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ON BOUNDEDNESS OF MAXIMAL FUNCTIONS IN SOBOLEV SPACES

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Abstract. Kinnunen and Lindqvist proved in [9] that the local Hardy–Littlewood maximal function, defined in an open set Ω in the Euclidean space, is a bounded operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega)$, provided $p>1$. In this note we give a simpler argument which leads to a slightly more general result. Furthermore, the argument also allows us to prove boundedness of the spherical maximal function in the Sobolev space $W^{1,p}(\Omega)$, $p>n/(n-1)$. We also raise many questions concerning boundedness of maximal operators in Sobolev spaces.

1. Introduction

The theory of Sobolev spaces and the Hardy–Littlewood maximal function, one of the most important tools in analysis, have been developed a great deal for more than seven decades now. However, surprisingly, no one had raised a question about boundedness of the maximal function in the Sobolev space $W^{1,p}$, until quite recently, when Kinnunen, [7], proved that the Hardy–Littlewood maximal operator is bounded in $W^{1,p}(\mathbf{R}^n)$, $p>1$. His argument was very nice and simple, but even a simpler argument leading to more general results is available now. The result of Kinnunen has been employed and generalized in [3], [8], [9], [10], [11], [12], [13] and [18].

Let us start with review of some basic definitions and facts. The Hardy–Littlewood maximal function of $u\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ is defined by

$$\mathscr{M}u(x)=\sup_{r>0}\fint_{B(x,r)}|u(z)|\,dz. \tag{1}$$

Here and in what follows, $g_A=\fint_A g=|A|^{-1}\int_A g$ denotes the integral average of g over a set A. The maximal function is sub-linear in the sense that $\mathscr{M}(u+v)\le\mathscr{M}u+\mathscr{M}v$.

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More generally we say that an operator $A\colon X \to Y$ acting between linear function spaces is sub-linear if $Au \geq 0$ a.e. for $u \in X$ and $A(u+v) \leq Au + Av$ a.e. for $u, v \in X$.(1) If X and Y are normed spaces, then A is called bounded provided $\|Au\|_Y \leq C\|u\|_X$ for some $C > 0$ and all $u \in X$. Note that if A is not linear, then its boundedness need not imply continuity.

The main classical result concerning the maximal operator $\mathscr{M}$ is its boundedness (even continuity) in L^p for $1 < p \leq \infty$ and the weak type estimate for $p = 1$, see [15, Chapter 1].

The maximal function has an important property of being commutative with translations. More precisely for a function u and $y \in \mathbf{R}^n$ we define $u_y(x) = u(x-y)$. Then $\mathscr{M}(u_y)(x) = (\mathscr{M}u)_y(x)$ for all $x, y \in \mathbf{R}^n$.

The Sobolev space $W^{1,p}(\Omega)$, where $\Omega \subset \mathbf{R}^n$ is an open set and $1 \leq p < \infty$, is defined as the class of all $u \in L^p(\Omega)$ such that the distributional gradient of u satisfies $|\nabla u| \in L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is a Banach space with respect to the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$.

It turns out that any sub-linear operator which commutes with translations and is bounded in L^p for some $1 < p < \infty$ forms a bounded operator in $W^{1,p}(\mathbf{R}^n)$.

Theorem 1. *Assume that* $A\colon L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$, $1 < p < \infty$, *is bounded and sub-linear. If* $A(u_y) = (Au)_y$ *for all* $u \in L^p(\mathbf{R}^n)$ *and every* $y \in \mathbf{R}^n$, *then* A *is bounded in* $W^{1,p}(\mathbf{R}^n)$.

Proof. According to [5, Section 7.11], $u \in W^{1,p}(\mathbf{R}^n)$, $1 < p < \infty$, if and only if $u \in L^p(\mathbf{R}^n)$ and $\limsup_{y\to 0} \|u_y - u\|_p/|y| < \infty$. Combining this characterization with the following inequality

$$\|(Au)_y - Au\|_p = \|A(u_y) - Au\|_p \leq \|A(u-u_y)\|_p + \|A(u_y-u)\|_p \leq C\,\|u_y - u\|_p,$$

yields the claim. □

Theorem 1 and its proof are folklore now. As a special case it gives Kinnunen's theorem:

$$\mathscr{M}\colon W^{1,p}(\mathbf{R}^n) \to W^{1,p}(\mathbf{R}^n) \text{ is bounded for } 1 < p < \infty.$$

Another interesting application concerns the spherical maximal function which is defined by

$$\mathscr{S}u(x) = \sup_{0<r\leq 1} \fint_{S^{n-1}(x,r)} |u(z)|\,d\sigma(z).$$

Here the integral average of $|u|$ is taken with respect to the $(n-1)$-dimensional surface measure. It is a deep result of Stein, [16], when $n \geq 3$, and Bourgain, [2], when $n = 2$, that the spherical maximal function is bounded in L^p for $p >$

(1) We do not assume that $A(\lambda u) = |\lambda| Au$.

$n/(n-1)$, see also [17]. This operator is clearly sub-linear and commutes with translations. Hence we conclude from Theorem 1 that

$$\text{(2)} \qquad \mathscr{S}\colon W^{1,p}(\mathbf{R}^n) \to W^{1,p}(\mathbf{R}^n) \text{ is bounded for } n/(n-1) < p < \infty,$$

see also [10]. All this raise several natural questions.

Is the Hardy–Littlewood maximal function bounded in $W^{1,1}(\mathbf{R}^n)$? No, it is not. Since the maximal operator of the characteristic function of the unit ball is not integrable over $\mathbf{R}^n$, a smooth bump function shows that the operator $\mathscr{M}$ is not bounded in $W^{1,1}(\mathbf{R}^n)$. Of course, we are more interested in weak differentiability properties of $\mathscr{M}u$ rather than in its global integrability, so we may inquire whether $\mathscr{M}u$ is in $W^{1,1}_{\mathrm{loc}}$ with $|\nabla \mathscr{M}u| \in L^1$ for $u \in W^{1,1}$.

Question 1. *Is the operator* $u \mapsto |\nabla \mathscr{M}u|$ *bounded from* $W^{1,1}(\mathbf{R}^n)$ *to* $L^1(\mathbf{R}^n)$?

Tanaka, [18], gave a positive answer to this question in the case of the non-centered maximal function and provided that $n = 1$. The non-centered maximal function is defined as the supremum taken over all balls containing $x \in \mathbf{R}$. However, his proof does not suggest how to approach the case $n > 1$.

If $1 < p < n$, $n \ge 2$, then by the Sobolev embedding theorem, [19], $W^{1,p}(\mathbf{R}^n) \subset L^{p^*}(\mathbf{R}^n)$, $p^* = np/(n-p) > n/(n-1)$, so the Bourgain–Stein maximal theorem (mentioned earlier) implies that

$$\mathscr{S}\colon W^{1,p}(\mathbf{R}^n) \to L^{p^*}(\mathbf{R}^n) \text{ is bounded for } 1 < p < n.$$

Such a result would also be a consequence of (2) together with the positive answer to

Question 2. *Is* $\mathscr{S}\colon W^{1,p}(\mathbf{R}^n) \to W^{1,p}(\mathbf{R}^n)$ *bounded for* $1 < p \le n/(n-1)$?

There are easy examples showing that $\mathscr{S}$ is not bounded on $W^{1,1}(\mathbf{R}^n)$ and even that the operator $u \mapsto |\nabla \mathscr{S}u|$ is not bounded from $W^{1,1}(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$. Indeed, if u is a smooth extension of $|x|^{1-n}\log^{-1-(n-1)/n}(e/|x|)$ from the unit ball to a compactly supported function, then $u \in W^{1,1}(\mathbf{R}^n)$, but there is no constant c such that $(\mathscr{S}u) - c \in L^{n/(n-1)}$, which contradicts the integrability of $|\nabla \mathscr{S}u|$, see e.g. [6] and also Remark 4 following Theorem 3. To see that $(\mathscr{S}u) - c \notin L^{n/(n-1)}$ for any choice of c we estimate

$$\mathscr{S}u(x) \ge v(x) := \fint_{S^{n-1}(x,|x|)} |u|\, d\sigma$$

and easily show that $v \notin L^{n/(n-1)}$ in any neighborhood of the origin.

In general, bounded non-linear operators need not be continuous. An example of this type is a surprising result of Almgren and Lieb [1] who proved that the (known to be bounded) symmetric decreasing rearrangement $\mathscr{R}\colon W^{1,p}(\mathbf{R}^n) \to W^{1,p}(\mathbf{R}^n)$ is not continuous when $1 < p < n$ and $n > 1$. The following question was asked by Iwaniec.

Question 3. *Is the Hardy–Littlewood maximal operator continuous in* $W^{1,p}(\mathbf{R}^n)$, $1 < p < \infty$?

The same question can be formulated for the spherical maximal function. However, one should try first the case $n/(n-1) < p < \infty$, because we know that the operator is bounded in this instance.

Recently Kinnunen and Lindqvist [9] have proved that the Hardy–Littlewood maximal function forms a bounded operator in $W^{1,p}(\Omega)$, where $1 < p < \infty$ and $\Omega \subset \mathbf{R}^n$ is an arbitrary open set. Here we need a local maximal function defined for $u \in L^1_{\mathrm{loc}}(\Omega)$ by [(2)]

$$\mathscr{M}_\Omega u(x) = \sup\left\{ \fint_{B(x,r)} |u(z)|\, dz : 0 < r < \operatorname{dist}(x, \partial\Omega) \right\}.$$

Their result reads as follows.

Theorem 2. *Let* $\Omega \subset \mathbf{R}^n$ *be open and* $1 < p < \infty$. *If* $u \in W^{1,p}(\Omega)$, *then* $\mathscr{M}_\Omega u \in W^{1,p}(\Omega)$ *and*

$$|\nabla \mathscr{M}_\Omega u(x)| \le 2\mathscr{M}_\Omega |\nabla u|(x) \tag{3}$$

for almost every $x \in \Omega$.

It turns out that this result does not follow from a simple modification of Kinnunen's proof for the case of $\Omega = \mathbf{R}^n$. Also the proof of Theorem 1 does not easily modify to cover Theorem 2. Indeed, there is no group structure in Ω which would allow for translations $f \mapsto f_y$. The argument of Kinnunen and Lindqvist was neat and tricky.

Theorem 2 raises more questions. Which results related to Questions 1–3 do extend to the case of a general domain? In particular we may ask

Question 4. *Is the spherical maximal function bounded in* $W^{1,p}(\Omega)$, *when* $n/(n-1) < p < \infty$?

Here we mean a local spherical maximal function defined by

$$\mathscr{S}_\Omega u(x) = \sup\left\{ \fint_{S^{n-1}(x,r)} |u(z)|\, d\sigma(z) : 0 < r < \min\{\operatorname{dist}(x, \partial\Omega), 1\} \right\}.$$

The main results of this paper are Theorems 3 and 4 below. Theorem 3 generalizes the result of Kinnunen and Lindqvist and Theorem 4 gives a positive answer to Question 4.

(2) We say that $u \in L^p_{\mathrm{loc}}(\Omega)$ if u is in L^p on every bounded and measurable subset of Ω. In particular $L^p_{\mathrm{loc}}(\Omega) = L^p(\Omega)$, if Ω is bounded. Clearly, our definition differs from the standard one. We also define $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ by assuming that $|u|$ and $|\nabla u|$ are both in $L^p_{\mathrm{loc}}(\Omega)$.

Theorem 3. *Let $\Omega \subset \mathbf{R}^n$ be an arbitrary open set and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. If $\mathscr{M}_\Omega u < \infty$ a.e. and $\mathscr{M}_\Omega |\nabla u| \in L^1_{\mathrm{loc}}(\Omega)$, then $\mathscr{M}_\Omega u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and*

$$|\nabla \mathscr{M}_\Omega u(x)| \leq 2\mathscr{M}_\Omega |\nabla u|(x) \tag{4}$$

for almost every $x \in \Omega$.

Remarks. (1) Theorem 2 follows immediately from Theorem 3. Moreover, Theorem 3 applies to the case in which $|\nabla u|\big(\log(e+|\nabla u|)\big) \in L^1_{\mathrm{loc}}(\Omega)$, $\Omega \neq \mathbf{R}^n$.(3) Indeed, in this case $\mathscr{M}_\Omega |\nabla u| \in L^1_{\mathrm{loc}}(\Omega)$ by the Stein theorem [14], and hence $\mathscr{M}_\Omega u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ by Theorem 3.

(2) We will actually show that for $\Omega = \mathbf{R}^n$, inequality (4) holds with the constant 1 in place of 2.

(3) If $u(x) = x$ and $\Omega = \mathbf{R}^n$, then $\mathscr{M} u = \infty$ everywhere, so the assumption $\mathscr{M}_\Omega u < \infty$ a.e. is needed.

(4) If $\Omega = \mathbf{R}^n$ and $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$ with $|\nabla u| \in L^1(\mathbf{R}^n)$ then, $\mathscr{M} u < \infty$ a.e. Indeed, if $n = 1$, then integrability of u' implies boundedness of u and hence boundedness of $\mathscr{M} u$. If $n > 1$, then there is a constant c, such that $u - c \in L^{n/(n-1)}$, see e.g. [6], so the claim follows from the boundedness of the maximal function in $L^{n/(n-1)}$.

(5) If $\Omega \neq \mathbf{R}^n$, then $\mathscr{M}_\Omega u < \infty$ a.e. for $u \in W^{1,1}_{\mathrm{loc}}$. Indeed, u is in L^1 on bounded sets and on each such set the radii that are used in the definition of the maximal function $\mathscr{M}_\Omega u$ are uniformly bounded, so finiteness of $\mathscr{M}_\Omega u$ follows from the finiteness of $\mathscr{M} v$ for $v \in L^1(\mathbf{R}^n)$.

(6) Note that Theorem 3 cannot be concluded from the method of Kinnunen and Lindqvist, which is based on the reflexivity of the Sobolev space $W^{1,p}$, $p > 1$. For the same reason no argument related to Theorem 1 can be used here.

Theorem 4. *Let $\Omega \subset \mathbf{R}^n$ be an arbitrary open set. If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, and $\mathscr{S}_\Omega |\nabla u| \in L^1_{\mathrm{loc}}(\Omega)$, then $\mathscr{S}_\Omega u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and*

$$|\nabla \mathscr{S}_\Omega u(x)| \leq 2\mathscr{S}_\Omega |\nabla u|(x)$$

for almost every $x \in \Omega$.

Combining Theorem 4 with the Bourgain–Stein theorem, we conclude the positive answer to Question 4.

Corollary 5. *If $\Omega \subset \mathbf{R}^n$ is open, $n \geq 2$, then $\mathscr{S}_\Omega\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is bounded for $p > n/(n-1)$.*

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(3) A suitable version holds also for $\Omega = \mathbf{R}^n$.

2. Theorems 3 and 4 in the entire space

The case of a general domain is pretty similar to the case in which the underlying domain is $\mathbf{R}^n$. However, in the general case the technical details are more difficult which makes the main geometric idea less visible. For this reason we will present the global case separately.

Proof of Theorem 3 *in the case* $\Omega = \mathbf{R}^n$. Since $|u| \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$ and $\bigl|\nabla |u|\bigr| = |\nabla u|$ a.e., we can assume that the function u is non-negative. We will prove absolute continuity of $\mathscr{M} u$ on segments $\overline{xy}$, for a.e. $x, y \in \mathbf{R}^n$. More precisely, we will prove the estimate

$$\bigl|\mathscr{M} u(x) - \mathscr{M} u(y)\bigr| \le \int_{\overline{xy}} \mathscr{M}|\nabla u| \tag{5}$$

for a.e. $x, y \in \mathbf{R}^n$. Fix $x, y \in \mathbf{R}^n$ such that $\mathscr{M} u(x) < \infty$ and $\mathscr{M} u(y) < \infty$. By symmetry, we can assume that $\mathscr{M} u(x) \ge \mathscr{M} u(y)$. Pick a sequence $(r_n)_{n=1}^{\infty}$, $0 < r_n < \infty$, such that

$$\lim_{n\to\infty} \fint_{B(x,r_n)} u(z)\,dz = \mathscr{M} u(x).$$

We write

$$u_{r_n}(x) = \fint_{B(x,r_n)} u(z)\,dz$$

for all $n \in \mathbf{N}$. Since

$$\begin{aligned} |\mathscr{M} u(x) - \mathscr{M} u(y)| &= \mathscr{M} u(x) - u_{r_n}(y) + u_{r_n}(y) - \mathscr{M} u(y) \\ &\le \mathscr{M} u(x) - u_{r_n}(y) \\ &= \bigl(\mathscr{M} u(x) - u_{r_n}(x)\bigr) + \bigl(u_{r_n}(x) - u_{r_n}(y)\bigr) \end{aligned}$$

for all $n \in \mathbf{N}$, we have

$$|\mathscr{M} u(x) - \mathscr{M} u(y)| \le \limsup_{n\to\infty} \bigl(u_{r_n}(x) - u_{r_n}(y)\bigr).$$

Combining this with the following estimate

$$\begin{aligned} |u_{r_n}(x) - u_{r_n}(y)| &= \left| \fint_{B(x,r_n)} u(z)\,dz - \fint_{B(y,r_n)} u(z)\,dz \right| \\ &= \left| \fint_{B(x,r_n)} \bigl(u(z) - u\bigl(z + (y-x)\bigr)\bigr)\,dz \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \fint_{B(x,r_n)} \int_0^1 \frac{d}{dt} u\big(z + t(y-x)\big)\, dt\, dz \right| \\
&\le |y-x| \int_0^1 \fint_{B(x,r_n)} \big|\nabla u\big(z + t(y-x)\big)\big|\, dz\, dt \\
&= |y-x| \int_0^1 \fint_{B(x+t(y-x),r_n)} |\nabla u(z)|\, dz\, dt \\
&\le \int_{\overline{xy}} \mathscr{M}|\nabla u|,
\end{aligned}
$$

yields the desired inequality (5).

Now, fix a direction $\nu \in S^{n-1}$. By Fubini's theorem $\mathscr{M}|\nabla u|$ is locally integrable on almost all lines parallel to the direction ν. Combining this with inequality (5), we find that

$$|D_\nu \mathscr{M} u(x)| \le \mathscr{M}|\nabla u|(x)$$

for almost every $x \in \mathbf{R}^n$ (cf. [4, Section 4.9]). Taking the supremum over a countable and dense set of directions $\nu \in S^{n-1}$ yields $|\nabla \mathscr{M} u| \le \mathscr{M}|\nabla u|$ a.e., as desired. □

Proof of Theorem 4 *in the case* $\Omega = \mathbf{R}^n$. We notice that the same method extends to the case of the spherical maximal operator provided the spherical maximal operator is finite almost everywhere in $\mathbf{R}^n$. This follows from the well-known trace theorem:

Lemma 6. *If* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, *then* $\mathscr{S}_\Omega u$ *is finite almost everywhere*[4] *in* Ω.

Remark. Although we prove Theorem 4 in $\mathbf{R}^n$ now, we prove the lemma for a general domain as it will be needed in the next section.

Proof. If $u \in W^{1,1}\big(B(0,1)\big)$, then the trace of u on the boundary $S^{n-1}(0,1)$ satisfies

$$\fint_{S^{n-1}(0,1)} |u|\, d\sigma \le C\bigg(\fint_{B^n(0,1)} |u| + \fint_{B^n(0,1)} |\nabla u| \bigg),$$

with C depending on n only, see [4, Section 4.3]. Hence a standard scaling argument (affine change of variables) shows that

$$\fint_{S^{n-1}(x,r)} |u|\, d\sigma \le C\bigg(\fint_{B^n(x,r)} |u| + r \fint_{B^n(x,r)} |\nabla u| \bigg)$$

[4] While dealing with the definition of $\mathscr{S}u$, for u in the Sobolev space, the restriction of u to a sphere is understood as a trace of a Sobolev function.

for $u \in W^{1,1}_{\rm loc}(\Omega)$, whenever $B(x,r) \subset \Omega$. Taking the supremum over $r \leq \min\bigl\{{\rm dist}\,(x,\partial\Omega),1\bigr\}$ yields

$$\mathscr{S}_\Omega u \leq C(\mathscr{M}_\Omega u + \mathscr{M}_\Omega|\nabla u|) < \infty \text{ a.e.}$$

This completes the proof of Lemma 6 and hence that of Theorem 4 for $\Omega = \mathbf{R}^n$. □

Remark. A delicate point in the proof of Theorem 4 is that we defined the spherical maximal function of a Sobolev function using traces. Equivalently we could say that *there is* a representative of $u \in W^{1,1}_{\rm loc}$ (in the class of functions equal to u a.e.) such that $\mathscr{S}u$ defined now in the standard way (with restrictions to spheres instead of traces) has all desired properties. At the same time our proof was very elementary, avoiding the use of the Bourgain–Stein theorem. On the other hand it follows from the Bourgain–Stein theorem that for two different representatives of a given function spherical maximal functions are equal a.e.[5] Hence the conclusion of Theorem 4 and its corollary is true for *any* representative of u with $\mathscr{S}u$ defined in a standard way.

3. Proofs of Theorems 3 and 4

Let us first focus on the case of the Hardy–Littlewood maximal operator.

Proof of Theorem 3. We can assume that $\Omega \neq \mathbf{R}^n$. The argument will be similar to that in the case $\Omega = \mathbf{R}^n$. There is, however, a problem that has to be faced. Namely the line segments connecting z to $z+(y-x)$, for $x,y,z \in \Omega$ are not necessarily contained in the domain Ω.

It suffices to prove that for every $\varepsilon > 0$ inequality (3) holds a.e. in $\Omega_\varepsilon = \bigl\{x \in \Omega : {\rm dist}(x,\Omega^c) > \varepsilon\bigr\}$. To this end it suffices to prove inequality (7) for a.e. $x,y \in \Omega_\varepsilon$ which are sufficiently close one to another.

Let $x,y \in \Omega_\varepsilon$ be such that $\mathscr{M}_\Omega u(x)$, $\mathscr{M}_\Omega u(y) < \infty$ and $|x-y| < \frac{1}{2}\varepsilon$. By symmetry, we can assume that $\mathscr{M}_\Omega u(x) \geq \mathscr{M}_\Omega u(y)$. Similarly as in the case of $\Omega = \mathbf{R}^n$ we define $(r_n)_{n=1}^\infty$ as a sequence of positive radii $0 < r_n < {\rm dist}\,(x,\partial\Omega)$ such that $u_{r_n}(x) \to \mathscr{M}_\Omega u(x)$. Next we split our argument into two separate cases.

Case 1. If there is $N \in \mathbf{N}$ such that $r_n \leq |x-y|$ for all $n > N$, then $B(x, r_n + |x-y|) \subset B(x,\varepsilon) \subset \Omega$ for all $n > N$. Hence exactly the same argument as in the case $\Omega = \mathbf{R}^n$ shows that

$$|\mathscr{M}_\Omega u(x) - \mathscr{M}_\Omega u(y)| \leq \int_{\overline{xy}} \mathscr{M}_\Omega|\nabla u|. \tag{6}$$

[5] Because the spherical maximal function of a function which equals zero a.e. equals zero a.e.

Case 2. If there is a subsequence (r_{n_k}) such that $r_{n_k} > |x-y|$, then we define auxiliary radii $\tilde r_{n_k} = r_{n_k} - |x-y|$. Notice that $B(x, \tilde r_{n_k} + |x-y|) \subset \Omega$. As in the case $\Omega = \mathbf{R}^n$ we have

$$|\mathscr{M}_\Omega u(x) - \mathscr{M}_\Omega u(y)| \le \limsup_{k\to\infty} \big(u_{r_{n_k}}(x) - u_{\tilde r_{n_k}}(y)\big).$$

We now estimate the right-hand side

$$\begin{aligned}
|u_{r_{n_k}}(x) - u_{\tilde r_{n_k}}(y)| &= \left| \fint_{B(x,r_{n_k})} u(z)\,dz - \fint_{B(y,\tilde r_{n_k})} u(z)\,dz \right| \\
&= \left| \fint_{B(x,r_{n_k})} \left(u(z) - u\left(\frac{\tilde r_{n_k}}{r_{n_k}}(z-x) + y \right) \right) dz \right| \\
&= \left| \fint_{B(x,r_{n_k})} \int_0^1 \frac{d}{dt} u\left(z + t\left(\frac{\tilde r_{n_k}}{r_{n_k}}(z-x) + y - z \right) \right) dt\,dz \right| \\
&\le 2|x-y| \fint_{B(x,r_{n_k})} \int_0^1 \left| \nabla u\left(z + t\left(\frac{\tilde r_{n_k}}{r_{n_k}}(z-x) + y - z \right) \right) \right| dt\,dz,
\end{aligned}$$

where the last inequality follows from

$$\begin{aligned}
\left| \frac{\tilde r_{n_k}}{r_{n_k}}(z-x) + y - z \right| &\le |y-x| + \left| \frac{\tilde r_{n_k}}{r_{n_k}}(z-x) + x - z \right| \\
&\le |y-x| + |z-x| \left| \frac{\tilde r_{n_k}}{r_{n_k}} - 1 \right| \le 2|x-y|.
\end{aligned}$$

Hence we have

$$\begin{aligned}
|u_{r_{n_k}}(x) - u_{\tilde r_{n_k}}(y)| &\le 2|x-y| \int_0^1 \fint_{B(x+t(y-x), r_{n_k} - t|x-y|)} |\nabla u(z)|\,dz\,dt \\
&\le 2|x-y| \int_0^1 \mathscr{M}_\Omega |\nabla u|\big(x + t(y-x)\big)\,dt \le 2 \int_{\overline{xy}} \mathscr{M}_\Omega |\nabla u|.
\end{aligned}$$

The above estimates give

$$(7) \qquad |\mathscr{M}_\Omega u(x) - \mathscr{M}_\Omega u(y)| \le 2 \int_{\overline{xy}} \mathscr{M}_\Omega |\nabla u|.$$

This inequality, in turn, implies, as in the case of $\Omega = \mathbf{R}^n$, that

$$|\nabla \mathscr{M}_\Omega u(x)| \le 2 \mathscr{M}_\Omega |\nabla u|(x)$$

for almost every $x \in \Omega_\varepsilon$. Finally, letting $\varepsilon \to 0$, the desired point-wise estimate (4) follows. □

Proof of Theorem 4. Since $\mathscr{S}u$ is finite a.e. in Ω by Lemma 6, the previous proof extends also to the case of the spherical maximal function. □

References

[1] Almgren, F. J., and E. H. Lieb: Symmetric decreasing rearrangement is sometimes continuous. - J. Amer. Math. Soc. 2, 1989, 683-773.

[2] Bourgain, J.: Averages in the plane over convex curves and maximal operators. - J. Anal. Math. 47, 1986, 69–85.

[3] Buckley, S. M.: Is the maximal function of a Lipschitz function continuous? - Ann. Acad. Sci. Fenn. Math. 24, 1999, 519–528.

[4] Evans, L. C., and R. F. Gariepy: Measure Theory and Fine Properties of Functions. - Stud. Adv. Math., CRC, Boca Raton, 1992.

[5] Gilbarg, D., and N. S. Trudinger: Elliptic Partial Differential Equations of Second Order. - Springer-Verlag, Berlin, 1983.

[6] Hajłasz, P., and A. Kałamajska: Polynomial asymptotics and approximation of Sobolev functions. - Studia Math. 113, 1995, 55–64.

[7] Kinnunen, J.: The Hardy–Littlewood maximal function of a Sobolev function. - Israel J. Math. 100, 1997, 117–124.

[8] Kinnunen, J., and V. Latvala: Lebesgue points for Sobolev functions on metric spaces. - Rev. Mat. Iberoamericana 18, 2002, 685–700.

[9] Kinnunen, J., and P. Lindqvist: The derivative of the maximal function. - J. Reine Angew. Math. 503, 1998, 161–167.

[10] Kinnunen, J., and E. Saksman: Regularity of the fractional maximal function. - Bull. London Math. Soc. (to appear).

[11] Korry, S.: Boundedness of Hardy–Littlewood maximal operator in the framework of Lizorkin–Triebel spaces. - Rev. Mat. Univ. Complut. Madrid 15, 2002, 401–416.

[12] MacManus, P.: The maximal function and Sobolev spaces. - Preprint.

[13] Martio, O.: The Hardy–Littlewood maximal function and Sobolev spaces on a metric space. - In: Function Spaces, Differential Operators and Nonlinear Analysis, 113–124, Prometheus, Prague, 1996.

[14] Stein, E. M.: Note on the class $L\log L$. - Studia Math. 32, 1969, 305–310.

[15] Stein, E. M.: Singular Integrals and Differentiability Properties of Functions. - Princeton Math. Ser. 30, Princeton Univ. Press, Princeton, NJ, 1970.

[16] Stein, E. M.: Maximal functions. I. Spherical means. - Proc. Nat. Acad. Sci. U.S.A. 73, 1976, 2174–2175.

[17] Stein, E. M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. (With the assistance of T. S. Murphy.) - Princeton Math. Ser. 43, Monographs in Harmonic Analysis, III. Princeton Univ. Press, Princeton, NJ, 1993.

[18] Tanaka, H.: A remark on the derivative of the one-dimensional Hardy–Littlewood maximal function. - Bull. Austral. Math. Soc. 65, 2002, 253–258.

[19] Ziemer, W.: Weakly Differentiable Functions. - Grad. Texts in Math. 120, Springer-Verlag, 1989.

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