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Formation of cracks under deformations with finite energy

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Abstract. With a map $f : \Omega \to \mathbf{R}^n$, $\Omega \subset \mathbf{R}^n$, that belongs to the John Ball class $A^+_{p,q}(\Omega)$ where $n-1 < p < n$ and $q \geq p/(p-1)$ one can associate a set valued map F whose values $F(x) \subset \mathbf{R}^n$ are subsets of $\mathbf{R}^n$ describing the topological character of the singularity of f at $x \in \Omega$. Šverak conjectured that $\mathcal{H}^{n-1}(F(S)) = 0$, where S is the set of points at which f is not continuous and $\mathcal{H}^{n-1}$ is the Hausdorff measure. The purpose of our paper is to confirm this expectation.

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1 Introduction

The study of elastic deformations leads to regularity questions of Sobolev mappings. When a body occupying a region $\Omega \subset \mathbf{R}^n$ is deformed, the change of the position of a particle $x \in \Omega$ determines a mapping $f : \Omega \to \mathbf{R}^n$. Such a mapping f is a minimizer of an energy integral of the form

$$I(f,\Omega) = \int_\Omega W(x, f(x), \nabla f(x))\, dx.$$

Here ∇f is the weak gradient of f. The competing functions naturally need to have finite energy and thus one is lead to inquire the regularity properties of mappings f for which $I(f,\Omega) < \infty$. In the fundamental work of Ball [1] a suitable competing class is recognized as

$$A^+_{p,q} = \{f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{R}^n) :\ \nabla f \in L^p(\Omega), \quad \text{adj}\nabla f \in L^q(\Omega), \det \nabla f > 0 \text{ a.e. in } \Omega\}, \tag{1}$$

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where $p > n-1$ and $q \geq p/(p-1)$. Here adj∇f is the adjugate of ∇f, consisting of all the $(n-1)$-minors of ∇f.

Šverak [13] showed that each $f \in A^+_{p,q}$, where $p > n-1$ and $q \geq p/(p-1)$, generates a set-valued mapping F on Ω; see Sect. 2 for the construction of F. If f is continuous at a point x, then $F(x)$ is a vector, i.e. it consists of a single point. If, however, f is not continuous at x, then $F(x) \subset \mathbf{R}^n$ is a compact connected set that describes the topological character of the singularity at x. He proved that f is continuous outside a set S of vanishing variational p-capacity (for $p \geq n$, f is continuous everywhere). Moreover, no holes are formed under f. The set $F(S)$ tells us how the discontinuity set S is deformed under the mapping f. Šverak showed that the volume of $F(S)$ is zero and that the $(n-1)$-dimensional measure of $F(x)$ is zero for each $x \in S$. Physically, the former conclusion means that there is no creation of matter in S and the latter that no point in S can result in a crack in $f(\Omega)$; these two physically relevant conclusions then hold in Ω. Šverak conjectures that, in fact, the $(n-1)$-dimensional measure of the entire $F(S)$ is zero (so that no cracks are formed), see [13, p. 119]. The purpose of this short note is to confirm this expectation by establishing the following result.

Theorem 1 *Let $\Omega \subset \mathbf{R}^n$ be a domain and suppose that $f \in A^+_{p,q}(\Omega)$, where $n-1 < p < n$ and $q \geq p/(p-1)$. Let F be the associated set-valued map and S the corresponding singular set. Then $\mathcal{H}^{n-p}(S) = 0$ and $\mathcal{H}^{n-1}(F(E)) = 0$ for each $E \subset \Omega$ with $\mathcal{H}^{n-p}(E) < \infty$. In particular $\mathcal{H}^{n-1}(F(S)) = 0$.*

Müller, Tang, and Yan [12] have shown that most of the theory of $A^+_{p,q}$-mappings developed by Šverak for $n-1 < p < n$ and $q \geq p/(p-1)$ extends, with the same conclusions, to a larger class of $A^+_{p,q}$-mappings for $n-1 < p < n$ and $q \geq n/(n-1)$. This includes construction of the set-valued mapping F associated with $f \in A^+_{p,q}$ and the fact that $\mathcal{H}^{n-1}(F(x)) = 0$ for every $x \in S$. It is natural to inquire if Theorem 1 could hold in this larger class of mappings. The answer turns out to be negative at least when the distortion of the dimension of general sets is considered.

Proposition 2 *Assume that we are given $n-1 < p < n$ and a nonempty open and bounded set $\Omega \subset \mathbf{R}^n$. Then there exists a homeomorphism $f \in A^+_{r,q}(\Omega) \subset A^+_{p,q}(\Omega)$ for some $r > n$ and $q > n/(n-1)$ which maps a set of vanishing $(n-p)$-dimensional measure onto a set of Hausdorff dimension larger than $n-1$.*

We do not know if $\mathcal{H}^{n-1}(F(S)) = 0$ for $f \in A^+_{p,q}(\Omega)$, $n-1 < p < n$ and $q \geq n/(n-1)$.

It turns out that a mapping constructed by Gehring and Väisälä, [6], has all properties that we need. Indeed, for given $0 < s < n$ and $0 < t < n$ they have constructed a quasiconformal homeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$ which maps a Cantor type set $K \subset \Omega$ of dimension s onto another Cantor type set $f(K)$ of dimension t. Take $s < n-p$ and $t > n-1$. Then $\mathcal{H}^{n-p}(K) = 0$ and the Hausdorff dimension of $f(K)$ is larger than $n-1$. Since $\det \nabla f > 0$ a.e. and, by the higher integrability of the gradient of a quasiconformal mapping, [5], $f \in W^{1,r}_{\mathrm{loc}}(\mathbf{R}^n)$ for some $r > n$, we conclude that adj$\nabla f \in L^q_{\mathrm{loc}}$, where $q = r/(n-1) > n/(n-1)$ and hence $f|_\Omega \in A^+_{r,q}(\Omega)$. This gives us the claim of the proposition.

2 Construction and basic properties of F

Let $f \in A^+_{p,q}(\Omega)$, where $p > n-1$ and $q \geq p/(p-1)$. For $x \in \Omega$ we define $r_x = \operatorname{dist}(x, \partial\Omega)$. Since $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbf{R}^n)$ and $p > n-1$, the Fubini and the Sobolev embedding theorems guarantee that there is a set $Z_x \subset (0, r_x)$ of measure zero such that for each $r \in (0, r_x) \setminus Z_x$ the trace $f_r = f|_{S^{n-1}(x,r)}$ belongs to the space $W^{1,p}(S^{n-1}(x,r))$ (and so is continuous) and $\operatorname{adj}\nabla f|_{S^{n-1}(x,r)} \in L^q(S^{n-1}(x,r))$.

Recall that for a bounded domain $G \subset \mathbf{R}^n$, and a continuous mapping $g : \partial G \to \mathbf{R}^n$ one can define the topological degree $\deg(g, \partial G, y)$ for all $y \in \mathbf{R}^n \setminus g(\partial G)$. The degree is integer-valued, constant on components of $\mathbf{R}^n \setminus g(\partial G)$ and equal to zero on the unbounded component of $\mathbf{R}^n \setminus g(\partial G)$, see e.g. [4]. Employing the notion of topological degree we define for $r \in (0, r_x) \setminus Z_x$

$$E(f,x,r) = \{y \in \mathbf{R}^n \setminus f_r(S^{n-1}(x,r)) :\ \deg(f_r, S^{n-1}(x,r), y) \geq 1\} \\ \cup f_r(S^{n-1}(x,r)).$$

Since $E(f,x,r)$ consists of the image of the sphere $f_r(S^{n-1}(x,r))$ plus some of the bounded components of $\mathbf{R}^n \setminus f_r(S^{n-1}(x,r))$ (those components where degree is ≥ 1), the set $E(f,x,r)$ is compact, connected and $\operatorname{diam}(E(f,x,r)) = \operatorname{osc}_{S^{n-1}(x,r)} f_r$.

Using the assumption $f \in A^+_{p,q}$, Šverak proves [13, Lemma 3] that the sets $E(f,x,r)$ are nested: for $r_1 \in (0, r_x) \setminus Z_x$, $r_2 \in (0, r_y) \setminus Z_y$, such that $B(x, r_1) \subset B(y, r_2)$ we have

$$E(f,x,r_1) \subset E(f,y,r_2). \tag{2}$$

The key point in establishing this property is that the topological degree can be represented using an integral of $\det \nabla f$ and that $\det \nabla f(x) > 0$ a.e. in Ω. It was later proven by Müller, Tang and Yan [12] that this also holds if we only assume that $p > n-1$ and $q \geq n/(n-1)$ above.

In particular, for $r_1, r_2 \in (0, r_x) \setminus Z_x$, $r_1 < r_2$, we have

$$E(f,x,r_1) \subset E(f,x,r_2).$$

We now simply define

$$F(x) = \bigcap_{r \in (0,r_x) \setminus Z_x} E(f,x,r), \qquad \text{and} \qquad F(A) = \bigcup_{x \in A} F(x).$$

Clearly $F(x)$ is compact, connected and

$$\operatorname{diam}(F(x)) = \lim_{r \to 0+} \operatorname{diam} E(f,x,r). \tag{3}$$

for every $x \in \Omega$. Moreover $\operatorname{diam}(F(x)) = 0$ and so $F(x)$ is a vector whenever f has a representative that is continuous at x.

It is essential for us that there is a representative of f that is continuous outside a set of finite (in fact zero) $(n-p)$-dimensional measure. This observation lead us to the correct track for the proof of Theorem 1. We define such a representative by the formula

$$f(x) := \limsup_{r\to 0} \fint_{B(x,r)} f(y)\,dy \qquad \text{for every } x \in \Omega, \tag{4}$$

where the $\limsup$ is taken coordinate-wise. Here and in what follows the barred integral denotes the integral average over the ball. The fact that this is a representative of f follows from the Lebesgue differentiation theorem [2, 1.7.1/Theorem 1]. From now on we will always assume that $f \in A^+_{p,q}$ coincides with its representative given by (4).

It was proved in [13, Corollary 1] that

$$f(y) \in E(f,x,r) \quad \text{for a.e. } y \in B(x,r) \text{ whenever } r \in (0,r_x) \setminus Z_x. \tag{5}$$

This and the choice of the representative (4) imply that for each $z \in B(x,r)$, $f(z)$ belongs to the smallest box with sides parallel to the coordinate axes containing $E(f,x,r)$, provided $r \in (0,r_x)\setminus Z_x$. Hence $\operatorname{diam}(F(x)) = 0$ implies that $F(x) = \{f(x)\}$ and that f is continuous at x. Thus $F(x)$ consists of a single point if and only if f is continuous at x.

Proposition 3 *Let $\Omega \subset \mathbf{R}^n$ be a domain and suppose that $f \in A^+_{p,q}(\Omega)$, where $n-1 < p < n$ and $q \geq p/(p-1)$. Then there is a set $S \subset \Omega$ so that $\mathcal{H}^{n-p}(S) = 0$ and f is continuous in $\Omega \setminus S$.*

Remarks.

1) Just to make sure that the statement is properly understood: we claim that f is continuous at every point of $\Omega \setminus S$, which is more than continuity of f restricted to $\Omega \setminus S$.
2) The proof of Proposition 3 is a minor variation of the argument of Šverak that gave continuity outside a set of vanishing variational p-capacity. This improvement on Šverak's result was observed by Müller and Spector [11, Theorem 7.4], see also [8, Theorem 4.1 and Theorem 4.5]. For the convenience of the reader we sketch a direct proof here.
3) Actually Proposition 3 and properties of the mapping F easily imply a stronger result: the set-valued mapping F is continuous at each point of $\Omega \setminus S$ with respect to the natural notion of convergence of sets (in the Hausdorff metric).

Proof. Since $|\nabla f|^p \in L^1$, it is a well known consequence of a covering argument that $\mathcal{H}^{n-p}(S) = 0$, where

$$S = \{x \in \Omega :\ \limsup_{r\to 0} r^{p-n} \int_{B(x,r)} |\nabla f|^p > 0\},$$

see [4, Proposition 4.37]. Clearly

$$\lim_{r\to 0} r^{p-n} \int_{B(x,r)} |\nabla f|^p = 0 \quad \text{for each } x \in \Omega \setminus S. \tag{6}$$

We will prove that f, given by (4), is continuous at every point of $\Omega \setminus S$. Fix $x \in \Omega \setminus S$ and $0 < r < \operatorname{dist}(x, \partial\Omega)/4$. Then $f \in W^{1,p}(S^{n-1}(x,s))$ for a.e. $r < s < 2r$ and by the Fubini theorem we find an allowable $r < s < 2r$ with

$$\int_{S^{n-1}(x,s)} |\nabla f|^p \, d\sigma \le r^{-1} \int_{B(x,2r)} |\nabla f|^p \, dx. \tag{7}$$

On the other hand, by the Sobolev embedding theorem on spheres,

$$\operatorname*{osc}_{S^{n-1}(x,s)} f \le C s^{1-(n-1)/p} \left(\int_{S^{n-1}(x,s)} |\nabla f|^p \, d\sigma \right)^{1/p}. \tag{8}$$

Here and in what follows C will denote a general constant whose value can change even in the same string of estimates. Combining inequalities (7) and (8) and the fact that $\operatorname{diam}(E(f,x,s)) = \operatorname{osc}_{S^{n-1}(x,s)} f_s$ we arrive at

$$(\operatorname{diam} E(f,x,s))^p \le C r^{p-n} \int_{B(x,2r)} |\nabla f|^p \, dx.$$

It thus follows from (6) and (3) that $\operatorname{diam}(F(x)) = 0$ and hence f is continuous at x. This completes the proof of the proposition.

3 Proof of Theorem 1

Proof of the fact that $\mathcal{H}^{n-1}(S) = 0$ is contained in Proposition 3. Now let $E \subset \Omega$ be a set of finite $(n-p)$-dimensional measure. We will show that $\mathcal{H}^{n-1}(F(E)) = 0$. Clearly we can assume that Ω is bounded. Take an arbitrary $m > 0$ such that $\mathcal{H}^{n-p}(E) \le m < \infty$. It follows from the definition of the Hausdorff measure and a standard covering argument, [9, Theorem 2.1], that for given $\varepsilon > 0$ there is a family of pairwise disjoint balls $\{B(x_i, r_i)\}_{i=1}^{\infty}$ such that $r_i < \varepsilon$ for all i and

$$E \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_i), \qquad \sum_{i=1}^{\infty} r_i^{n-p} \le C(n)m.$$

Pick, using the Fubini theorem for each r_i, an allowable $5r_i < s_i < 10r_i$, $s_i \in (0, r_{x_i}) \setminus Z_{x_i}$, so that

$$\int_{S^{n-1}(x_i,s_i)} |\mathrm{adj}\nabla f| \, d\sigma \le C r_i^{-1} \int_{B(x_i,10r_i)} |\mathrm{adj}\nabla f| \, dx.$$

This and the area formula, [13, (3) and Theorem 1], yield

$$\begin{aligned} &\mathcal{H}^{n-1}(f(S^{n-1}(x_i,s_i))) \\ &\le \int_{S^{n-1}(x_i,s_i)} |\mathrm{adj}\nabla f| \, d\sigma \le C r_i^{-1} \int_{B(x_i,10r_i)} |\mathrm{adj}\nabla f| \, dx. \end{aligned}$$

We will need now the Hardy–Littlewood maximal function

$$Mh(x) = \sup_{B:\, x \in B} \fint_B |h|,$$

where, as usual, B denotes a ball. We have

$$\begin{aligned}\int_{B(x_i,10r_i)} |\mathrm{adj}\nabla f|\,dx &= Cr_i^n \fint_{B(x_i,10r_i)} |\mathrm{adj}\nabla f|\,dx \\ &\le Cr_i^n \inf_{x\in B(x_i,r_i)} M|\mathrm{adj}\nabla f| \\ &\le Cr_i^n \fint_{B(x_i,r_i)} M|\mathrm{adj}\nabla f|\,dx \\ &= C\int_{B(x_i,r_i)} M|\mathrm{adj}\nabla f|\,dx.\end{aligned}$$

Thus Hölder's inequality applied twice yields

$$\begin{aligned}&\sum_{i=1}^{\infty}\mathcal{H}^{n-1}(f(S^{n-1}(x_i,s_i))) \le C\sum_{i=1}^{\infty} r_i^{-1}\int_{B(x_i,r_i)} M|\mathrm{adj}\nabla f|\,dx \\ &\le C\sum_{i=1}^{\infty} r_i^{\frac{n}{p}-1}\left(\int_{B(x_i,r_i)} (M|\mathrm{adj}\nabla f|)^{p/(p-1)}\,dx\right)^{(p-1)/p} \\ &\le C\left(\sum_{i=1}^{\infty} r_i^{n-p}\right)^{1/p}\left(\sum_{i=1}^{\infty}\int_{B(x_i,r_i)} (M|\mathrm{adj}\nabla f|)^{p/(p-1)}\,dx\right)^{(p-1)/p} \\ &\le Cm^{1/p}\left(\int_{\bigcup_{i=1}^{\infty} B(x_i,r_i)} (M|\mathrm{adj}\nabla f|)^{p/(p-1)}\,dx\right)^{(p-1)/p}. \qquad (9)\end{aligned}$$

Boundedness of Ω implies that $|\mathrm{adj}\nabla u| \in L^q(\Omega) \subset L^{p/(p-1)}(\Omega)$. Since, by the Hardy–Littlewood theorem, [9, Theorem 2.19], the maximal function forms a bounded operator in $L^{p/(p-1)}$, we conclude that the function $(M|\mathrm{adj}\nabla f|)^{p/(p-1)}$ is integrable. This, the estimate of the measure

$$\left|\bigcup_{i=1}^{\infty} B(x_i,r_i)\right| = C\sum_{i=1}^{\infty} r_i^n \le C\varepsilon^p\sum_{i=1}^{\infty} r_i^{n-p} \le Cm\varepsilon^p \to 0 \quad \text{as } \varepsilon\to 0,$$

and the absolute continuity of the integral imply that the right hand side of (9) goes to zero as $\varepsilon \to 0$.

Next, from (2) and the definition of F, we conclude that $F(E) \subset \bigcup_i E(f, x_i, s_i)$. Thus it suffices to show each $E(f,x_i,s_i)$ with $\mathcal{H}^{n-1}(E(f,x_i,s_i)) > 0$ can be covered by balls $\{B(y_j,R_j)\}_j$ so that

$$\sum_j R_j^{n-1} \le C(n)\mathcal{H}^{n-1}(f(S^{n-1}(x_i,s_i))). \qquad (10)$$

By definition, $E(f,x_i,s_i)$ is the union of $f(S^{n-1}(x_i,s_i))$ and a bounded open set whose boundary is contained in $f(S^{n-1}(x_i,s_i))$. Hence $\mathcal{H}^{n-1}(E(f,x_i,s_i)) > 0$ implies that $\mathcal{H}^{n-1}(f(S^{n-1}(x_i,s_i))) > 0$. Clearly we can cover $f(S^{n-1}(x_i,s_i))$ by balls satisfying (10). The existence of a cover like in (10) for the remaining open set in $E(f,x_i,s_i)$ is a consequence of the following result, due to Gustin, [7], and known as the boxing inequality. For an elementary proof see e.g. [3], [10, 1.2.1/Theorem 2].

Lemma 4 *If $U \subset \mathbf{R}^n$ is a bounded open set, then there is a covering of the closure $\overline{U}$ by a finite collection of balls $B(y_j, R_j)$, such that $\sum_j R_j^{n-1} \leq C(n)\mathcal{H}^{n-1}(\partial U)$.*

Remarks.

1) It is assumed in [10, 1.2.1/Theorem 2] that the boundary of U is smooth, but this assumption is never employed in the proof.
2) For a more general statement of Lemma 4 in which $\overline{U}$ is replaced by any compact set K of positive $(n-1)$-dimensional measure, see [7]. This result can easily be reduced to the above special case.

References

1. Ball, J. M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. **63**, 25–43, 337–403 (1978)
2. Evans, L. C., Gariepy, R. F.: Measure theory and fine properties of functions. Studies in Advanced Mathematics, CRC Press, Boca Raton, FL 1992
3. Federer, H.: The area of a nonparametric surface. Proc. Amer. Math. Soc. **11**, 436–439 (1960)
4. Fonseca, I., Gangbo, W.: Degree theory in analysis and applications. Oxford Lecture Series in Math. and Appl. Vol. 2. Clarendon Press, Oxford 1995
5. Gehring, F. W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. Acta Math. **130**, 265–277 (1973)
6. Gehring, F. W., Väisälä, J.: Hausdorff dimension and quasiconformal mappings. J. London Math. Soc. **6**, 504–512 (1973)
7. Gustin, W.: Boxing inequalities. J. Math. Mech. **9**, 229–239 (1960)
8. Hajłasz, P., Malý, J.: Approximation of nonlinear expressions involving gradient in Sobolev spaces. Ark. Math. **40**, 245–274 (2002)
9. Mattila, P.: Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge 1995
10. Maz'ja, V. G.: Sobolev spaces. Springer, Berlin Heidelberg New York 1985
11. Müller, S., Spector, S. J.: An existence theory for nonlinear elasticity that allows for cavitation. Arch. Rational Mech. Anal. **131**, 1–66 (1995)
12. Müller, S., Tang, Q., Yan, B. S.: On a new class of elastic deformations not allowing for cavitation, Analyse non lineaire **11**, 217–243 (1994)
13. Šverak, V.: Regularity properties of deformations with finite energy. Arch. Rational Mech. Anal. **100**, 105–127 (1988)