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WHITNEY'S EXAMPLE BY WAY OF ASSOUAD'S EMBEDDING

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Abstract. In this note we show how to use the Assouad embedding theorem (about almost bi-Lipschitz embeddings) to construct examples of C^m functions which are not constant on a critical set homeomorphic to the n-dimensional cube. This extends the famous example of Whitney. Our examples are shown to be sharp.

1. Introduction

In 1935 Whitney [20] published a surprising example of a C^{n-1} function in $\mathbb{R}^n$ which is not constant on some arc and has all partial derivatives of order $\leq n-1$ vanishing on that arc. The construction is sharp because it follows from Sard's theorem that every $C^n(\mathbb{R}^n)$ function with the first order partial derivatives vanishing on an arc is constant on that arc (see also [13, p. 374]). Since the Whitney example a great deal of work has been done in constructing similar examples and studying conditions under which such phenomena cannot occur; see e.g. [4], [7], [10], [13], [14], [15], [17], [21]. These are only some of the references and it is not our purpose to give a complete list of related papers here.

Most of the constructions (including the original one of Whitney and the one presented below) are based on the same idea. First, one constructs a compact set K and a suitable continuous function u on K which is not constant on K. Then employing Whitney's extension theorem [19] one shows that the function u extends to a $C^m(\mathbb{R}^n)$ smooth function (for some m) with all partial derivatives of order $\leq m$ equal to zero on K. Always the most tricky part of the proof is the construction of K and u. The application of Whitney's extension theorem is then straightforward.

Assouad [3] (see also [1], [2], [9]) proved that for any metric space (X, d) that is doubling (see the definition below) and any $s \in (0,1)$ the new metric space (X, d^s) admits a bi-Lipschitz embedding to the Euclidean space $\mathbb{R}^N$ for some N. We will employ Assouad's theorem to construct K and u. The application of the Whitney extension theorem will then lead to Whitney-type examples.

We will also show that there is a Whitney-type example with the critical set being the Van Koch snowflake.

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The purpose of this note is to show a new class of Whitney type examples as well as to show a new application of the wonderful result of Assouad. For other applications of Assouad's theorem, see e.g. [5], [9], [16].

1.1. **Notation and statement.** Now we explain notation and give the precise statement of the results.

We say that $u \in C^{m,\lambda}(\mathbb{R}^n)$, where m is a positive integer and $\lambda \in (0,1]$ if u is continuously m times differentiable with λ-Hölder continuous derivatives of order m. Given $u \in C^m(\mathbb{R}^n)$ we say that a point $x \in \mathbb{R}^n$ is *m-critical* if all partial derivatives of u of order $\leq m$ vanish at the point x. We set

$$\mathrm{Crit}_m(u) = \{x \in \mathbb{R}^n :\ D^\alpha u(x) = 0 \text{ for all } 0 < |\alpha| \leq m\}\,.$$

We call a set $K \subset \mathbb{R}^n$ *m-critical* for u if $K \subset \mathrm{Crit}_m(u)$.

Given $K \subset \mathbb{R}^n$, we say that a function $u : K \to \mathbb{R}$ is *m-critical* if there exists a function $U \in C^m(\mathbb{R}^n)$ such that K is m-critical for U and $U|_K = u$. Similarly we say that $u : K \to \mathbb{R}$ is (m,λ)-critical, $\lambda \in (0,1]$ is u is m-critical and U can be chosen in the class $C^{m,\lambda}(\mathbb{R}^n)$.

With the above notation we can state Whitney's result as follows: *There exists a set $K \subset \mathbb{R}^n$ homeomorphic to the interval $[0,1]$ and an $(n-1)$-critical function $u : K \to \mathbb{R}$ which is not constant.*

The arc K in the original Whitney example consists of infinitely many segments that connect a Cantor type set in $\mathbb{R}^n$. The function u is constant (and it has to be constant) in each of these segments. Thus u reminds us of the famous Cantor staircase function.

The theorem below, the main result of the paper, extends Whitney's example in several ways: arc is replaced by a higher-dimensional topological cube, and the class of critical functions u on K is much larger than the corresponding class in Whitney's example in the sense that u need not be constant in any subset of K.

The main result of the paper reads as follows.

Theorem 1. *Assume given positive integers m and n. Then there exists a positive integer N and a compact set $K \subset \mathbb{R}^N$ homeomorphic to the cube $[0,1]^n$ such that the class of m-critical functions on K form a dense subset of $C(K)$, the space of all continuous functions on K.*

Actually we will construct a homeomorphism $\Phi : [0,1]^n \to K$ such that for every Lipschitz function u on $[0,1]^n$, the function $u \circ \Phi^{-1}$ is m-critical on K.

The proof of the theorem also applies to the case in which we take the planar Van Koch snowflake as K (see [8]). We will denote the snowflake by Γ. This yields the following result.

Corollary 2. *The class of $(1,\alpha)$-critical functions on the Van Koch snowflake Γ, where $1+\alpha = \log 4/\log 3$, is dense in the space $C(\Gamma)$. However, if $\beta > \alpha$, then $(1,\beta)$-critical functions on Γ are constant.*

Note that $\log 4/\log 3$ is the Hausdorff dimension of the snowflake Γ.

In the next section we prove the theorem, show that the example constructed in the proof is sharp (Proposition 5) and then prove the corollary.

2. Proof of the main theorem and its corollary

First we state the Whitney extension theorem and the Assouad embedding theorem, the main ingredients in the proof of Theorem 1.

Given a positive integer m, $\lambda \in (0,1]$, and a compact set $K \subset \mathbb{R}^n$, Whitney's extension theorem [19] provides a necessary and sufficient condition for a continuous function $u : K \to \mathbb{R}$ to be the restriction of a function $U \in C^{m,\lambda}(\mathbb{R}^n)$ (or $U \in C^m(\mathbb{R}^n)$) to K. Actually Whitney dealt with the C^m case only. The case $C^{m,\lambda}$ is similar but slightly more difficult, and the proof can be found e.g. in [12]. The statement of the Whitney theorem is quite complicated as one may also prescribe all partial derivatives of u of order $\le m$ on K. However in the case we want all partial derivatives of order $\le m$ to be equal to zero on K, i.e. we want u to be (m,λ)-critical (or m-critical), the statement substantially simplifies. We will actually need this special case of Whitney's theorem and we state it as the following lemma.

Lemma 3. *Assume given a positive integer m, $\lambda \in (0,1]$, and a compact set $K \subset \mathbb{R}^n$. Then a continuous function $u : K \to \mathbb{R}$ is (m,λ)-critical if and only if there is a constant $M > 0$ such that $|u(x) - u(y)| \le M|x-y|^{m+\lambda}$ for all $x, y \in K$.*

Thus a necessary and sufficient condition for u to be (m,λ)-critical is the $(m+\lambda)$-Hölder continuity. It is well known that if u is not constant and γ-Hölder continuous in a regular set (e.g. smooth submanifold), then $\gamma \le 1$. Since we will need $m+\lambda$ to be arbitrarily large, the set K will have to be far from regular, say "fractal". Fractal sets admitting nontrivial $(m+\lambda)$-Hölder continuous functions will be obtained as images of the Assouad embedding that we now describe.

One of the most interesting problems of geometric analysis and geometric topology is a question of characterizing those compact metric spaces (X,d) that admit bi-Lipschitz embedding to some Euclidean space, i.e. a mapping $\Phi : X \to \mathbb{R}^N$, for some N, such that

$$C_1 d(x,y) \le |\Phi(x) - \Phi(y)| \le C_2 d(x,y)$$

for all $x, y \in X$ with some fixed constants $C_1, C_2 > 0$. An obvious necessary condition for X is the doubling condition which we state next.

We say that a metric space (X,d) is *doubling* if there is a constant M such that every ball B in X can be covered by at most M balls with half the radius of B.

Since every subset of the Euclidean space is doubling, and the doubling condition is invariant under bi-Lipschitz mappings, we immediately conclude that in order for X to be bi-Lipschitz embeddable to the Euclidean space, X has to be doubling. Unfortunately the doubling condition is not sufficient as there are examples of doubling metric spaces that do not admit any bi-Lipschitz embedding to any of the Euclidean spaces (cf. [9]). Assouad proved however an amazing result showing that a doubling metric space admits an "almost" bi-Lipschitz embedding. We state Assouad's theorem as the following lemma.

Lemma 4. *Let (X,d) be a doubling metric space. Then for any $s \in (0,1)$ there is N and a bi-Lipschitz embedding $\Phi : (X, d^s) \to \mathbb{R}^N$, i.e. a mapping such that*

$$C_1 d(x,y)^s \le |\Phi(x) - \Phi(y)| \le C_2 d(x,y)^s$$

for some constants $C_1, C_2 > 0$ and all $x, y \in X$.

Now we can prove the theorem.

Proof of Theorem 1. The unit cube $[0,1]^n$ is doubling as a metric space with the Euclidean metric. Hence Assouad's theorem (Lemma 4) asserts that for any $s \in (0,1)$ there is N and a mapping $\Phi : [0,1]^n \to \mathbb{R}^N$ such that

$$C_1 |x-y|^s \le |\Phi(x) - \Phi(y)| \le C_2 |x-y|^s \,. \tag{1}$$

If we take $1/s = m + \lambda$, $\lambda \in (0,1)$, then for any Lipschitz function u on $[0,1]^n$ the function $u \circ \Phi^{-1}$ is $(m+\lambda)$-Hölder continuous on $K = \Phi([0,1]^n)$ and hence by Lemma 3, the function $u \circ \Phi^{-1}$ is (m,λ)-critical. In particular $u \circ \Phi^{-1}$ is m-critical. The density of m-critical functions in $C(K)$ follows from the density of Lipschitz functions in $C([0,1]^n)$. The proof of the theorem is complete. □

Now we show that the above construction is sharp in the following sense.

Let $K = \Phi([0,1]^n)$ be the set constructed in the proof of Theorem 1 and let $1/s = m + \lambda$, $\lambda \in (0,1)$, be as above. Recall that for any Lipschitz function u on $[0,1]^n$ the function $u \circ \Phi^{-1}$ is (m,λ)-critical on K. In particular (m,λ)-critical functions form a dense subset of $C(K)$.

Proposition 5. *If u is (m,β)-critical on K, where $\beta > \lambda$, then $u = \text{const}$ on K.*

Proof. Assume by contradiction that $u(x) \neq u(y)$ for some $x, y \in K$. Divide the segment connecting $\Phi^{-1}(x)$ and $\Phi^{-1}(y)$ in $[0,1]^n$ into k segments of equal length. Denote the ends of the consecutive segments by z_i and z_{i+1}. Obviously $z_1 = \Phi^{-1}(x)$ and $z_{k+1} = \Phi^{-1}(y)$. Now we have

$$\begin{aligned}
0 &< |u(x) - u(y)| \leq \sum_{i=1}^{k} |u(\Phi(z_i)) - u(\Phi(z_{i+1}))| \\
&\leq M \sum_{i=1}^{k} |\Phi(z_i) - \Phi(z_{i+1})|^{(m+\beta)} \\
&\leq CM \sum_{i=1}^{k} |z_i - z_{i+1}|^{(m+\beta)/(m+\lambda)} \\
&= CMk \left(\frac{|\Phi^{-1}(x) - \Phi^{-1}(y)|}{k} \right)^{(m+\beta)/(m+\lambda)} \to 0\,,
\end{aligned}$$

as $k \to \infty$, because $(m+\beta)/(m+\lambda) > 1$. The contradiction proves the claim. The proof is complete. □

Note that in the proof of Theorem 1 we employed the left-hand side estimate of (1), while in the proof of Proposition 5 we employed the right-hand side estimate of (1).

Now we are left with the proof of the corollary.

Proof of Corollary 2. Following the iterative construction of the Van Koch snowflake, it is easy to produce its parametrization by the homeomorphism $\Phi : S^1 \to \Gamma$ satisfying $|\Phi(x) - \Phi(y)| \approx |x - y|^{\log 3/\log 4}$ for $x, y \in S^1$; see e.g. [18, p. 151]. Then the corollary follows as a direct consequence of the proof of Theorem 1 in which we replace Assouad's embedding by the above parametrization. The fact that $(1,\beta)$-critical functions, $\beta > \alpha$, are constant on Γ is a consequence of the proposition. The proof is complete. □

To construct more examples like that, one may also use [11, Lemma 2.3] or results of [6] to substitute Assouad's embedding in the proof of Theorem 1.

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