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Sobolev embeddings, extensions and measure density condition

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Abstract

There are two main results in the paper. In the first one, Theorem 1, we prove that if the Sobolev embedding theorem holds in Ω, in any of all the possible cases, then Ω satisfies the measure density condition. The second main result, Theorem 5, provides several characterizations of the $W^{m,p}$-extension domains for $1 < p < \infty$. As a corollary we prove that the property of being a $W^{1,p}$-extension domain, $1 < p \leqslant \infty$, is invariant under bi-Lipschitz mappings, Theorem 8.

Keywords: Sobolev spaces; Sobolev embedding; Extension operator; Measure density condition

1. Introduction

In this paper we deal with various properties of the Sobolev space $W^{m,p}(\Omega)$ of functions on a domain $\Omega \subset \mathbb{R}^n$ whose distributional partial derivatives of all orders up to m are L^p-integrable. This is a Banach space with the norm $\|u\|_{p,m;\Omega} = \sum_{|\alpha|\leqslant m} \|D^\alpha u\|_{p;\Omega}$. Here and in what follows we write $\|f\|_{p;\Omega} = \|f\|_{L^p(\Omega)}$. There are two main results in the paper. In the first one, Theorem 1, we prove that if the Sobolev embedding theorem holds in Ω, in any of all the possible

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cases, then Ω satisfies the *measure density condition*, i.e. there exists a constant $c > 0$ such that for all $x \in \Omega$ and all $0 < r \leqslant 1$

$$\left|B(x,r) \cap \Omega\right| \geqslant cr^n. \tag{1}$$

(Note that sets satisfying such a condition are sometimes called in the literature *regular* sets or n-sets.) We use the notation $|A|$ for the Lebesgue measure of a set A. In particular, if Ω is a $W^{m,p}$*-extension domain*, i.e. there is a bounded linear operator

$$\mathcal{E}: W^{m,p}(\Omega) \to W^{m,p}\left(\mathbb{R}^n\right) \tag{2}$$

such that $\mathcal{E}u|_\Omega = u$ for each $u \in W^{m,p}(\Omega)$, then the Sobolev embedding is satisfied in the space $W^{m,p}(\Omega)$ and hence Ω satisfies the measure density condition. The fact that Sobolev extension domains satisfy the measure density condition has been known previously for $W^{1,p}$-extension domains, where $p > n - 1$, see [14] and references therein. Notice that the measure density condition along with the Lebesgue differentiation theorem imply that the boundary of a $W^{1,p}$-extension domain is necessarily of volume zero. This answers the separate inquiries by Markus Biegert, Dagmar Medkova and Bill Ziemer.

Theorem 1 together with a recent result of Shvartsman [23] (see also [22,24]) leads to the second main result, Theorem 5, which provides several characterizations of the $W^{m,p}$-extension domains for $1 < p < \infty$. In Theorem 7 we provide a similar characterization for $m = 1$ and $p = \infty$. As a particular application of these characterizations we prove that the property of being a $W^{1,p}$-extension domain, $1 < p \leqslant \infty$, is invariant under bi-Lipschitz mappings (Theorem 8), and that Ω is a $W^{m,p}$-extension domain for $p > 1$ if and only if the *trace operator*

$$\mathcal{T}: W^{m,p}\left(\mathbb{R}^n\right) \to W^{m,p}(\Omega), \quad \mathcal{T}u = u|_\Omega \tag{3}$$

is onto. The last result is a far reaching generalization of [9, Theorem 9].

See [10,15,17] and references therein for known results about Sobolev extension domains.

Notation used in the paper is standard. We write χ_E for the characteristic function of a set E and ω_n is the measure of the unit ball in $\mathbb{R}^n$. The symbol c will be used to designate a general constant whose value may change even within a single string of estimates. To show explicit dependence of c on parameters we write e.g. $c = c(n,m)$. We write $\nabla^j u$ to denote the vector of all partial derivatives of u of order j. As usual, $\|\cdot\|_\infty$ stands for the supremum norm.

2. Main results

In this section we state our results and prove all of them but Theorems 1 and 2.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $p \geqslant 1$, and m a positive integer.*

(a) *If $mp < n$ and $W^{m,p}(\Omega) \subset L^{p^*}(\Omega)$, where $p^* = np/(n - mp)$, then Ω satisfies* (1).
(b) *If $mp = n$, $p > 1$, and there are constants $A, s, M > 0$ such that for every $x \in \Omega$, every $0 < r \leqslant 1$ and every $u \in W^{m,p}(\Omega)$*

$$\inf_{\gamma\in\mathbb{R}} \int\limits_{B(x,r)\cap\Omega} \exp\left(\frac{A|u-\gamma|}{\|u\|_{m,p;\Omega}}\right)^s \leqslant Mr^n, \tag{4}$$

then Ω satisfies (1).

(c) *If $m = n$, $p = 1$ and there is a constant $M > 0$ such that*

$$\big|u(x) - u(y)\big| \leqslant M\|u\|_{m,p;\Omega} \quad \textit{for } u \in W^{m,p}(\Omega),$$

whenever $x, y \in \Omega$, $|x - y| \leqslant 1$, then Ω satisfies (1).

(d) *Let $mp > n$ and let k be the smallest integer such that $kp > n$. Then $m = k + j$, $j \geqslant 0$. We have three cases*:

(α) *If $n > (k-1)p$ and there is a constant $M > 0$ such that*

$$\big|\nabla^j u(x) - \nabla^j u(y)\big| \leqslant M|x - y|^{k-n/p}\|u\|_{m,p;\Omega} \quad \textit{for } u \in W^{m,p}(\Omega),$$

whenever $x, y \in \Omega$, $|x - y| \leqslant 1$, then Ω satisfies (1).

(β) *If $n = k - 1$, $p = 1$ and there is $M > 0$ such that*

$$\big|\nabla^j u(x) - \nabla^j u(y)\big| \leqslant M|x - y|\|u\|_{m,p;\Omega} \quad \textit{for } u \in W^{m,p}(\Omega),$$

whenever $x, y \in \Omega$, $|x - y| \leqslant 1$, then Ω satisfies (1).

(γ) *If $n = (k-1)p$, $p > 1$ and there are constant $A, s, M > 0$ and a multi-index α, $|\alpha| = j + 1$ such that for every $x \in \Omega$, every $0 < r \leqslant 1$ and every $u \in W^{m,p}(\Omega)$*

$$\inf_{\gamma \in \mathbb{R}} \int\limits_{B(x,r)\cap\Omega} \exp\left(\frac{A|D^\alpha u - \gamma|}{\|u\|_{m,p;\Omega}}\right)^s \leqslant M r^n, \tag{5}$$

then Ω satisfies (1).

The above theorem together with the corresponding Sobolev-type embeddings in $\mathbb{R}^n$ give the following result.

Theorem 2. *If $\Omega \subset \mathbb{R}^n$ is a domain, $1 \leqslant p < \infty$ and integer $m \geqslant 1$ are such that the trace operator* (3) *is surjective, then Ω satisfies the measure density condition* (1). *In particular the measure density condition is satisfied by all $W^{m,p}$-extension domains.*

Calderón [4] (see also [5]) characterized the Sobolev space $W^{m,p}(\mathbb{R}^n)$, $1 < p \leqslant \infty$, in terms of the *fractional sharp maximal function*. Let $\mathcal{P}^m = \mathcal{P}^m(\mathbb{R}^n)$, where m is a nonnegative integer, be the linear space of polynomials on $\mathbb{R}^n$ of degree less than or equal to m. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $r > 0$ we set

$$\mathcal{E}_m(f; x, r) = \inf_{P \in \mathcal{P}^{m-1}} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f - P|\, dy,$$

and define the fractional sharp maximal function by

$$f_m^{\#}(x) = \sup_{r>0} r^{-m}\mathcal{E}_m(f; x, r).$$

The result of Calderón reads as follows.

Proposition 3. *Let $1 < p \leqslant \infty$ and m be a positive integer. Then $f \in W^{m,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $f_m^{\#} \in L^p(\mathbb{R}^n)$. Moreover*

$$\|f\|_{m,p;\mathbb{R}^n} \approx \|f\|_{p;\mathbb{R}^n} + \left\|f_m^{\#}\right\|_{p;\mathbb{R}^n},$$

up to a constant depending on n, m and p only.

We write $A \approx B$ if there is a constant $c \geqslant 1$ such that $c^{-1}B \leqslant A \leqslant cB$.

Shvartsman [23] used the fractional sharp maximal function to characterize the space of traces of $W^{m,p}(\mathbb{R}^n)$ functions on a measurable set $E \subset \mathbb{R}^n$ satisfying the measure density condition

$$\left|E \cap B(x,r)\right| \geqslant cr^n \quad \text{for } x \in E,\ 0 < r \leqslant 1. \tag{6}$$

Recall that if $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a Banach space of measurable functions on $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is a measurable set of positive Lebesgue measure, then $\mathcal{A}|_E$ is the *trace space* defined as

$$\mathcal{A}|_E = \{f : E \to \mathbb{R}\text{: there exists } F \in \mathcal{A} \text{ such that } F|_E = f \text{ a.e.}\}.$$

This space is equipped with the norm

$$\|f\|_{\mathcal{A}|_E} = \inf\left\{\|F\|_{\mathcal{A}}\text{: } F \in \mathcal{A},\ F|_E = f \text{ a.e.}\right\}.$$

Denoting the trace operator by $\mathcal{T}F = F|_E$ we see that the space $\mathcal{A}|_E$ is isomorphic to the quotient space $\mathcal{A}/\ker\mathcal{T}$. The above construction applies, in particular, to the Sobolev space $\mathcal{A} = W^{m,p}(\mathbb{R}^n)$.

For a set $E \subset \mathbb{R}^n$ of positive Lebesgue measure denote

$$C^{m,p}(E) = \left\{f \in L^p(E)\text{: } f_{m,E}^{\#} \in L^p(E)\right\}, \quad \|f\|_{C^{m,p}(E)} = \|f\|_{p;E} + \left\|f_{m,E}^{\#}\right\|_{p;E},$$

where

$$f_{m,E}^{\#}(x) = \sup_{r>0} r^{-m}\mathcal{E}_{m,E}(f;x,r),$$

$$\mathcal{E}_{m,E}(f;x,r) = \inf_{P \in \mathcal{P}^{m-1}} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)\cap E} |f-P|\,dy.$$

With this notation Calderón's result states that

$$C^{m,p}\left(\mathbb{R}^n\right) = W^{m,p}\left(\mathbb{R}^n\right)$$

provided $1 < p \leqslant \infty$ and m is a positive integer.

Shvartsman [23] generalized an earlier result of Rychkov [22] and proved the following characterization of traces of $W^{m,p}$-functions.

Proposition 4. *Let $E \subset \mathbb{R}^n$ be a measurable set satisfying the measure density condition* (6). *Then*

$$W^{m,p}(\mathbb{R}^n)\big|_E = C^{m,p}(E)$$

as sets and the norms are equivalent. Moreover there is a bounded linear extension operator

$$\mathcal{E} : C^{m,p}(E) \to C^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n).$$

Actually Shvartsman constructed an extension operator explicitly as a variant of the Whitney–Jones extension.

With the help of this result we can prove the second main result of the paper which reads as follows.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain, $1 < p < \infty$ and m a positive integer. Then the following conditions are equivalent.*

(a) *For every $f \in W^{m,p}(\Omega)$ there exists $F \in W^{m,p}(\mathbb{R}^n)$ such that $F|_\Omega = f$ a.e.*
(b) *The trace operator* (3) *is surjective.*
(c) *There exists a bounded linear extension operator* (2).
(d) *Ω satisfies the measure density condition* (1) *and $C^{m,p}(\Omega) = W^{m,p}(\Omega)$ as sets and the norms are equivalent.*

Proof. The equivalence between (a) and (b) is obvious. The implication from (d) to (c) follows from Proposition 4 and the implication from (c) to (b) is obvious again. Finally the implication from (b) to (d) is a direct consequence of Theorem 2 and Proposition 4. The proof is complete. □

Note that the equivalence between the conditions (b) and (c) is obvious when $p = 2$. Indeed, it is a direct consequence of the Hilbert structure of the space $W^{m,2}$. Namely, if the trace operator is surjective, then $\mathcal{T}|_{(\ker \mathcal{T})^\perp} : (\ker \mathcal{T})^\perp \to W^{m,2}(\Omega)$ is an isomorphism and hence

$$\mathcal{E} = (\mathcal{T}|_{(\ker \mathcal{T})^\perp})^{-1} : W^{m,2}(\Omega) \to (\ker \mathcal{T})^\perp \subset W^{m,2}(\mathbb{R}^n)$$

defines a bounded linear extension operator.

This argument cannot be applied for $p \neq 2$ as not every subspace of $W^{m,p}$ for $p \neq 2$ is complemented. Recall that a closed subspace Y of a Banach space X is *complemented* if there is another closed subspace Z of X such that $X = Y \oplus Z$. That is, $Y \cap Z = \{0\}$ and every element $x \in X$ can be written as $x = y + z$, with $y \in Y$ and $z \in Z$. The following result is a standard exercise in functional analysis and is left to the reader.

Proposition 6. *Let $\Omega \subset \mathbb{R}^n$ be a domain such that, for some $1 \leqslant p \leqslant \infty$ and a positive integer m, every $u \in W^{m,p}(\Omega)$ admits an extension to $W^{m,p}(\mathbb{R}^n)$. Then there exists a bounded linear extension operator* (2) *if and only if the subspace* $\ker \mathcal{T}$ *is complemented in $W^{m,p}(\mathbb{R}^n)$.*

Note that, for $1 < p < \infty$, the space $W^{m,p}(\mathbb{R}^n)$ is isomorphic to $L^p(\mathbb{R}^n)$ [26, Chapter 5]. Accordingly, the equivalence between the conditions (b) and (c) is not obvious for $p \neq 2$ because not every subspace of $L^p(\mathbb{R}^n)$, $p \neq 2$, is complemented. Actually, the property of being complemented is rather rare, see e.g. [2,12,13,16,20,21,25].

Question 1. *Are the conditions* (b) *and* (c) *from Theorem* 5 *equivalent for* $p = 1$?

In this context, a particularly relevant result is due to Peetre [18]. According to a theorem of Gagliardo [6], there is a bounded and surjective trace operator $\mathcal{T}: W^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^{n-1})$, and hence every $u \in L^1(\mathbb{R}^{n-1})$ admits an extension to $W^{1,1}(\mathbb{R}^n)$. However, as was proven by Peetre [18] (cf. [19]), there is no bounded linear extension operator $\mathcal{E}: L^1(\mathbb{R}^{n-1}) \to W^{1,1}(\mathbb{R}^n)$.

For an arbitrary closed set $F \subset \mathbb{R}^n$, let $\mathrm{Lip}_\infty(F) = \mathrm{Lip}(F) \cap L^\infty(F)$ be the space of bounded Lipschitz functions on F. It is a Banach space with the norm

$$\|f\|_L = \|f\|_\infty + \mathrm{Lip}(f) = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Every bounded Lipschitz function on a domain $\Omega \subset \mathbb{R}^n$ uniquely extends to a bounded Lipschitz function on the closure, so we can consider $\mathrm{Lip}_\infty(\Omega)$ to be equal to $\mathrm{Lip}_\infty(\overline{\Omega})$. It is well known that $W^{1,\infty}(\mathbb{R}^n) = \mathrm{Lip}_\infty(\mathbb{R}^n)$ and for an arbitrary domain Ω, $\mathrm{Lip}_\infty(\Omega) \subset W^{1,\infty}(\Omega)$ is a linear subspace.

We say that a domain $\Omega \subset \mathbb{R}^n$ is *uniformly locally quasiconvex* if there are constants $C > 0$ and $R > 0$ such that for every $x, y \in \Omega$ satisfying $|x - y| < R$ there is a rectifiable curve γ connecting x and y in Ω such that the length of γ is bounded from above by $C|x - y|$.

For $p = \infty$ and $m = 1$ we have the following counterpart of Theorem 5.

Theorem 7. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain. Then the following conditions are equivalent*:

(a) *For every $u \in W^{1,\infty}(\Omega)$ there exists $v \in W^{1,\infty}(\mathbb{R}^n)$, such that $v|_\Omega = u$.*
(b) *The trace operator $\mathcal{T}: W^{1,\infty}(\mathbb{R}^n) \to W^{1,\infty}(\Omega)$ is surjective.*
(c) *There exists a bounded linear extension operator $\mathcal{E}: W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n)$.*
(d) $W^{1,\infty}(\Omega) = \mathrm{Lip}_\infty(\Omega)$.
(e) *Ω is uniformly locally quasiconvex.*

As it follows from the proof, it is not only that an extension operator exists, but such an operator can be constructed explicitly (Whitney's extension).

Note that the measure density condition does not appear in Theorem 7. In fact, there are obvious examples of quasiconvex domains that do not satisfy that condition. Hence the existence of a bounded extension operator for $p = \infty$ does not imply the measure density condition, differently as in the case $1 \leqslant p < \infty$.

If $p = \infty$ and $m > 1$ the situation is more complicated because it was shown by Zobin [30], that local uniform quasiconvexity is not necessary for the existence of an extension operator.

Proof of Theorem 7. Equivalence of the conditions (a) and (b) and the implication from (c) to (b) are obvious. Now we prove the implication from (b) to (d). As a restriction of a Lipschitz function to Ω is Lipschitz we conclude that

$$\mathcal{T}: W^{1,\infty}(\mathbb{R}^n) \to \mathrm{Lip}_\infty(\Omega) \subset W^{1,\infty}(\Omega).$$

Accordingly, surjectivity of the trace operator implies that $\mathrm{Lip}_\infty(\Omega) = W^{1,\infty}(\Omega)$ as sets, and hence the norms are equivalent by the Banach open mapping theorem. To prove the implication from (d) to (c) we just recall the well-known fact that for an arbitrary closed set F, the Whitney extension operator defines a bounded linear extension operator $\mathcal{E} : \mathrm{Lip}_\infty(F) \to \mathrm{Lip}_\infty(\mathbb{R}^n)$. (A familiar way to extend a Lipschitz function $f : \mathbb{R}^n \supset F \to \mathbb{R}$ to a Lipschitz function on $\mathbb{R}^n$ is by way of McShane's formula $\tilde{f}(x) = \inf_{y\in E}\{f(y) + \mathrm{Lip}(f)|x-y|\}$. Note, however, that this does not give a linear extension.) To prove the implication from (e) to (d) we need to show that $W^{1,\infty}(\Omega) \subset \mathrm{Lip}_\infty(\Omega)$. Let $f \in W^{1,\infty}(\Omega)$. If $|x-y| < R$ and γ is as in the definition of a uniformly locally quasiconvex domain, then

$$\left|f(x) - f(y)\right| \leqslant \int_\gamma \|\nabla f\|_\infty \leqslant C\|\nabla f\|_\infty |x-y|.$$

If $|x-y| \geqslant R$, then $|f(x) - f(y)| \leqslant 2\|f\|_\infty R^{-1}|x-y|$ and hence $f \in \mathrm{Lip}_\infty(\Omega)$. To complete the proof of the theorem, it suffices to verify the implication from (c) to (e). For $x, y \in \Omega$, let $\varphi_x(y)$ be the infimum of lengths of curves that join x and y in Ω. Note that $\tilde{\varphi}_x = \min\{\varphi_x, 1\}$ satisfies $\tilde{\varphi}_x \in W^{1,\infty}(\Omega)$, $\|\nabla\tilde{\varphi}_x\|_\infty \leqslant 1$. Now (c) yields that $\{\mathcal{E}\tilde{\varphi}_x\}_{x\in\Omega}$ is a bounded family of functions in $\mathrm{Lip}_\infty(\mathbb{R}^n)$ and hence

$$\tilde{\varphi}_x(y) = \left|\tilde{\varphi}_x(x) - \tilde{\varphi}_x(y)\right| = \left|\mathcal{E}\tilde{\varphi}_x(x) - \mathcal{E}\tilde{\varphi}_x(y)\right| \leqslant C|x-y|,$$

whenever $x, y \in \Omega$. Now if $|x-y| \leqslant R = C^{-1}$, then $1 \geqslant C|x-y| \geqslant \tilde{\varphi}_x(y) = \varphi_x(y)$ and hence (e) follows. The proof is complete. □

If $1 < p \leqslant \infty$ and $E \subset \mathbb{R}^n$ is a measurable set satisfying the measure density condition (1), then the space $C^{1,p}(E)$ is equivalent to the space $M^{1,p}(E)$ which is defined as follows:

$$M^{1,p}(E) = \left\{f \in L^p(E)\colon \exists 0 \leqslant g \in L^p(E)\ \left|f(x) - f(y)\right| \leqslant |x-y|\big(g(x) + g(y)\big) \text{ a.e.}\right\},$$
$$\|f\|_{M^{1,p}(E)} = \|f\|_{p;E} + \inf_g \|g\|_{p;E},$$

where the infimum is taken over the class of all functions g that appear in the definition of the space $M^{1,p}(E)$. For a proof see [8, Theorem 3.4] (the theorem is true in the general setting of metric spaces with doubling measures).

In particular, if $m = 1$ and $1 < p < \infty$, then there is one more condition equivalent to conditions (a)–(d) of Theorem 5.

(e) *Ω satisfies the measure density condition* (1) *and $W^{1,p}(\Omega) = M^{1,p}(\Omega)$ as sets and the norms are equivalent.*

Theorem 8. *Let $\Omega, G \subset \mathbb{R}^n$ be two domains that are bi-Lipschitz homeomorphic. Then Ω is a $W^{1,p}$-extension domain for some $1 < p \leqslant \infty$ if and only if G is a $W^{1,p}$-extension domain.*

If $p = \infty$, the claim easily follows from Theorem 7, but if $1 < p < \infty$ the theorem is far from being obvious. If we knew that there were a bi-Lipschitz homeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\Omega) = G$, the claim would easily follow even for $p = 1$. However, in general, a bi-Lipschitz homeomorphism $T:\Omega \to G$ cannot be extended beyond Ω (cf. [28,29]), and accordingly, we do not know the answer to the following question.

Question 2. *Is Theorem* 8 *true for* $p = 1$?

Proof of Theorem 8. We may assume that $1 < p < \infty$. Let $T:\Omega \to G$ be a bi-Lipschitz homeomorphism. Suppose that one of the domains, say Ω, is a $W^{1,p}$-extension domain. By Theorem 5(e), Ω satisfies (1) and $W^{1,p}(\Omega) = M^{1,p}(\Omega)$. Now G satisfies (1) as bi-Lipschitz homeomorphisms preserve the measure density condition. Moreover, the transformation $\Phi(u) = u \circ T$ induces isomorphisms of spaces, $\Phi : W^{1,p}(G) \to W^{1,p}(\Omega)$, and $\Phi : M^{1,p}(G) \to M^{1,p}(\Omega)$. Therefore $W^{1,p}(G) = M^{1,p}(G)$ and again we can apply Theorem 5 to construct an extension for $W^{1,p}(G)$. The proof is complete. $\square$

3. Proof of Theorem 2

If the trace operator (3) is surjective, then the space $W^{m,p}(\Omega)$ is isomorphic to $W^{m,p}(\mathbb{R}^n)/\ker\mathcal{T}$ and hence there is $c > 0$ such that for every $u \in W^{m,p}(\Omega)$ there is $v \in W^{m,p}(\mathbb{R}^n)$ satisfying $v|_\Omega = u$ and

$$\|v\|_{m,p;\mathbb{R}^n} \leqslant c\|u\|_{m,p;\Omega}.$$

If $mp < n$, then $W^{m,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ and hence

$$\|u\|_{p^*;\Omega} \leqslant \|v\|_{p^*;\mathbb{R}^n} \leqslant c\|v\|_{m,p;\mathbb{R}^n} \leqslant c'\|u\|_{m,p;\Omega}.$$

This proves the embedding $W^{m,p}(\Omega) \subset L^{p^*}(\Omega)$ and hence the measure density condition by Theorem 1.

If $mp = n$, $p > 1$, then the gradient of a $W^{m,p}(\mathbb{R}^n)$ function belongs to $L^n(\mathbb{R}^n)$ by the Sobolev embedding, $\|\nabla v\|_{n,\mathbb{R}^n} \leqslant c\|v\|_{m,p;\mathbb{R}^n} \leqslant c'\|u\|_{m,p;\Omega}$. Let us recall the inequality of Judovič [11] and Trudinger [27].

Lemma 9. *There exist positive constants $c_1(n)$ and $c_2(n)$ such that if $u \in W^{1,n}(B)$, where $B \subset \mathbb{R}^n$ is an arbitrary ball, then*

$$\int_B \exp\left(\frac{c_1|u - u_B|}{\|\nabla u\|_{n;B}}\right)^{n/(n-1)} \leqslant c_2|B|.$$

For a proof see e.g. [1,7,27]. Now for every ball $B \subset \mathbb{R}^n$ we have

$$\begin{aligned} c_2|B| &\geqslant \int_B \exp\left(\frac{c_1|v - v_B|}{\|\nabla v\|_{n;B}}\right)^{n/(n-1)} \\ &\geqslant \int_{B\cap\Omega} \exp\left(\frac{c'|u - v_B|}{\|u\|_{m,p;\Omega}}\right)^{n/(n-1)} \\ &\geqslant \inf_{\gamma\in\mathbb{R}} \int_{B\cap\Omega} \exp\left(\frac{c'|u - \gamma|}{\|u\|_{m,p;\Omega}}\right)^{n/(n-1)}, \end{aligned}$$

which implies the condition (b) of Theorem 1 and hence the measure density condition.

If m, n and p are such as in the cases (c), (d)(α), (d)(β) of Theorem 1, then the measure density condition follows from Theorem 1 and the corresponding embedding for $\mathbb{R}^n$, see [1, Lemma 5.8, Theorem 5.4C′,C″] (or part II of Theorem 4.12 in the second edition of the book).

If m, n and p are as in the case (d)(γ) of Theorem 1, then the proof of the measure density condition is similar to that in the case $mp = n$, $p > 1$. We leave details to the reader. The proof is complete. □

Remark. It follows from Theorem 5.4C″ in [1] that in the case (d)(γ), $\nabla^j u$ is Hölder continuous with any exponent $0 < \lambda < 1$ for $u \in W^{m,p}(\Omega)$, but this embedding seems too weak to guarantee the measure density condition. Actually in this case stronger embedding theorems of Brezis and Wainger [3] hold, and it would be interesting to see if embeddings of this type are sufficient for the measure density condition.

4. Proof of Theorem 1

We will need the following well-known and easy to prove result.

Lemma 10. *Given $0 < a < b$, there is a function $\varphi_{a,b} \in C_0^\infty(\mathbb{R}^n)$ such that:*

1. $0 \leqslant \varphi_{a,b}(x) \leqslant 1$ *for* $x \in \mathbb{R}^n$.
2. $\varphi_{a,b}(x) = 1$ *if* $|x| \leqslant a$, $\varphi_{a,b}(x) = 0$ *if* $|x| \geqslant b$.
3. *For every positive integer m there is a constant $c = c(n,m)$ such that $\|D^\alpha \varphi_{a,b}\|_\infty \leqslant c|b-a|^{-|\alpha|}$ for all multi-indices α with $|\alpha| \leqslant m$.*

Proof. Choose $\psi \in C_0^\infty(B(0,1))$ such that $\psi \geqslant 0$, $\int \psi\, dx = 1$, and define $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$,

$$\varphi_{a,b}(x) = (\chi_{B(0,\frac{a+b}{2})} * \psi_{\frac{b-a}{2}})(x).$$

Here $\chi_{B(0,\frac{a+b}{2})}$ is the characteristic function of the ball $B(0, \frac{a+b}{2})$. The claim follows. □

In the proof of the theorem, we will frequently use the following notation. For $x \in \Omega$ and $0 < r \leqslant 1$, there exists a unique $0 < \tilde{r} < r$ such that

$$\left|B(x,\tilde{r}) \cap \Omega\right| = \left|A(r,\tilde{r}) \cap \Omega\right| = \frac{1}{2}\left|B(x,r) \cap \Omega\right|, \tag{7}$$

where

$$A(r,\tilde{r}) = B(x,r) \setminus B(x,\tilde{r}).$$

Case $mp < n$. The following lemma is a crucial step in the proof.

Lemma 11. *There is a constant $c > 0$ such that*

$$r - \tilde{r} \leqslant c\left|B(x,r) \cap \Omega\right|^{1/n} \tag{8}$$

for all $x \in \Omega$ and all $0 < r \leqslant 1$.

Proof. For $x \in \Omega$ and $0 < r \leqslant 1$ define a function in Ω by the formula

$$u(y) = \varphi_{\tilde{r},r}(y - x) \quad \text{for } y \in \Omega,$$

where the function $\varphi_{\tilde{r},r}$ is as in Lemma 10. According to the closed graph theorem, the inclusion $W^{m,p}(\Omega) \subset L^{p^*}(\Omega)$ is a bounded operator and hence there is a constant $c > 0$ (independent of u) such that $\|u\|_{p^*;\Omega} \leqslant c\|u\|_{m,p;\Omega}$. This inequality yields

$$\begin{aligned} \left|B(x,\tilde{r}) \cap \Omega\right|^{1/p^*} &\leqslant c\left(\left|B(x,r) \cap \Omega\right|^{1/p} + \sum_{k=1}^{m} (r - \tilde{r})^{-k} \left|A(r,\tilde{r}) \cap \Omega\right|^{1/p}\right) \\ &\leqslant \frac{c'|B(x,\tilde{r}) \cap \Omega|^{1/p}}{(r - \tilde{r})^m}, \end{aligned}$$

and since $p^* = np/(n - mp)$, the lemma follows easily. $\square$

To prove the case $mp < n$ of the theorem, let $x \in \Omega$ and $0 < r \leqslant 1$. Define a sequence $r_0 > r_1 > r_2 > \cdots > 0$ by induction:

$$r_0 = r, \qquad r_{j+1} = \tilde{r}_j.$$

Clearly $|B(x,r_j) \cap \Omega| = 2^{-j}|B(x,r) \cap \Omega|$. Hence $r_j \to 0$ and

$$r_j - r_{j+1} \leqslant c2^{-j/n}\left|B(x,r) \cap \Omega\right|^{1/n}$$

by (8). This in turn yields

$$r = \sum_{j=0}^{\infty} (r_j - r_{j+1}) \leqslant c\left(\sum_{j=0}^{\infty} 2^{-j/n}\right)\left|B(x,r) \cap \Omega\right|^{1/n} \leqslant c'\left|B(x,r) \cap \Omega\right|^{1/n},$$

which implies the measure density condition (1).

Case $mp = n$, $p > 1$. For $x \in \Omega$ and $0 < r \leqslant 1$, we choose $0 < \tilde{\tilde{r}} < \tilde{r} < r$ such that

$$\left|B(x,\tilde{\tilde{r}}) \cap \Omega\right| = \frac{1}{2}\left|B(x,\tilde{r}) \cap \Omega\right| = \frac{1}{4}\left|B(x,r) \cap \Omega\right|. \tag{9}$$

Note that

$$\left|A(\tilde{r},\tilde{\tilde{r}}) \cap \Omega\right| = \left|B(x,\tilde{\tilde{r}}) \cap \Omega\right| \quad \text{and} \quad \left|A(r,\tilde{r}) \cap \Omega\right| = \left|B(x,\tilde{r}) \cap \Omega\right|. \tag{10}$$

Now we define a function in Ω by the formula

$$u(y) = \varphi_{\tilde{\tilde{r}},\tilde{r}}(y - x) \quad \text{for } y \in \Omega,$$

where $\varphi_{\tilde{\tilde{r}},\tilde{r}}$ is the function from Lemma 10. We have

$$\|u\|_{m,p;\Omega} \leqslant c(\tilde{r} - \tilde{\tilde{r}})^{-m}\left|B(x,\tilde{r}) \cap \Omega\right|^{1/p}.$$

Obviously, for each $\gamma \in \mathbb{R}$, inequality $|u-\gamma| \geqslant 1/2$ is satisfied on at least one of the sets $B(x,\tilde{\tilde{r}}) \cap \Omega$ and $A(r,\tilde{r}) \cap \Omega$. Hence inequality (4) yields

$$\min\{|B(x,\tilde{\tilde{r}}) \cap \Omega|, |A(r,\tilde{r}) \cap \Omega|\} \exp\bigl(c(\tilde{r}-\tilde{\tilde{r}})^m |B(x,\tilde{r}) \cap \Omega|^{-1/p}\bigr)^s \leqslant Mr^n.$$

After elementary calculations we obtain

$$(\tilde{r}-\tilde{\tilde{r}})^m \leqslant c|B(x,\tilde{r}) \cap \Omega|^{1/p}\left(\ln\left(\frac{2Mr^n}{|B(x,\tilde{r}) \cap \Omega|}\right)\right)^{1/s}.$$

We can assume that $2M > \omega_n$, where ω_n is volume of the unit ball (otherwise we replace M by a larger constant). This condition is needed to ensure positivity of the logarithm. Since $mp = n$ the above estimate proves the following lemma.

Lemma 12. *There exist constants $c_1 = c_1(m,n,p,\alpha) > 0$ and $c_2 = c_2(M,n) > \omega_n$, such that for every $x \in \Omega$ and $0 < r \leqslant 1$ we have*

$$\tilde{r}-\tilde{\tilde{r}} \leqslant c_1|B(x,\tilde{r}) \cap \Omega|^{1/n}\left(\ln\left(\frac{c_2 r^n}{|B(x,\tilde{r}) \cap \Omega|}\right)\right)^{1/sm}. \tag{11}$$

Lemma 13. *If the measure density condition* (1) *holds for all $x \in \Omega$ and all $r \leqslant 1$ such that $r \leqslant 10\tilde{r}$, where $\tilde{r}$ is defined by* (9)*, then* (1) *holds for all $x \in \Omega$ and all $r \leqslant 1$.*[1]

Proof. Let $r \leqslant 1$. If $\Omega \subset B(x,r)$, then

$$|B(x,r) \cap \Omega| = |\Omega| \geqslant |\Omega| r^n$$

and hence (1) is satisfied. If $r \leqslant 10\tilde{r}$, then (1) is also satisfied. Thus we may assume that $\Omega \setminus B(x,r) \neq \emptyset$ and $r > 10\tilde{r}$. Take $x' \in B(x,r) \cap \Omega$ such that $|x-x'| = \tilde{r} + r/5$. Such an x' exists because $\Omega \setminus B(x,r) \neq \emptyset$ and Ω is connected. Let $R = 2\tilde{r} + r/5$. We have

$$B(x,\tilde{r}) \subset B(x',R) \subset B(x,r)$$

and

$$B(x',R/2) \subset B(x',r/5) \subset A(r,\tilde{r}).$$

Hence $B(x,\tilde{r})$ and $B(x',R/2)$ are disjoint subsets of $B(x',R)$ and thus

$$\begin{aligned}
|B(x',R/2) \cap \Omega| &\leqslant \frac{1}{2}\bigl(|A(r,\tilde{r}) \cap \Omega| + |B(x',R/2) \cap \Omega|\bigr) \\
&= \frac{1}{2}\bigl(|B(x,\tilde{r}) \cap \Omega| + |B(x',R/2) \cap \Omega|\bigr) \\
&\leqslant \frac{1}{2}|B(x',R) \cap \Omega|.
\end{aligned}$$

[1] Perhaps with a different constant c.

This, in turn, implies that $\tilde{R} \geqslant R/2$, and so the measure density condition is satisfied for the ball $B(x', R)$. Hence

$$\left|B(x,r) \cap \Omega\right| \geqslant \left|B(x', R) \cap \Omega\right| \geqslant cR^n \geqslant c5^{-n} r^n.$$

The proof of the lemma is complete. □

We are ready now to complete the proof of the theorem in the case $mp = n$, $p > 1$. We need to prove (1) for all $x \in \Omega$ and all $0 < r \leqslant 1$. According to Lemma 13 we may assume that $r \leqslant 10\tilde{r}$. Define a sequence by setting

$$r_0 = r, \qquad r_{j+1} = \tilde{r}_j.$$

Lemma 12 yields

$$r_{j+1} - r_{j+2} \leqslant c_1 \left|B(x, r_{j+1}) \cap \Omega\right|^{1/n} \left(\ln\left(\frac{c_2 r_j^n}{|B(x, r_{j+1}) \cap \Omega|}\right)\right)^{1/sm}.$$

Since

$$\left|B(x, r_{j+1}) \cap \Omega\right| = 2^{-j} \left|B(x, \tilde{r}) \cap \Omega\right|, \tag{12}$$

we conclude that

$$r_{j+1} - r_{j+2} \leqslant c_1 2^{-j/n} \left|B(x, \tilde{r}) \cap \Omega\right|^{1/n} \left(\ln\left(\frac{c_2 2^j r_j^n}{|B(x, \tilde{r}) \cap \Omega|}\right)\right)^{1/sm}.$$

It follows from (12) that $r_j \to 0$ as $j \to \infty$, and hence

$$\tilde{r} = \sum_{j=0}^{\infty} (r_{j+1} - r_{j+2}) \leqslant c_1 \left|B(x, \tilde{r}) \cap \Omega\right|^{1/n} \sum_{j=0}^{\infty} 2^{-j/n} \left(\ln\left(\frac{c_2 2^j r^n}{|B(x, \tilde{r}) \cap \Omega|}\right)\right)^{1/sm}.$$

The sum on the right-hand side is bounded (up to a constant factor depending on sm only) by

$$\sum_{j=0}^{\infty} 2^{-j/n} j^{1/sm} (\ln 2)^{1/sm} + \left(\sum_{j=0}^{\infty} 2^{-j/n}\right) \left(\ln\left(\frac{c_2 r^n}{|B(x, \tilde{r}) \cap \Omega|}\right)\right)^{1/sm}.$$

The two sums in the above expression converge to some constants depending on n, m and s only, and hence we obtain

$$\tilde{r} \leqslant c \left|B(x, \tilde{r}) \cap \Omega\right|^{1/n} \left[1 + \left(\ln\left(\frac{c_2 r^n}{|B(x, \tilde{r}) \cap \Omega|}\right)\right)^{1/sm}\right]. \tag{13}$$

Write

$$\left|B(x, \tilde{r}) \cap \Omega\right| = \varepsilon \tilde{r}^n.$$

Since

$$\left|B(x,r)\cap\Omega\right| = 2\left|B(x,\tilde r)\cap\Omega\right| = 2\varepsilon\tilde r^n \geqslant 2\cdot 10^{-n}\varepsilon r^n,$$

it suffices to show that ε is bounded from below by some positive constant depending on m, n, p, α, M and s only. Inequality (13) gives

$$c\varepsilon^{1/n}\left(1+\left(\ln\left(c_2 10^n\varepsilon^{-1}\right)\right)^{1/sm}\right)\geqslant 1.$$

Now it suffices to observe that the expression on the left-hand side converges to 0 if $\varepsilon\to 0$, and since it is bounded from below by a positive constant, ε must also be bounded from below by a positive constant. This ends the proof of the theorem in the given case.

Case $m = n$, $p = 1$. Let $\varphi\in C_0^\infty(B(0,1))$, $\varphi(0)=1$ and let $\varphi_r(x)=\varphi(x/r)$. For $x\in\Omega$ and $0<r\leqslant 1$ we define a function in Ω by setting

$$u(y)=\varphi_r(y-x)\quad\text{for } y\in\Omega.$$

If $\Omega\subset B(x,r)$, then $|B(x,r)\cap\Omega| = |\Omega|\geqslant|\Omega|r^n$ and the condition (1) follows. If $\Omega\setminus B(x,r)\neq\emptyset$, then there is $y\in B(x,r)\cap\Omega$ such that $u(y)=0$ and hence

$$1=\left|u(x)-u(y)\right|\leqslant M\|u\|_{n,1;\Omega}\leqslant MCr^{-n}\left|B(x,r)\cap\Omega\right|.$$

This, in turn, implies the measure density condition.

Case $mp > n$. In the proofs for the subcases (d)(α) and (d)(β) we will need the following auxiliary results.

Lemma 14. *Let $\varphi\in C_0^\infty(-1,1)$, $\varphi(0)=1$. Then for every nonnegative integer j there is $x\in(-1,1)$ such that*

$$\varphi^{(j)}(x)\geqslant 1\quad(j\text{th derivative}).$$

Proof. By induction. □

Corollary 15. *Let $\varphi\in C_0^\infty(B(0,1))$, $\varphi(0)=1$, $\varphi_r(x)=\varphi(x/r)$. Then for every nonnegative integer j there is a point $x\in B(0,r)$ on the x_1-axis such that*

$$\frac{\partial^j\varphi_r}{\partial x_1^j}(x)\geqslant r^{-j}.$$

Proof. Direct application of the lemma. □

Let $\varphi\in C_0^\infty(B(0,1))$ be radially symmetric, and such that $\varphi\equiv 1$ on $B(0,1/2)$. We set $\varphi_r(x)=\varphi(x/r)$. For $x\in\Omega$ and $0<r\leqslant 1$ we define a function in Ω by setting

$$u(y)=\varphi_r(y-x)\quad\text{for } y\in\Omega.$$

The fact that φ is radially symmetric and Corollary 15 yield that after a suitable rotation of the coordinate system there is $y \in B(x,r) \cap \Omega$ such that

$$\left| \frac{\partial^j u}{\partial x_1^j}(y) - \frac{\partial^j u}{\partial x_1^j}(x) \right| \geqslant r^{-j}.$$

This is obvious when $j = 0$ (because $u(x) = 1$ and hence we may take y with $u(y) = 0$) and it follows from the corollary when $j > 0$ (because $\partial^j u / \partial_{x_1 j}(x) = 0$).

Case ($\boldsymbol{\alpha}$). We have

$$r^{-j} \leqslant M|x-y|^{k-n/p} \|u\|_{m,p;\Omega} \leqslant M r^{k-n/p} c r^{-m} \left| B(x,r) \cap \Omega \right|^{1/p}$$

and the claim follows easily.

Case ($\boldsymbol{\beta}$). We may apply the same argument as in Case (α).

Case ($\boldsymbol{\gamma}$). Proof is similar to that for Case (b), but more difficult. The main difference is in the construction of test functions.

For $x \in \Omega$ and $0 < r \leqslant 1$ we choose $0 < \tilde{\tilde{r}} < \tilde{r} < r$ such that

$$\left| B(x, \tilde{\tilde{r}}) \cap \Omega \right| = \frac{1}{2} \left| B(x, \tilde{r}) \cap \Omega \right| = \frac{1}{4} \left| B(x, r) \cap \Omega \right|.$$

Once we prove the following lemma, the remaining part of the proof is the same as in the case $mp = n$, $p > 1$.

Lemma 16. *There exist constants $c_1 > 0$ and $c_2 > \omega_n$, such that for every $x \in \Omega$ and $0 < r \leqslant 1$ we have*

$$\tilde{r} - \tilde{\tilde{r}} \leqslant c_1 \left| B(x, \tilde{r}) \cap \Omega \right|^{1/n} \left(\ln \left(\frac{c_2 r^n}{|B(x, \tilde{r}) \cap \Omega|} \right) \right)^{p/sn}.$$

Proof. First we need to construct appropriate functions. Let $\psi \in C_0^\infty(B(0,1))$ be such that $\psi \geqslant 0$, $\int \psi \, dx = 1$ and set $\psi_\varepsilon(x) = \varepsilon^{-n} \psi(x/\varepsilon)$. For $0 < a < b \leqslant 1$ we define

$$f_{a,b}(x) = \frac{x^\alpha}{\alpha!} \chi_{B(0, \frac{a+b}{2})}(x)$$

and

$$\varphi_{a,b,\alpha}(x) = (f_{a,b} * \psi_{\frac{b-a}{2}})(x).$$

Clearly $\varphi_{a,b,\alpha} \in C_0^\infty(B(0,b))$ and

$$\varphi_{a,b,\alpha}(x) = (f * \psi_{\frac{b-a}{2}})(x) \quad \text{for } |x| \leqslant a,$$

where $f(x) = x^\alpha / \alpha!$ This implies that $D^\alpha \varphi_{a,b,\alpha}(x) = 1$ for $x \in B(0,a)$. For $|\beta| \leqslant m$ we have

$$\left|D^\beta \varphi_{a,b,\alpha}\right| = \left|f_{a,b} * \left(D^\beta \psi\right)_{\frac{b-a}{2}}\right|\left(\frac{b-a}{2}\right)^{-|\beta|} \leqslant \|f_{a,b}\|_\infty \left(\frac{b-a}{2}\right)^{-|\beta|} \int \left|D^\beta \psi\right|_{\frac{b-a}{2}}$$
$$= \|f_{a,b}\|_\infty \left\|D^\beta \psi\right\|_1 \left(\frac{b-a}{2}\right)^{-|\beta|} \leqslant c b^{j+1}(b-a)^{-m}$$

with a constant $c = c(m,n)$. Hence

$$\left|D^\beta \varphi_{a,b,\alpha}\right| \leqslant c b^{j+1}(b-a)^{-m} \chi_{B(0,b)}. \tag{14}$$

This immediately implies the following estimate.

Lemma 17. *For $x \in \Omega$ and $0 < a < b \leqslant 1$ we define*

$$u(y) = \varphi_{a,b,\alpha}(y-x) \quad \textit{for } y \in \Omega.$$

Then

$$\|u\|_{m,p;\Omega} \leqslant c b^{j+1}(b-a)^{-m} \left|B(x,b) \cap \Omega\right|^{1/p}.$$

In particular the function

$$u(y) = \varphi_{\tilde{\tilde{r}},\tilde{r},\alpha}(y-x) \quad \text{for } y \in \Omega$$

satisfies

$$\|u\|_{m,p;\Omega} \leqslant c \tilde{r}^{j+1}(\tilde{r} - \tilde{\tilde{r}})^{-m} \left|B(x,\tilde{r}) \cap \Omega\right|^{1/p}.$$

Since $D^\alpha u = 1$ on $B(x,\tilde{\tilde{r}}) \cap \Omega$ and $D^\alpha u = 0$ on $A(\tilde{r}, r) \cap \Omega$ for every $\gamma \in \mathbb{R}$ we have that $|D^\alpha u - \gamma| \geqslant 1/2$ on at least one of the sets

$$B(x,\tilde{\tilde{r}}) \cap \Omega \quad \text{or} \quad A(\tilde{r}, r) \cap \Omega$$

and hence the Trudinger-type inequality (5) yields

$$\left|B(x,\tilde{\tilde{r}}) \cap \Omega\right| \exp\!\left(c \tilde{r}^{-(j+1)}(\tilde{r} - \tilde{\tilde{r}})^m \left|B(x,\tilde{r}) \cap \Omega\right|^{-1/p}\right)^s \leqslant M r^n.$$

Replacing M by a constant $c' > \max\{M, \omega_n\}$, after elementary calculations we arrive at

$$\tilde{r} - \tilde{\tilde{r}} \leqslant c\left(\frac{\tilde{r}}{\tilde{r} - \tilde{\tilde{r}}}\right)^{(j+1)p/n} \left|B(x,\tilde{r}) \cap \Omega\right|^{1/n} \left(\ln\left(\frac{c' r^n}{|B(x,\tilde{\tilde{r}}) \cap \Omega|}\right)\right)^{p/sn}. \tag{15}$$

We need to consider two cases.

Case I. $\tilde{r} - \tilde{\tilde{r}} > \tilde{r}/2$. In this case the estimate (15) is exactly the same as the estimate from Lemma 16 that we needed to prove (note that $|B(x,\tilde{\tilde{r}}) \cap \Omega| = |B(x,\tilde{r}) \cap \Omega|/2$).

Case II. $\tilde{r} - \tilde{\tilde{r}} \leqslant \tilde{r}/2$. If $\tilde{r} - \tilde{\tilde{r}}$ is much smaller than $\tilde{r}$, then the factor $(\tilde{r}/(\tilde{r} - \tilde{\tilde{r}}))^{(j+1)p/n}$ is very large and the estimate (15) is much worse than the one we want to prove. To handle this problem we need to construct a different test function. We will need the following easy geometric observation.

Lemma 18. *There is a constant $\kappa = \kappa(n)$ and a finite number of balls $B(x_i, \frac{\tilde{r}-\tilde{\tilde{r}}}{4})$, $i = 1, 2, \ldots, \ell$, such that*:

- $B(x_i, \frac{\tilde{r}-\tilde{\tilde{r}}}{2}) \subset B(x, \tilde{r})$ *for* $i = 1, 2, \ldots, \ell$,
- *the balls* $B(x_i, \frac{\tilde{r}-\tilde{\tilde{r}}}{2})$, $i = 1, 2, \ldots, \ell$ *are pairwise disjoint,*
- $$\left| \left(\bigcup_{i=1}^{\ell} B\left(x_i, \frac{\tilde{r} - \tilde{\tilde{r}}}{4}\right) \right) \cap B(x, \tilde{\tilde{r}}) \cap \Omega \right| \geqslant \kappa \left| B(x, \tilde{\tilde{r}}) \cap \Omega \right|.$$

Proof. Consider the family $\mathcal{F}$ of all balls of radius $(\tilde{r} - \tilde{\tilde{r}})/4$ centered at the points of $\mathbb{R}^n$ whose all coordinates are integer multiples of $\tilde{r} - \tilde{\tilde{r}}$, i.e. the balls are centered at the points of the rescaled integer lattice $(\tilde{r} - \tilde{\tilde{r}})\mathbb{Z}^n$. Clearly, the balls in the collection with the same centers and twice the radii are also pairwise disjoint. A finite number $c(n)$ of parallel translations of the family $\mathcal{F}$ covers all of $\mathbb{R}^n$. Hence at least one of the translated families intersected with $B(x, \tilde{\tilde{r}}) \cap \Omega$ covers at least $\kappa(n) = 1/c(n)$-fraction of the measure of the set $B(x, \tilde{\tilde{r}}) \cap \Omega$. Define the balls $\{B(x_i, (\tilde{r} - \tilde{\tilde{r}})/4)\}_{i=1}^{\ell}$ to be the balls from this translated family that have a nonempty intersection with $B(x, \tilde{\tilde{r}}) \cap \Omega$. Obviously $B(x_i, (\tilde{r} - \tilde{\tilde{r}})/2) \subset B(x, \tilde{r})$. The proof of the lemma is complete. $\square$

Now we are ready to define a new function. Let

$$u(y) = \sum_{i=1}^{\ell} \varphi_{\frac{\tilde{r}-\tilde{\tilde{r}}}{4}, \frac{\tilde{r}-\tilde{\tilde{r}}}{2}, \alpha}(y - x_i) \quad \text{for } y \in \Omega.$$

The functions that appear in the sum have disjoint supports and $\operatorname{supp} u \subset B(x, \tilde{r})$. Estimate (14) yields

$$\left| D^{\beta} u \right| \leqslant c(\tilde{r} - \tilde{\tilde{r}})^{-(k-1)} \chi_{B(x,\tilde{r}) \cap \Omega} \quad \text{for } |\beta| \leqslant m$$

and hence

$$\|u\|_{m,p;\Omega} \leqslant c(\tilde{r} - \tilde{\tilde{r}})^{-(k-1)} \left| B(x, \tilde{r}) \cap \Omega \right|^{1/p}.$$

On the other hand,

$$D^{\alpha} u(y) = 1 \quad \text{for } y \in \bigcup_{i=1}^{\ell} B\left(x_i, \frac{\tilde{r} - \tilde{\tilde{r}}}{4}\right) \cap \Omega,$$

and hence $D^{\alpha} u(y) = 1$ on a subset of $B(x, \tilde{\tilde{r}}) \cap \Omega$ whose measure is at least $\kappa |B(x, \tilde{\tilde{r}}) \cap \Omega|$. Since $D^{\alpha} u = 0$ in $A(\tilde{r}, r) \cap \Omega$, for every $\gamma \in \mathbb{R}$ the inequality $|D^{\alpha} u - \gamma| \geqslant 1/2$ holds on a subset of

$B(x,r) \cap \Omega$ whose measure is at least $\kappa|B(x,\tilde{\tilde{r}}) \cap \Omega|$. Hence the Trudinger type inequality (5) yields

$$\kappa\left|B(x,\tilde{\tilde{r}}) \cap \Omega\right| \exp\bigl(c(\tilde{r}-\tilde{\tilde{r}})^{k-1}\left|B(x,\tilde{r}) \cap \Omega\right|^{-1/p}\bigr)^{s} \leqslant Mr^{n}$$

which easily implies the estimate from Lemma 16. The proof of Lemma 16 and hence the proof of the theorem are complete. □

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