setminus \{0\}$, which is identity outside a sufficiently large ball and has the property that Lipschitz mappings $\text{Lip}(M, \widetilde{M})$ are not dense in $W^{1,n}(M, \widetilde{M})$, where $\widetilde{M} = \Phi^{-1}(M)$.*

Clearly \widetilde{M} cannot be a Lipschitz neighborhood retract. The derivative of the mapping Φ is zero at 0 and hence derivative of Φ^{-1} is unbounded in a neighborhood of 0. This causes \widetilde{M} to have highly oscillating smooth "wrinkles" which accumulate at one point. In a neighborhood of that point \widetilde{M} is a graph of a continuous function which is smooth everywhere except for this point. Actually construction is done in such a way that \widetilde{M} is $W^{1,n}$ -homeomorphic to M , but due to high oscillations there is no Lipschitz

mapping from M onto \widetilde{M} , and one proves that this $W^{1,n}$ -homeomorphism cannot be approximated by Lipschitz mappings. This actually shows that there is a continuous Sobolev mapping from M onto \widetilde{M} which cannot be approximated by Lipschitz mappings, a situation which will never occur in the case of approximation of mappings between smooth manifolds, see Proposition 1.2.

Another interesting question is stability of the density of Lipschitz mappings with respect to bi-Lipschitz modifications of the target.

Assume that X and Y are compact subsets of \mathbb{R}^ν that are bi-Lipschitz homeomorphic. Assume that M is a closed n -dimensional manifold and that Lipschitz mappings $\text{Lip}(M, X)$ are dense in $W^{1,p}(M, X)$ for some $1 \leq p < \infty$. Are the Lipschitz mappings $\text{Lip}(M, Y)$ dense in $W^{1,p}(M, Y)$?

Since bi-Lipschitz invariance is a fundamental principle in geometric analysis on metric spaces, one expects basic theorems and definitions to remain unchanged when the ambient space is subject to a bi-Lipschitz transformation. Although the composition with a Lipschitz mapping is not continuous in the Sobolev norm, there are several reasons to expect the positive answer to remain in accordance with the principle.

Firstly, if $\Phi : X \rightarrow Y$ is a bi-Lipschitz mapping, then $T(f) = \Phi \circ f$ induces bijections

$$T : W^{1,p}(M, X) \rightarrow W^{1,p}(M, Y), \quad T : \text{Lip}(M, X) \rightarrow \text{Lip}(M, Y).$$

Secondly, have the following positive result, [31].

Theorem 2.9. *If Lipschitz mappings $\text{Lip}(M, X)$ are dense in $W^{1,p}(M, X)$ in the following strong sense: for every $\varepsilon > 0$ there is $f_\varepsilon \in \text{Lip}(M, X)$ such that $|\{x \mid f_\varepsilon(x) \neq f(x)\}| < \varepsilon$, and $\|f - f_\varepsilon\|_{1,p} < \varepsilon$, then Lipschitz mappings are dense in $W^{1,p}(M, Y)$.*

The strong approximation property described in the theorem is quite natural, because if $f \in W^{1,p}(M, \mathbb{R}^\nu)$, then for every $\varepsilon > 0$ there is a Lipschitz mapping $f_\varepsilon \in \text{Lip}(M, \mathbb{R}^\nu)$ such that $|\{x \mid f_\varepsilon(x) \neq f(x)\}| < \varepsilon$ and $\|f - f_\varepsilon\|_{1,p} < \varepsilon$. Such an approximation argument was employed in the proof of Theorem 2.7.

The above facts are convincing reasons to believe that the answer to the stability question should be positive. Surprisingly it is not. The following counterexample was constructed in [30].

Theorem 2.10. *Fix an integer $n \geq 2$. There is a compact and connected set $X \subset \mathbb{R}^{n+2}$ and a global bi-Lipschitz homeomorphism $\Phi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+2}$ with the property that for any closed n -dimensional manifold M smooth mappings $C^\infty(M, X)$ are dense in $W^{1,n}(M, X)$, but Lipschitz mappings $\text{Lip}(M, Y)$ are not dense in $W^{1,n}(M, Y)$, where $Y = \Phi(X)$.*

By smooth mappings $C^\infty(M, X)$ we mean smooth mappings from M to \mathbb{R}^{n+2} with the image contained in X .

The space X constructed in the proof is quite irregular: it is a closure of a carefully constructed sequence of smooth submanifolds that converges to a manifold with a point singularity and all the manifolds are connected by a fractal curve. The space X looks like a stack of pancakes. The proof involves also a construction of a mapping $f \in W^{1,p}(M, X)$ which can be approximated by Lipschitz mappings, but the mappings that approximate f do not coincide with f at any point, so the strong approximation property from Theorem 2.9 is not satisfied.

3. SOBOLEV SPACES ON METRIC-MEASURE SPACES

In order to define the space of Sobolev mappings between metric spaces we need first define Sobolev spaces on metric spaces equipped with so called doubling measures. By the end of 1970s it was discovered that a substantial part of harmonic analysis could be generalized such spaces, [14]. This included the study of maximal functions, Hardy spaces and BMO, but it was only the zeroth order analysis in the sense that no derivatives were involved. The study of first order analysis with suitable generalizations of derivatives, fundamental theorem of calculus, and Sobolev spaces, in the setting of metric spaces with a doubling measure has been developed since 1990s. This area is growing and plays an important role in many areas of the contemporary mathematics, [43].

We recommend to the reader a beautiful expository paper of Juha Heinonen [44] where the significance and broad scope of applications of the first order analysis on metric spaces is carefully explained.

We will precede the definition of the Sobolev space with auxiliary definitions and results. The material of Sections 3.1–3.5 is standard by now. In our presentation we will follow [29], where the reader will find detailed proofs.

3.1. Integration on rectifiable curves. Let (X, d) be a metric space. By a *curve* in X we will mean any continuous mapping $\gamma : [a, b] \rightarrow X$. The *image of the curve* will be denoted by $|\gamma| = \gamma([a, b])$. The *length* of γ is defined as

$$\ell(\gamma) = \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \right\},$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \dots < t_n = b$. We say that the curve is *rectifiable* if $\ell(\gamma) < \infty$. The *length function* associated with a rectifiable curve $\gamma : [a, b] \rightarrow X$ is $s_\gamma : [a, b] \rightarrow [0, \ell(\gamma)]$, given by

$s_\gamma(t) = \ell(\gamma|_{[a,t]})$. Not surprisingly the length function is nondecreasing and continuous.

It turns out that every rectifiable curve admits the *arc-length* parametrization.

Theorem 3.1. *If $\gamma : [a, b] \rightarrow X$ is a rectifiable curve, then there is a unique curve $\tilde{\gamma} : [0, \ell(\gamma)] \rightarrow X$ such that*

$$(3.1) \quad \gamma = \tilde{\gamma} \circ s_\gamma.$$

Moreover $\ell(\tilde{\gamma}|_{[0,t]}) = t$ for every $t \in [0, \ell(\gamma)]$. In particular $\tilde{\gamma} : [0, \ell(\gamma)] \rightarrow X$ is a 1-Lipschitz mapping.

We call $\tilde{\gamma}$ parametrized by the *arc-length* because $\ell(\tilde{\gamma}|_{[0,t]}) = t$ for $t \in [0, \ell(\gamma)]$.

Now we are ready to define the integrals along the rectifiable curves. Let $\gamma : [a, b] \rightarrow X$ be a rectifiable curve and $\varrho : |\gamma| \rightarrow [0, \infty]$ be a Borel measurable function, where $|\gamma| = \gamma([a, b])$. Then we define

$$\int_\gamma \varrho := \int_0^{\ell(\gamma)} \varrho(\tilde{\gamma}(t)) dt,$$

where $\tilde{\gamma} : [0, \ell(\gamma)] \rightarrow X$ is the arc-length parametrization of γ .

It turns out that we can nicely express this integral in any Lipschitz parametrization of γ .

Theorem 3.2. *For every Lipschitz curve $\gamma : [a, b] \rightarrow X$ the speed*

$$|\dot{\gamma}|(t) := \lim_{h \rightarrow 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|},$$

exists a.e. and

$$(3.2) \quad \ell(\gamma) = \int_a^b |\dot{\gamma}|(t) dt.$$

Theorem 3.3. *Let $\gamma : [a, b] \rightarrow X$ be a Lipschitz curve and let $\varrho : |\gamma| \rightarrow [0, \infty]$ be Borel measurable. Then*

$$\int_\gamma \varrho = \int_a^b \varrho(\gamma(t)) |\dot{\gamma}|(t) dt.$$

3.2. Modulus. In the study of geometric properties of Sobolev functions on Euclidean spaces absolute continuity on almost all lines plays a crucial role. Thus there is a need to define a notion of almost all curves also in the setting of metric spaces. This leads to the notion of the modulus of the family of rectifiable curves, which is a kind of a measure in the space of all rectifiable curves.

Let (X, d, μ) be a *metric-measure space*, i.e. a metric space with a Borel measure that is positive and finite on every ball.

Let \mathfrak{M} denote the family of all nonconstant rectifiable curves in X . It may happen that $\mathfrak{M} = \emptyset$, but we are interested in metric spaces for which the space \mathfrak{M} is sufficiently large.

For $\Gamma \subset \mathfrak{M}$, let $F(\Gamma)$ be the family of all Borel measurable functions $\varrho : X \rightarrow [0, \infty]$ such that

$$\int_{\gamma} \varrho \geq 1 \quad \text{for every } \gamma \in \Gamma.$$

Now for each $1 \leq p < \infty$ we define

$$\text{Mod}_p(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_X \varrho^p d\mu.$$

The number $\text{Mod}_p(\Gamma)$ is called *p-modulus* of the family Γ .

The following result is easy to prove.

Theorem 3.4. *Mod_p is an outer measure on \mathfrak{M} .*

If some property holds for all curves $\gamma \in \mathfrak{M} \setminus \Gamma$, where $\text{Mod}_p(\Gamma) = 0$, then we say that the property holds for *p-a.e. curve*.

The notion of *p-a.e. curve* is consistent with the notion of almost every line parallel to a coordinate axis. Indeed, if $E \subset [0, 1]^{n-1}$ is Borel measurable and we consider straight segments passing through E

$$\Gamma_E = \{\gamma_{x'} : [0, 1] \rightarrow [0, 1]^n : \gamma_{x'}(t) = (t, x'), x' \in E\}$$

then $\text{Mod}_p(\Gamma_E) = 0$ if and only if the $(n-1)$ -dimensional Lebesgue measure of E is zero. This fact easily follows from the definition of the modulus and the Fubini theorem.

3.3. Upper gradient. As before we assume that (X, d, μ) is a metric-measure space. Let $u : X \rightarrow \mathbb{R}$ be a Borel function. Following [46] we say that a Borel function $g : X \rightarrow [0, \infty]$ is an *upper gradient* of u if

$$(3.3) \quad |u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g$$

for every rectifiable curve $\gamma : [a, b] \rightarrow X$. We say that g is a *p-weak upper gradient* of u if (3.3) holds on *p-a.e. curve* $\gamma \in \mathfrak{M}$.

If g is an upper gradient of u and $\tilde{g} = g$, μ -a.e., is another nonnegative Borel function, then it may be that \tilde{g} is no longer upper gradient of u . However, we have

Lemma 3.5. *If g is a p -weak upper gradient of u and \tilde{g} is another nonnegative Borel function such that $\tilde{g} = g$ μ -a.e., then \tilde{g} is a p -weak upper gradient of u too.*

It turns out that p -weak upper gradients can be approximated in the L^p norm by upper gradients.

Lemma 3.6. *If g is a p -weak upper gradient of u which is finite a.e., then for every $\varepsilon > 0$ there is an upper gradient g_ε of u such that*

$$g_\varepsilon \geq g \text{ everywhere,} \quad \text{and} \quad \|g_\varepsilon - g\|_{L^p} < \varepsilon.$$

We do not require here that $g \in L^p$.

The following result shows that the notion of an upper gradient is a natural generalization of the length of the gradient to the setting of metric spaces, see also Theorem 3.10.

Proposition 3.7. *If $u \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$, then $|\nabla u|$ is an upper gradient of u . This upper gradient is the least one in the sense that if $g \in L^1_{\text{loc}}(\Omega)$ is another upper gradient of u , then $g \geq |\nabla u|$ a.e.*

3.4. Sobolev spaces $N^{1,p}$. Let $\tilde{N}^{1,p}(X, d, \mu)$, $1 \leq p < \infty$, be the class of all L^p integrable Borel functions on X for which there exists a p -weak upper gradient in L^p . For $u \in \tilde{N}^{1,p}(X, d, \mu)$ we define

$$\|u\|_{\tilde{N}^{1,p}} = \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all p -weak upper gradients g of u .

Lemma 3.6 shows that in the definition of $\tilde{N}^{1,p}$ and $\|\cdot\|_{\tilde{N}^{1,p}}$, p -weak upper gradients can be replaced by upper gradients.

We define an equivalence relation in $\tilde{N}^{1,p}$, $u \sim v$ iff $\|u - v\|_{\tilde{N}^{1,p}} = 0$. Then the space $N^{1,p}(X, d, \mu)$ is defined as the quotient $\tilde{N}^{1,p}(X, d, \mu) / \sim$ and is equipped with the norm

$$\|u\|_{N^{1,p}} := \|u\|_{\tilde{N}^{1,p}}.$$

The space $N^{1,p}$ was introduced by Shanmugalingam in [77].

Theorem 3.8. *$N^{1,p}(X, d, \mu)$, $1 \leq p < \infty$ is a Banach space.*

One can prove that functions $u \in N^{1,p}(X, d, \mu)$ are absolutely continuous on almost all curves in the sense that for p -a.e. $\gamma \in \mathfrak{M}$, $u \circ \tilde{\gamma}$ is absolutely continuous, where $\tilde{\gamma}$ is the arc-length parametrization of γ . This fact, Proposition 3.7 and the characterization of the classical Sobolev space $W^{1,p}(\Omega)$ by absolute continuity on lines lead to the following result.

Theorem 3.9. *If $\Omega \subset \mathbb{R}^n$ is open and $1 \leq p < \infty$, then*

$$N^{1,p}(\Omega, |\cdot|, \mathcal{L}^n) = W^{1,p}(\Omega)$$

and the norms are equal.

Here we consider the space $N^{1,p}$ on Ω regarded as a metric space with respect to the Euclidean metric $|\cdot|$ and the Lebesgue measure \mathcal{L}^n . The following result supplements the above theorem.

Theorem 3.10. *Any function $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, has a representative for which $|\nabla u|$ is a p -weak upper gradient. On the other hand if $g \in L^1_{\text{loc}}$ is a p -weak upper gradient of u , then $g \geq |\nabla u|$ a.e.*

Both of the above theorems hold also when Ω is replaced by a Riemannian manifold, and also in this case $|\nabla u|$ is the least p -weak upper gradient of $u \in W^{1,p}$. Actually, one can prove that there always exists a minimal p -weak upper gradient.

Theorem 3.11. *For every $u \in N^{1,p}(X, d, \mu)$, $1 \leq p < \infty$, there exists the least p -weak upper gradient $g_u \in L^p$ of u . It is smallest in the sense that if $g \in L^p$ is another p -weak upper gradient of u , then $g \geq g_u$ μ -a.e.*

3.5. Doubling measures. We say that the measure μ is *doubling* if there is a constant $C_d \geq 1$ (called *doubling constant*) such that $0 < \mu(2B) \leq C_d \mu(B) < \infty$ for every ball $B \subset X$.

We say that a metric space X is *metric-doubling* if there is a constant $M > 0$ such that every ball in X can be covered by at most M balls of half the radius.

If μ is a doubling measure on X , then it easily follows that X is metric-doubling. In particular bounded sets in X are totally bounded. Hence if X is a complete metric space equipped with a doubling measure, then bounded and closed sets are compact.

The following beautiful characterization of metric spaces supporting doubling measures was proved by Volberg and Konyagin, [62], [82].

Theorem 3.12. *Let X be a complete metric space. Then there is a doubling measure on X if and only if X is metric-doubling.*

The doubling condition implies a lower bound for the measure of a ball.

Lemma 3.13. *If the measure μ is doubling with the doubling constant C_d , and $s = \log_2 C_d$, then*

$$(3.4) \quad \frac{\mu(B(x, r))}{\mu(B_0)} \geq 4^{-s} \left(\frac{r}{r_0} \right)^s$$

whenever B_0 is a ball of radius r_0 , $x \in B_0$ and $r \leq r_0$.

The lemma easily follows from the iteration of the doubling inequality. The exponent s is sharp as the example of the Lebesgue measure shows.

Metric spaces equipped with a doubling measure are called *spaces of homogeneous type* and $s = \log_2 C_d = \log C_d / \log 2$ is called *homogeneous dimension*.

An important class of doubling measures is formed by so called *n-regular* measures⁵, which are measures for which there are constants $C \geq 1$ and $s > 0$ such that $C^{-1}r^s \leq \mu(B(x, r)) \leq Cr^s$ for all $x \in X$ and $0 < r < \text{diam } X$. The s -regular measures are closely related to the Hausdorff measure \mathcal{H}^s since we have

Theorem 3.14. *If μ is a an s -regular measure, then there is a constant $C \geq 1$ so that $C^{-1}\mu(E) \leq \mathcal{H}^s(E) \leq C\mu(E)$ for every Borel set $E \subset X$. In particular \mathcal{H}^s is s -regular too.*

The proof is based on the so called $5r$ -covering lemma.

For a locally integrable function $g \in L^1_{\text{loc}}(\mu)$ we define the *Hardy-Littlewood maximal function*

$$\mathcal{M}g(x) = \sup_{r>0} \int_{B(x,r)} |g| d\mu.$$

Theorem 3.15. *If μ is doubling, then*

- (1) $\mu(\{x : \mathcal{M}g(x) > t\}) \leq Ct^{-1} \int_X |g| d\mu$ for every $t > 0$.
- (2) $\|\mathcal{M}g\|_{L^p} \leq C\|g\|_{L^p}$, for $1 < p < \infty$.

3.6. Other spaces of Sobolev type. There are many other definitions of Sobolev type spaces on metric spaces that we describe now, see [19], [27], [29], [32], [33]. Let (X, d, μ) be a metric-measure space with a doubling measure.

Following [27] for $0 < p < \infty$ we define $M^{1,p}(X, d, \mu)$ to be the set of all functions $u \in L^p(\mu)$ for which there is $0 \leq g \in L^p(\mu)$ such that

$$(3.5) \quad |u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \quad \mu\text{-a.e.}$$

Then we set

$$\|u\|_{M^{1,p}} = \|u\|_p + \inf_g \|g\|_p$$

where the infimum is taken over the class of all g satisfying (3.5). For $p \geq 1$, $\|\cdot\|_{1,p}$ is a norm and $M^{1,p}(X, d, \mu)$ is a Banach space.

⁵Called also *Ahlfors–David* regular measures.

For a locally integrable function u we define the *Calderón maximal function*

$$u_1^\#(x) = \sup_{r>0} r^{-1} \int_{B(x,r)} |u - u_B| d\mu.$$

Following [32] we define $C^{1,p}(X, d, \mu)$ to be the class of all $u \in L^p(\mu)$ such that $u_1^\# \in L^p(\mu)$. Again, for $p \geq 1$, $C^{1,p}(X, d, \mu)$ is a Banach space with respect to the norm

$$\|u\|_{C^{1,p}} = \|u\|_p + \|u_1^\#\|_p.$$

Following [33], for $0 < p < \infty$ we say that a locally integrable function $u \in L^1_{\text{loc}}$ belongs to the space $P^{1,p}(X, d, \mu)$ if there are $\sigma \geq 1$ and $0 \leq g \in L^p(\mu)$ such that

$$(3.6) \quad \int_B |u - u_B| d\mu \leq r \left(\int_{\sigma B} g^p d\mu \right)^{1/p} \quad \text{for every ball } B \text{ of radius } r.$$

We do not equip the space $P^{1,p}$ with a norm.

To motivate the above definitions observe that $u \in W^{1,p}(\mathbb{R}^n)$ satisfies the pointwise inequality

$$|u(x) - u(y)| \leq C|x - y|(\mathcal{M}|\nabla u|(x) + \mathcal{M}|\nabla u|(y)) \quad \text{a.e.},$$

where $\mathcal{M}|\nabla u|$ is the Hardy-Littlewood maximal function, so $g = \mathcal{M}|\nabla u| \in L^p$ for $p > 1$, and actually one can prove, [27], that $u \in W^{1,p}(\mathbb{R}^n)$, $p > 1$, if and only if $u \in L^p$ and there is $0 \leq g \in L^p$ such that $|u(x) - u(y)| \leq |x - y|(g(x) + g(y))$ a.e. Moreover $\|u\|_{1,p} \approx \|u\|_p + \inf_g \|g\|_p$. Thus for $p > 1$, $W^{1,p}(\mathbb{R}^n) = M^{1,p}(\mathbb{R}^n)$. For the case $p = 1$, $M^{1,1}(\mathbb{R}^n)$ is not equivalent with $W^{1,1}(\mathbb{R}^n)$, [28], see, however, [57] and Theorem 3.16 below.

The classical Poincaré inequality

$$(3.7) \quad \int_{B(x,r)} |u - u_{B(x,r)}| \leq Cr \int_{B(x,r)} |\nabla u|$$

implies that for $u \in W^{1,p}(\mathbb{R}^n)$ the Calderón maximal function is bounded by the maximal function of $|\nabla u|$ and hence it belongs to L^p for $p > 1$. Calderón [10] proved that for $p > 1$, $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p$ and $u_1^\# \in L^p$. Moreover $\|u\|_{1,p} \approx \|u\|_p + \|u_1^\#\|_p$. Thus for $p > 1$, $W^{1,p}(\mathbb{R}^n) = C^{1,p}(\mathbb{R}^n)$.

The inequality (3.7) also implies that for $p \geq 1$, $\sigma \geq 1$ and $u \in W^{1,p}(\mathbb{R}^n)$ we have

$$\int_B |u - u_B| dx \leq Cr \left(\int_{\sigma B} |\nabla u|^p dx \right)^{1/p}.$$

Thus $W^{1,p}(\mathbb{R}^n) \subset P^{1,p} \cap L^p$. On the other hand it was proved in [56], [19], [28] that for $p \geq 1$, $W^{1,p}(\mathbb{R}^n) = P^{1,p} \cap L^p$.

In the case of general metric spaces we have

Theorem 3.16. *If the measure μ is doubling, and $1 \leq p < \infty$, then*

$$C^{1,p}(X, d, \mu) = M^{1,p}(X, d, \mu) \subset P^{1,p}(X, d, \mu) \cap L^p(\mu) \subset N^{1,p}(X, d, \mu).$$

For a proof see [29, Corollary 10.5 and Theorem 9.3], [32, Theorem 3.4 and Lemma 3.6] and [71].

The so called telescoping argument (infinite iteration of the inequality (3.6) on a decreasing sequence of balls) shows that if $u \in P^{1,p}(X, d, \mu)$, then

$$(3.8) \quad |u(x) - u(y)| \leq Cd(x, y)((\mathcal{M}g^p(x))^{1/p} + (\mathcal{M}g^p(y))^{1/p}) \quad \text{a.e.}$$

see [33]. A version of the same telescoping argument shows also that for $u \in L^1_{\text{loc}}$ we have

$$|u(x) - u(y)| \leq Cd(x, y)(u_1^\#(x) + u_1^\#(y)) \quad \text{a.e.}$$

see [32, Lemma 3.6]. This implies that $C^{1,p} \subset M^{1,p}$ for $p \geq 1$. On the other hand if $u \in M^{1,p}$, $|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$, then a direct integration with respect to x and y yield

$$\int_B |u - u_B| d\mu \leq 4r \int_B g d\mu \leq 4r \left(\int_B g^p d\mu \right)^{1/p}.$$

Hence $M^{1,p} \subset P^{1,p} \cap L^p$ and also

$$u_1^\# \leq 4\mathcal{M}g$$

which shows that $M^{1,p} \subset C^{1,p}$ for $p > 1$. Thus for $p > 1$, $C^{1,p} = M^{1,p}$. The case $p = 1$ of this equality is more difficult, see [29, Theorem 9.3] and [71].

For the proof of the remaining inclusion $P^{1,p} \cap L^p \subset N^{1,p}$, see [29, Corollary 10.5].

If a metric space X has no non-constant rectifiable curves, then $g = 0$ is an upper gradient of any $u \in L^p$ and hence $N^{1,p}(X, d, \mu) = L^p(\mu)$. On the other hand the theory of Sobolev spaces $M^{1,p}$, $C^{1,p}$ and $P^{1,p}$ is not trivial in this case. Indeed, a variant of the above telescoping argument leads to the estimate of $|u - u_B|$ by a generalized Riesz potential, [33], and hence the fractional integration theorem implies Sobolev embedding theorems. Many results of the classical theory of Sobolev spaces extend to this situation, see e.g. [27], [33], [29], and we state just one of them.

Theorem 3.17. *Let μ be a doubling measure and $s = \log C_d / \log 2$ be as in Lemma 3.13. If $u \in L^1_{\text{loc}}(\mu)$, $\sigma \geq 1$ and $0 \leq g \in L^p(\mu)$, $0 < p < s$ are such that the p -Poincaré inequality*

$$\int_B |u - u_B| d\mu \leq r \left(\int_{\sigma B} g^p d\mu \right)^{1/p}$$

holds on every ball B of radius r , then for any $p < q < s$ the Sobolev-Poincaré inequality

$$\left(\int_B |u - u_B|^{q^*} d\mu \right)^{1/q^*} \leq Cr \left(\int_{5\sigma B} g^q d\mu \right)^{1/q}$$

holds on every ball B of radius r , where $q^* = sq/(s - q)$ is the Sobolev exponent.

This result implies Sobolev embedding for the spaces $C^{1,p}$, $M^{1,p}$ and $P^{1,p}$ but not for $N^{1,p}$.

Other results for $C^{1,p}$, $M^{1,p}$ and $P^{1,p}$ spaces available in the general case of metric spaces with doubling measure include Sobolev embedding into Hölder continuous functions, Trudinger inequality, compact embedding theorem, embedding on spheres and extension theorems.

3.7. Spaces supporting the Poincaré inequality. Metric spaces equipped with doubling measures are too general for the theory of $N^{1,p}$ spaces to be interesting. Indeed, if there are no non-constant rectifiable curves in X , then as we have already observed, $N^{1,p}(X, d, \mu) = L^p(\mu)$. Thus we need impose additional conditions on the metric space that will imply, in particular, existence of many rectifiable curves. Such a condition was discovered by Heinonen and Koskela [46].

We say that (X, d, μ) supports a p -Poincaré inequality, $1 \leq p < \infty$, if the measure μ is doubling and there exist constants C_p and $\sigma \geq 1$ such that for every ball $B \subset X$, every Borel measurable function $u \in L^1(\sigma B)$ and every upper gradient $0 \leq g \in L^p(\sigma B)$ of u on σB , the following Poincaré type inequality is satisfied

$$(3.9) \quad \int_B |u - u_B| d\mu \leq C_p r \left(\int_{\sigma B} g^p d\mu \right)^{1/p}.$$

Note that this condition immediately implies existence of rectifiable curves. Indeed, if u is not constant, then $g = 0$ cannot be an upper gradient of u as otherwise the inequality (3.9) would not be satisfied. More precisely we have, see e.g. [33, Proposition 4.4].

Theorem 3.18. *If a space X supports a p -Poincaré inequality, then there is a constant $C > 0$ such that any two points $x, y \in X$ can be connected by a curve of length less than or equal to $Cd(x, y)$.*

Clearly, \mathbb{R}^n supports the p -Poincaré inequality for all $1 \leq p < \infty$. Another example of spaces supporting Poincaré inequalities is provided by Riemannian manifolds of nonnegative Ricci curvature, [8], [72]. There are however, many examples of spaces supporting Poincaré inequalities which carry some mild geometric structure, but do not resemble Riemannian manifolds, [7],

[45], [46], [59], [60], [75]. An important class of spaces that support the p -Poincaré inequality is provided by so called *Carnot groups* [23], [68], [9] and more general Carnot-Carathéodory spaces [22], [23]. For the simplicity sake only the simplest case of the Heisenberg group will be described here.

The Heisenberg group \mathbb{H}_1 can be identified with $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ equipped with the noncommutative group law $(z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\text{Im}(z_1 \bar{z}_2))$. It is equipped with a nonriemannian metric $d(x, y) = \|a^{-1} \cdot b\|$, where $\|(z, t)\| = (|z|^4 + t^2)^{1/2}$. This metric is bi-Lipschitz equivalent to another so called Carnot-Carathéodory metric. The metric d is quite exotic because the Hausdorff dimension of (\mathbb{H}_1, d) is 4, while topological dimension is 3. The applications of the Heisenberg group include several complex variables, subelliptic equations and noncommutative harmonic analysis [78]. More recently it has been a subject of an intense study from the perspective of geometric measure theory [21], [9].

If a space (X, d, μ) supports the p -Poincaré inequality, $u \in N^{1,p}(X, d, \mu)$ and $0 \leq g \in L^p(\mu)$ is an upper gradient of u , then the p -Poincaré inequality (3.9) is satisfied and hence the Sobolev embedding, Theorem 3.17, holds. One can actually prove that in this case we can take $q = p$, i.e.

$$\left(\int_B |u - u_B|^{p^*} d\mu \right)^{1/p^*} \leq Cr \left(\int_{5\sigma B} g^p d\mu \right)^{1/p}$$

on every ball B of radius r , where $1 \leq p < s$ and $p^* = sp/(s - p)$, see [33].

A direct application of the Hölder inequality shows that if a space supports a p -Poincaré inequality, then it also supports a q -Poincaré inequality for all $q > p$. On the other hand we have the following important result of Keith and Zhong [54].

Theorem 3.19. *If a complete metric-measure space supports a p -Poincaré inequality for some $p > 1$, then it also supports a q -Poincaré inequality for some $1 \leq q < p$.*

This important result implies that in the case of spaces supporting the p -Poincaré inequality other approaches to Sobolev spaces described in the previous section are equivalent.

Theorem 3.20. *If the space supports the p -Poincaré inequality, $1 < p < \infty$, then $C^{1,p}(X, d, \mu) = M^{1,p}(X, d, \mu) = P^{1,p}(X, d, \mu) \cap L^p(\mu) = N^{1,p}(X, d, \mu)$.*

Indeed, prior to the work of Keith and Zhong it was known that the spaces are equal provided the space supports the q -Poincaré inequality for some $1 \leq q < p$, see e.g., [29, Theorem 11.3].

Spaces supporting Poincaré inequalities play a fundamental role in the modern theory of quasiconformal mappings [46], [47], in the geometric rigidity problems, [7], in nonlinear subelliptic equations, see, e.g. [11], [22], [20], [33], [34] and in the nonlinear potential theory [1], [6].

Although the known examples show that spaces supporting a Poincaré inequality can be very exotic, surprisingly, one can prove that such spaces are always equipped with a weak differentiable structure [13], [53].

4. SOBOLEV MAPPINGS BETWEEN METRIC SPACES

Throughout this section we assume that (X, d, μ) is a metric measure space equipped with a doubling measure. Let Y be another metric space. The construction of the space of Sobolev mappings between metric spaces $N^{1,p}(X, Y)$ is similar to that in Section 2 with the difference that the classical Sobolev space is replaced by the Sobolev space $N^{1,p}$. The space $N^{1,p}(X, Y)$ was introduced in [47].

Let V be a Banach space. Following [47] we say that $F \in \tilde{N}^{1,p}(X, V)$ if $F \in L^p(X, V)$ (in the Bochner sense) and there is a Borel measurable function $0 \leq g \in L^p(\mu)$ such that

$$\|F(\gamma(a)) - F(\gamma(b))\| \leq \int_{\gamma} g$$

for every rectifiable curve $\gamma : [a, b] \rightarrow X$. We call g an upper gradient of F . We also define

$$\|F\|_{1,p} = \|F\|_p + \inf_g \|g\|_p,$$

where the infimum is taken over all upper gradients of F . Now we define $N^{1,p}(X, V) = \tilde{N}^{1,p}(X, V) / \sim$, where $F_1 \sim F_2$, when $\|F_1 - F_2\|_{1,p} = 0$.

As in the case of $N^{1,p}(X, d, \mu)$ spaces, the p -upper gradient can be replaced by p -weak upper gradient in the above definition. The following two results were proved in [47] (cf. [31] for Theorem 4.2).

Theorem 4.1. $N^{1,p}(X, V)$ is a Banach space.

Theorem 4.2. Suppose that the space (X, d, μ) supports the p -Poincaré inequality for some $1 \leq p < \infty$. Then, for every Banach space V , the pair (X, V) supports the p -Poincaré inequality in the following sense: there is a constant $C > 0$ such that, for every ball $B \subset X$, for every $F \in L^1(6\sigma B, V)$ and for every $0 \leq g \in L^p(6\sigma B)$ being a p -weak upper gradient of F on $6\sigma B$, the inequality

$$(4.1) \quad \int_B \|F - F_B\| d\mu \leq C(\text{diam } B) \left(\int_{6\sigma B} g^p d\mu \right)^{1/p}.$$

is satisfied.

The Poincaré inequality (4.1) and the standard telescoping argument implies the pointwise inequality: if $F \in N^{1,p}(X, V)$ and $0 \leq g \in L^p(\mu)$ is a p -weak upper gradient of F , then $\|F(x) - F(y)\| \leq Cd(x, y)((\mathcal{M}g^p(x))^{1/p} + (\mathcal{M}g^p(y))^{1/p})$ a.e., where on the right hand side we have the maximal function, just like in the case of the equality (3.8).

In particular F restricted to the set $E_t = \{x : \mathcal{M}g^p < t^p\}$ is Lipschitz continuous with the Lipschitz constant Ct . Using the Lipschitz extension of $F|_{E_t}$ to the entire space X (McShane extension) one can prove, [31]

Theorem 4.3. *Suppose the space (X, d, μ) supports the p -Poincaré inequality for some $1 \leq p < \infty$ and V is a Banach space. If $F \in N^{1,p}(X, V)$, then for every $\varepsilon > 0$ there is a Lipschitz mapping $G \in \text{Lip}(X, V)$ such that $\mu\{x : F(x) \neq G(x)\} < \varepsilon$ and $\|F - G\|_{1,p} < \varepsilon$.*

As we have seen in the previous section the Poincaré inequality plays a crucial role in the development of the theory of Sobolev spaces on metric spaces. Since such an inequality is also valid for $N^{1,p}(X, V)$ spaces, Theorem 4.2, many results true for $N^{1,p}(X, d, \mu)$ like e.g. Sobolev embedding theorems can be generalized to $N^{1,p}(X, V)$ spaces, see [47].

Now if Y is a metric space isometrically embedded into a Banach space V , $Y \subset V$, we define

$$N^{1,p}(X, Y) = \{F \in N^{1,p}(X, V) : F(X) \subset Y\}.$$

Since $N^{1,p}(X, V)$ is a Banach space, $N^{1,p}(X, Y)$ is equipped with a norm metric.

If X is an open set in \mathbb{R}^n or if it is a compact manifold, the space $N^{1,p}(X, d, \mu)$ is equivalent with the classical Sobolev space, see Theorem 3.9. Hence in this case the definition of $N^{1,p}(\Omega, Y)$ (or $N^{1,p}(M, Y)$) is equivalent with that of $R^{1,p}(\Omega, Y)$ (or $R^{1,p}(M, Y)$), described in Section 2, see [47].

If $F \in N^{1,p}(X, Y)$, then according to Theorem 4.3, F can be approximated by Lipschitz mappings $\text{Lip}(X, V)$ and the question is under what conditions F can be approximated by $\text{Lip}(X, Y)$ mappings.

This is a question about extension of the theory described in Section 1 to the case of Sobolev mappings between metric spaces and it was formulated explicitly by Heinonen, Koskela, Shanmugalingam and Tyson [47, Remark 6.9].

The answer to this problem cannot be easy, because as soon as we leave the setting of manifolds we have many unpleasant counterexamples like those in Section 2. A particularly dangerous situation is created by the lack of stability with respect to bi-Lipschitz deformations of the target, Theorem 2.10.

Indeed, in most of the situations there is no canonical way to choose a metric on Y and we are free to choose any metric in the class of bi-Lipschitz equivalent metrics.

An example of spaces supporting the p -Poincaré inequality is provided by the Heisenberg group and more generally by Carnot groups and Carnot-Carathéodory spaces. In this setting Gromov [23, Section 2.5.E] stated as an open problem the extension of the results from Section 1 to the case of mappings from Carnot-Carathéodory spaces to Riemannian manifolds. Thus the question of Heinonen, Koskela, Shanmugalingam and Tyson can be regarded as a more general form of Gromov's problem.

The following result was proved in [31], (cf. Theorem 2.9 above).

Theorem 4.4. *Suppose that (X, d, μ) is a doubling metric-measure space of finite measure $\mu(X) < \infty$, and Y_1, Y_2 are two bi-Lipschitz homeomorphic metric spaces of finite diameter isometrically embedded into Banach spaces V_1 and V_2 respectively. Suppose that Lipschitz mappings $\text{Lip}(X, Y_1)$ are dense in $N^{1,p}(X, Y_1)$, $1 \leq p < \infty$, in the following strong sense: for every $f \in N^{1,p}(X, Y_1)$ and every $\varepsilon > 0$ there is $f_\varepsilon \in \text{Lip}(X, Y_1)$ such that $\mu(\{x : f(x) \neq f_\varepsilon(x)\}) < \varepsilon$ and $\|f - f_\varepsilon\|_{1,p} < \varepsilon$. Then the Lipschitz mappings $\text{Lip}(X, Y_2)$ are dense in $N^{1,p}(X, Y_2)$.*

This result shows that in the case in which we can prove strong density, there is no problem with the bi-Lipschitz invariance of the density.

It turns out that also White's theorem (Theorem 1.5) and the density result of Schoen and Uhlenbeck (Theorem 1.1 and Theorem 2.7) can be generalized to the setting of mappings between metric spaces. Theorem 4.4 plays a crucial role in the proof.

Theorem 4.5. *Let (X, d, μ) be a metric-measure space of finite measure $\mu(X) < \infty$ supporting the p -Poincaré inequality. If $p \geq s = \log C_d / \log 2$ and Y is a compact metric-doubling space which is bi-Lipschitz homeomorphic to a Lipschitz neighborhood retract of a Banach space, then for every isometric embedding of Y into a Banach space Lipschitz mappings $\text{Lip}(X, Y)$ are dense in $N^{1,p}(X, Y)$. Moreover, for every $f \in N^{1,p}(X, Y)$, there is $\varepsilon > 0$ such that if $f_1, f_2 \in \text{Lip}(X, Y)$ satisfy $\|f - f_i\|_{1,p} < \varepsilon$, $i = 1, 2$, then the mappings f_1 and f_2 are homotopic.*

4.1. Lipschitz polyhedra. By a *simplicial complex* we mean a finite collection K of simplexes in some Euclidean space \mathbb{R}^ν such that:

- (1) If $\sigma \in K$ and τ is a face of σ , then $\tau \in K$;
- (2) If $\sigma, \tau \in K$, then either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a common face of σ and τ .

The set $|K| = \bigcup_{\sigma \in K} \sigma$ is called a *rectilinear polyhedron*. By a *Lipschitz polyhedron* we mean any metric space which is bi-Lipschitz homeomorphic to a rectilinear polyhedron. The main result of [31] reads as follows.

Theorem 4.6. *Let Y be a Lipschitz polyhedron and $1 \leq p < \infty$. Then the class of Lipschitz mappings $\text{Lip}(X, Y)$ is dense in $N^{1,p}(X, Y)$ for every metric-measure space X of finite measure that supports the p -Poincaré inequality if and only if $\pi_1(Y) = \pi_2(Y) = \dots = \pi_{[p]}(Y) = 0$.*

Observe that the density of Lipschitz mappings does not depend on the particular choice of the metric in Y in the class of bi-Lipschitz equivalent metrics, only on the topology of Y . This is because in the proof of Theorem 4.6 one shows the strong approximation property described in Theorem 4.4. Theorem 4.6 can be regarded as a partial answer to problems of Heinonen, Koskela, Shanmugalingam and Tyson and also to the problem of Gromov.

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