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POINTWISE HARDY INEQUALITIES

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ABSTRACT. If $\Omega \subset \mathbb{R}^n$ is an open set with the sufficiently regular boundary, then the Hardy inequality $\int_\Omega |u|^p \varrho^{-p} \le C \int_\Omega |\nabla u|^p$ holds for $u \in C_0^\infty(\Omega)$ and $1 < p < \infty$, where $\varrho(x) = \operatorname{dist}(x, \partial\Omega)$. The main result of the paper is a pointwise inequality $|u| \le \varrho M_{2\varrho}|\nabla u|$, where on the right hand side there is a kind of maximal function. The pointwise inequality combined with the Hardy–Littlewood maximal theorem implies the Hardy inequality. This generalizes some recent results of Lewis and Wannebo.

1. INTRODUCTION

The classical Hardy inequality reads as follows

$$\int_0^\infty |u(x)|^p x^{-p+a}\, dx \le \left(\frac{p}{p-1-a}\right)^p \int_0^\infty |u'(x)|^p x^a\, dx,$$

where $1 < p < \infty$, $a < p - 1$, u is absolutely continuous on $[0, \infty)$, and $u(0) = 0$. It seems that the first to generalize this inequality to domains in $\mathbb{R}^n$ was Nečas [14] (cf. [10, 8.8]), who proved that if Ω is a bounded domain with the Lipschitz boundary, $1 < p < \infty$ and $a < p - 1$, then for $u \in C_0^\infty(\Omega)$ the inequality

$$\int_\Omega |u(x)|^p \varrho(x)^{-p+a}\, dx \le C \int_\Omega |\nabla u(x)|^p \varrho(x)^a\, dx \tag{1}$$

holds, with $\varrho(x) = \operatorname{dist}(x, \partial\Omega)$. This inequality was generalized later by Kufner, [10, Theorem 8.4], to domains with Hölder boundary, and then by Wannebo [18], to domains with generalized Hölder condition.

There is a rich literature concerning one dimensional variants of the Hardy inequality as well as the multidimensional variants, where one assumes a kind of regularity for the boundary like Lipschitz or Hölder type condition. We refer the reader to books by Opic and Kufner [15], Maz'ya [12], the paper by Wannebo [18], and references therein.

However, it is possible to obtain a far reaching generalization of some of these results by assuming a much weaker condition about the boundary. Instead of a

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regularity condition (like, e.g., Hölder continuity), it suffices to assume that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is "fat" enough in the sense of capacity.

We need to recall the definition of capacity. Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open set. For a compact set $K \subset \Omega$ the capacity is defined as

$$C_{1,p}(K,\Omega) = \inf \left\{ \int_\Omega |\nabla u(x)|^p \, dx : \; u \in C_0^\infty(\Omega), \; u \geq 1 \text{ on } K \right\}.$$

The definition of the capacity can be extended to an arbitrary set, but, we will need the capacity only for compact sets. For basic properties of the capacity; see [6, Chapter 2].

We say that a closed set $E \subset \mathbb{R}^n$ is *locally uniformly p-thick*, $1 < p < \infty$, if there exist $b > 0$ and $0 < r_0 \leq \infty$ such that

$$C_{1,p}(\overline{B}(x,r) \cap E, B(x,2r)) \geq b\, C_{1,p}(\overline{B}(x,r), B(x,2r)), \tag{2}$$

for $x \in E$ and $0 < r < r_0$. If $r_0 = \infty$, then we call E *uniformly p-thick*. Our definition slightly differs from that formulated in [6, p. 127], [8]; their definition of uniformly p-thick set coincides with our definition of locally uniformly p-thick set.

Note that by a scaling argument we obtain

$$C_{1,p}(\overline{B}(x,r), B(x,2r)) = C(n,p) r^{n-p}.$$

Independently Lewis [11] and Wannebo [17], proved that the Hardy inequality (1) holds in a proper open subset of $\mathbb{R}^n$ provided its complement is uniformly p-thick (Lewis: $a = 0$; Wannebo: $a \geq 0$, small). For a more precise statement, see [11], [17] and also Section 3. Their results extend earlier works of Ancona [2], [3]. The result is pretty sharp: Ancona [3], for $n = p = 2$, and, in the general case, Lewis [11, Theorem 3], proved that for $p = n$, the condition (2) is also necessary for the validity of the Hardy inequality. However, in the case $p < n$, the condition (2) fails to be necessary; see the remark following Theorem 2 in [11].

The class of open sets whose complement is uniformly p-thick is quite large. Here are some examples.

If $p > n$, then each nonempty closed set is locally uniformly p-thick.

Any closed, arc-wise connected set (containing at least two points) is locally uniformly p-thick for $p > n-1$. In particular the complement of any proper simply connected subdomain of $\mathbb{R}^2$ is uniformly p-thick for any $p > 1$.

If a closed set $E \subset \mathbb{R}^n$ satisfies the condition $|B(x,r) \cap E| \geq C|B(x,r)|$ for all $x \in E$ and all $r > 0$, then E is uniformly p-thick for any $p > 1$.

We want to point out the importance of the p-thickness condition in the theory of boundary regularity of A-harmonic functions (i.e., solutions to the A-harmonic equation $\operatorname{div} A(x, \nabla u) = 0$; see [6]). Condition (2) is stronger than Wiener's criterion for the continuity up to the boundary of A-harmonic functions (see [6, 6.16], [9]). This condition leads not only to the continuity up to the boundary (like the Wiener criterion does), but to the Hölder continuity; see [6, 6.41].

We also want to point out that under condition (2), Kilpeläinen and Koskela [8] proved global higher integrability of the gradients of A-harmonic functions.

Uniform thickness condition appeared recently in many other papers in the area of mathematical analysis; see e.g. [7], [16].

Now we give a rough statement of the main result of the paper (Theorem 2, Theorem 1). We prove that given an open and proper subset $\Omega \subset \mathbb{R}^n$ with the uniformly p-thick complement, there exists q with $1 < q < p$ such that the following pointwise inequality holds:

$$|u(x)|\varrho(x)^{-1} \le C M_{2\varrho(x),q}|\nabla u|(x), \tag{3}$$

for $u \in C_0^\infty(\Omega)$, where

$$M_{R,q}g(x) = \sup_{r\le R}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|g(z)|^q\,dz\right)^{1/q}$$

is a maximal operator.

Now taking the L^p norm on both sides of this inequality and applying the Hardy–Littlewood maximal theorem, we deduce immediately the Hardy inequality (1) for $a = 0$ and then that for small $a > 0$. As we mentioned before, under the same assumptions, Hardy inequality (1) was obtained by Lewis and Wannebo. We will manage, however, to obtain a "local version near the boundary" of the Hardy inequality (8) that does not follow from the results of Lewis and Wannebo.

It is reasonable to call (3) pointwise Hardy inequality.

Notation. The Lebesgue measure of the set E will be denoted by $|E|$, and the average value of a function u over a set E by $u_E = \fint_E u(x)\,dx = |E|^{-1}\int_E u(x)\,dx$. We will also use the symbols $\varrho(x)$ and $M_{R,q}g$ in the sense defined above. If $R=\infty$, then we simply write $M_q g$; if $q = 1$, then we write $M_R g$. By Ω we will always denote an open subset of $\mathbb{R}^n$ and by $\Omega^c = \mathbb{R}^n\setminus\Omega$ its complement. Symbols B, $B(x,r)$, $B(r)$ etc. will be reserved to denote a ball. By $2B$ we will denote the ball concentric with B and with radius twice that of B. By χ_E we will denote the characteristic function of the set E. Finally, by C we denote a general positive constant; it can change its value even in a single line. Writing $C = C(n,p)$ we indicate that the constant C depends on n and p only.

2. Elementary case

The main result of this paper (Theorem 2) involves in the statement the notion of capacity, so, first, for the sake of simplicity, we will prove a particular case of this theorem (Proposition 1), which is much easier to formulate and to prove. It does not make use of the notion of capacity, neither in the statement nor in the proof.

Proposition 1. *Let Ω be an arbitrary open and proper subset of $\mathbb{R}^n$. For every $x\in\Omega$, choose $\overline{x}\in\partial\Omega$ satisfying $|x-\overline{x}| = \varrho(x)$. Then for any $q > n$, $u\in C_0^\infty(\Omega)$ and any $x \in\Omega$,*

$$|u(x)| \le C(n,q)\varrho(x)M_{2\varrho(x),q}\left(|\nabla u|\chi_{B(\overline{x},\varrho(x))}\right)(x). \tag{4}$$

If we assume, in addition, that there exist constants $b > 0$ and $r_0\in(0,\infty]$ such that for every $z \in\partial\Omega$ and $0 < r < r_0$,

$$|B(z,r)\cap\Omega^c| \ge b|B(z,r)|, \tag{5}$$

then we can prove a stronger inequality,

$$|u(x)| \le C(n,b)\varrho(x)M_{2\varrho(x)}\left(|\nabla u|\chi_{B(\overline{x},\varrho(x))}\right)(x) \tag{6}$$

for all $x\in\Omega$ with $\varrho(x) < r_0$.

In particular, inequalities (4) and (6) imply the inequality of the form (3).

The "pointwise Hardy inequalities" stated in the above proposition, together with the Hardy–Littlewood maximal theorem lead to the generalization of the integral Hardy inequality (1).

Theorem 1. *Let $r_0 \in (0,\infty]$, $1 \le q < \infty$ and let Ω be an open and proper subset of $\mathbb{R}^n$. For every $x \in \Omega$ choose $\overline{x} \in \partial\Omega$ satisfying $|x-\overline{x}| = \varrho(x)$. If the inequality*

$$|u(x)| \le C_1 \varrho(x) M_{2\varrho(x),q}\left(|\nabla u| \chi_{B(\overline{x},\varrho(x))}\right)(x) \tag{7}$$

holds for any $u \in C_0^\infty(\Omega)$ and any $x \in \Omega$ with $\varrho(x) < r_0$, then for every $p > q$ there exists $a_0 = a_0(C_1, n, p, q) > 0$ such that for $0 < t \le r_0$ and $0 \le a < a_0$,

$$\int_{\Omega_t} |u(x)|^p \varrho(x)^{-p+a}\, dx \le C(C_1, p, q, n, a_0) \int_{\Omega_t} |\nabla u(x)|^p \varrho(x)^a\, dx, \tag{8}$$

where $\Omega_t = \{x \in \Omega :\ \varrho(x) < t\}$. In particular if $r_0 = \infty$, the inequality (8) holds with Ω_t replaced by Ω.

I do not know if it is possible to obtain inequality (8) directly from the Hardy inequality (1).

Note that in particular, Proposition 1 and Theorem 1 imply that for $p > n$, inequality (8) holds in any domain with $0 < t \le \infty$. The case $\Omega_t = \Omega$ is due to Ancona [2].

Proof of Proposition 1. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open subset and $u \in C_0^\infty(\Omega)$. Fix $x \in \Omega$ and denote $B = B(\overline{x}, \varrho(x))$. Then for any $y \in B \cap \Omega^c$ we have (see [4, Lemma 7.16])

$$\begin{aligned} |u(x)| &= |u(x) - u(y)| \le |u(x) - u_B| + |u(y) - u_B| \\ &\le C(n)\left(\int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz + \int_B \frac{|\nabla u(z)|}{|y-z|^{n-1}}\, dz\right) = \diamondsuit. \end{aligned}$$

Assuming $q > n$ and applying Hölder's inequality we obtain

$$\diamondsuit \le C(n,q)\varrho(x)^{1-n/q}\left(\int_B |\nabla u(z)|^q\, dz\right)^{1/q},$$

and hence (4) follows. (In fact we repeated an argument from the proof of the Sobolev imbedding $W^{1,q}(B) \subset C^{0,1-n/q}(B)$.)

Now assume the condition (5). If we prove the inequality

$$\inf_{y \in \Omega^c \cap B} \int_B \frac{|\nabla u(z)|}{|y-z|^{n-1}}\, dz \le C(n,b) \int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz, \tag{9}$$

then an application of the following well known inequality (see [19, Lemma 2.8.3]),

$$\int_{B(r)} \frac{|g(z)|}{|x-z|^{n-1}}\, dz \le C(n) r M_{2r} g(x), \tag{10}$$

where $B(r)$ is any ball containing x, will complete the proof.

Thus it remains to prove (9). To this end it suffices to show that

$$⨍_{\Omega^c \cap B} \int_B \frac{|\nabla u(z)|}{|y-z|^{n-1}}\, dz\, dy \le C(n,b) \int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz.$$

This, however, follows immediately by changing the order of integration on the LHS and by employing condition (5) together with the elementary inequality (see [19, Lemma 3.11.3]),

$$⨍_{B(x,r)} |y-z|^{1-n}\, dy \le C(n)|x-z|^{1-n},$$

which holds for all $x, z \in \mathbb{R}^n$ and all $r > 0$. The proof is complete.

Proof of Theorem 1. Fix $u \in C_0^\infty(\Omega)$ and $0 < t \le r_0$. Set $g = |\nabla u|\chi_{\Omega_t}$. Then according to (7) the inequality

$$|u(x)| \le C_1 \varrho(x) M_q g(x)$$

holds for all $x \in \Omega_t$. Since $q < p$, the Hardy–Littlewood maximal theorem implies

$$\|M_q g\|_{L^p(\mathbb{R}^n)} \le C(n, p/q)\|g\|_{L^p(\mathbb{R}^n)},$$

and hence the inequality (8) follows with $a = 0$.

Now we employ one trick previously used in [17, p. 93] and [5, Lemma 6] to deduce the inequality (8) with a small, positive a from the case $a = 0$.

Fix $\varepsilon > 0$ and set $v = |u|\varrho^\varepsilon$. Using the fact that the Lipschitz constant of ϱ is equal to 1 we obtain

$$|\nabla v| \le |\nabla u|\varrho^\varepsilon + \varepsilon \varrho^{\varepsilon-1}|u|.$$

Applying the inequality (8) with $a = 0$ to v we obtain

$$\int_{\Omega_t} |u(x)|^p \varrho(x)^{-p+p\varepsilon}\,dx \le C\left(\int_{\Omega_t} |\nabla u|^p \varrho^{p\varepsilon} + \varepsilon^p \int_{\Omega_t} |u|^p \varrho^{p(\varepsilon-1)}\right).$$

If $C\varepsilon^p < 1$, then

$$\int_{\Omega_t} |u|^p \varrho^{-p+p\varepsilon}\,dx \le \frac{C}{1 - C\varepsilon^p}\int_{\Omega_t} |\nabla u|^p \varrho^{p\varepsilon}\,dx.$$

This completes the proof.

3. General case

The main result of the paper reads as follows

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be an open and proper subset. For every $x \in \Omega$ choose $\overline{x} \in \partial\Omega$ satisfying $|x - \overline{x}| = \varrho(x)$. If the complement Ω^c is locally uniformly p-thick, $r_0 \in (0, \infty]$, $b > 0$ are as in (2) and $1 < p < \infty$, then there exists $1 < q < p$ such that the inequality*

$$|u(x)| \le C(n, p, q, b)\varrho(x) M_{2\varrho(x),q}\left(|\nabla u|\chi_{B(\overline{x},\varrho(x))}\right)(x)$$

holds for all $u \in C_0^\infty(\Omega)$ and all $x \in \Omega$ with $\varrho(x) < r_0$.

This theorem combined with Theorem 1, leads to the Hardy inequality (8), that in the case $\Omega_t = \Omega$ has previously been obtained by Lewis [11] and Wannebo [17]. Note, however, that if we assume local thickness only, then, in general, we cannot prove the inequality (8) with $\Omega_t = \Omega$.

Proof of Theorem 2. We need the following result due to Lewis [11, Theorem 1] (cf. [8, Proposition 2.3]). Another proof of Lewis' result was recently obtained by Mikkonen [13, Theorem 8.2].

Proposition 2. *Let $1 < p \le n$. If a closed set $E \subset \mathbb{R}^n$ is locally uniformly p-thick ($r_0 \in (0, \infty]$, b are as in (2)), then there exists $1 < q < p$ (depending on n, p and b only) such that E is locally uniformly q-thick (with the same r_0).*

It is well known that a Sobolev function vanishing on a subset of a positive capacity has almost the same properties as a function which vanishes on a set of positive Lebesgue measure (cf. [1, Theorem 8.2.1], [19, 4.5], [8, Lemma 3.1]). The following proposition is a special case of the theorem of Hedberg [1, Theorem 8.2.1]. For the sake of completness we give a proof.

Proposition 3. *Let $K \subset \overline{B}$ be a compact set with $C_{1,q}(K, 2B) > 0$. Then for every $u \in C^\infty(\overline{B})$ with $u|_K = 0$ and every $x \in \overline{B}$,*

$$|u(x)| \leq C(n) \int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz + C(n,q) \left(\frac{1}{C_{1,q}(K,2B)} \int_B |\nabla u(z)|^q \, dz \right)^{1/q}.$$

Proof. We can assume that $\|\nabla u\|_{L^q(B)} < \infty$, otherwise the inequality is trivial.

If $u_B = 0$, then (see [4, Lemma 7.16])

$$|u(x)| = |u(x) - u_B| \leq C(n) \int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz,$$

and the proposition follows.

Assume now that $u_B \neq 0$. By the homogeneity we can assume $u_B = 1$. Reflecting the function $u_B - u$ across the boundary and then multiplying by a cutoff function, we obtain $v \in W^{1,q}_0(2B)$ such that $v = 1$ on K and

$$\int_{2B} |\nabla v|^q \, dx \leq C(n,q) \int_B |\nabla u|^q \, dx. \tag{11}$$

Since v is a "test function" for the capacity, we readily get the inequality

$$1 \leq \left(\frac{1}{C_{1,q}(K,2B)} \int_{2B} |\nabla v|^q \, dx \right)^{1/q}.$$

Now for any $x \in B$ we have

$$\begin{aligned} |u(x)| &\leq |u(x) - u_B| + |u_B| = |u(x) - u_B| + 1 \\ &\leq C(n) \int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}} + \left(\frac{1}{C_{1,q}(K,2B)} \int_{2B} |\nabla v|^q \, dx \right)^{1/q}. \end{aligned}$$

The proposition follows directly by an application of the inequality (11).

Now we can finish the proof of the theorem. Fix $x \in \Omega$ with $\varrho(x) < r_0$, set $\overline{B} = \overline{B}(\overline{x}, \varrho(x))$ and $K = \Omega^c \cap \overline{B}$. By the assumption and Proposition 2, there exists $1 < q < p$ such that

$$C_{1,q}(K, 2B) \geq b\, C \varrho(x)^{n-q},$$

and hence a direct application of Proposition 3, inequality (10) and Hölder's inequality completes the proof.

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References

1. Adams, D. A., Hedberg, L. I.: *Function spaces and potential theory*, A Series of Comprehensive Studies in Mathematics vol. 314, Springer-Verlag, 1996. MR **97j:**46024
2. Ancona, A.: Une propriété des espaces se Sobolev. *C. R. Acad. Sci. Paris* 292 (1981), 477–480. MR **82d:**46051
3. Ancona, A.: On strong barriers and inequality of Hardy for domains in $\mathbb{R}^n$. *J. London Math. Soc.* 34 (1986), 274–290. MR **87k:**31004
4. Gilbarg, D., Trudinger, N.: *Elliptic Partial Differential Equations of Second Order*, Springer–Verlag 1983. MR **86c:**35035
5. Hajłasz, P., Koskela, P.: Isoperimetric inequalities and imbedding theorems in irregular domains, *J. London Math. Soc.* (to appear).

6. Heinonen, J., Kilpeläinen, T., Martio, O.: *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Univ. Press, 1993. MR **94e:**31003
7. Jones, P. W., Wolff, T. H.: Hausdorff dimension of harmonic measure in the plane. *Acta Math.* 161 (1988), 131–144. MR **90j:**31001
8. Kilpeläinen, T., Koskela, P.: Global integrability of the gradient of solutions to partial differential equations, *Nonl. Anal. T. M. A.* 23 (1994), 899–909. MR **95i:**35117
9. Kilpeläinen, T., Malý, J.: The Wiener test and potential estimates for quasilinear elliptic equations. *Acta Math.* 172 (1994), 137–161. MR **95a:**35050
10. Kufner, A.: *Weighted Sobolev spaces.* John Wiley & Sons, 1985. MR **86m:**46033
11. Lewis, J.L.: Uniformly fat sets, *Trans. Amer. Math. Soc.* 308 (1988), 177–196. MR **89e:**31012
12. Maz'ya, V. G.: *Sobolev spaces.* Springer-Verlag, 1985. MR **87g:**46056
13. Mikkonen, P.: On the Wolff potential and quasilinear elliptic equations involving measures, *Ann. Acad. Sci. Fenn. Ser. A I. Math. Dissertationes* 104 (1996). MR **97e:**35069
14. Nečas, J.: Sur une méthode pour résourde les équations aux dérivées partielle du type elliptique, voisine de la variationelle, *Ann. Scuola Norm. Sup. Pisa* 16 (1962), 305–326. MR **29:**357
15. Opic, B., Kufner, A.: *Hardy-type inequalities*, Research Notes in Mathematics 219, Pitman 1990. MR **92b:**26028
16. Pommerenke, Ch.: On uniformly perfect sets and Fuchsian groups. *Analysis* 4 (1984), 299–321. MR **86e:**30044
17. Wannebo, A.: Hardy inequalities. *Proc. Amer. Math. Soc.* 109 (1990), 85–95. MR **90h:**26025
18. Wannebo, A.: Hardy inequalities and imbedding in domains generalizing $C^{0,\lambda}$ domains, *Proc. Amer. Math. Soc.* 122 (1994), 1181–1190. MR **95b:**46053
19. Ziemer, W.: *Weakly differentiable functions.* Graduate Texts in Mathematics 120, Springer-Verlag, 1989. MR **91e:**46046

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