manuscripta math. 91, 61 - 72 (1996)

manuscripta mathematica


# On approximate differentiability of functions with bounded deformation

Piotr Hajłasz[1]

Institute on Mathematics, Warsaw University, ul. Banacha 2, 02-097 Warszawa, Poland
E-mail: hajlasz@mimuw.edu.pl

Received February 27, 1996;
in revised form June 11, 1996

We prove that functions with bounded deformation $u : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, i.e., such mappings that the symmetric part of the gradient $\frac{1}{2}(\nabla u + (\nabla u)^T)$ is a measure, are approximately differentiable a.e. Then we generalize the result to a more general class of functions.

1991 *Mathematics Subject Classification* 46E35. Secondary 73E99, 42B20.

## 1. Introduction

In the paper we deal with the space $BD(\Omega)$ of functions with bounded deformation. Let us recall the definition.

To a vector function $u = (u_1, \ldots, u_n) : \Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is an open set, we associate the deformation tensor $\varepsilon$ defined as a symmetric part of the gradient of $u$, i.e., $\varepsilon = \frac{1}{2}(\nabla u + (\nabla u)^T)$, or in terms of components,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

$BD(\Omega)$ is the space of all vector functions $u \in L^1(\Omega)^n$ such that $\varepsilon_{ij}$ (defined in the distributional sense) are measures with finite total variation. By a measure

[1]This research was carried out while the author stayed in the ICTP in Trieste in 1995. He wishes to thank the ICTP for the hospitality. The author was partially supported by KBN grant no. 2-PO3A-034-08.

we will always mean a signed Borel measure $\mu$ on an open set $\Omega \subset \mathbb{R}^n$ i.e., $\mu = \mu_+ - \mu_-$, where $\mu_+$ and $\mu_-$ are positive Borel measures on $\Omega$, supported on disjoint sets. As usual, $|\mu| = \mu_+ + \mu_-$ denotes the measure of variation and $\|\mu\| = |\mu|(\Omega)$ the total variation of $\mu$. We also use notation $X(\Omega)^M$ for the space of vector functions $u = (u_1, \ldots, u_M)$, $u_i \in X(\Omega)$, where $X(\Omega)$ is a given function space.

For the basic properties of $BD(\Omega)$, see [24]. Note that if $u \in BD(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, then $u\varphi \in BD(\mathbb{R}^n)$.

The class of functions with bounded deformation $BD(\Omega)$ was introduced by Matthies, Strang and Christiansen [16] in connection with the variational problems of perfect plasticity, and investigated by Temam and Strang, [24].

For the recent development of the theory of $BD(\Omega)$ functions and its applications to the calculus of variations see, e.g. [24], [25], [15], [3], [1], [4], [2].

It is well known that the class $BD(\Omega)$ is larger than the class of vector functions with bounded variation $BV(\Omega)$, see [16], [15]. This fact follows from the result of Ornstein, [20].

Kohn, [15], was first to prove that many fine properties (related to geometric measure theory) of $BD(\Omega)$ functions are similar to those of $BV(\Omega)$ functions.

Since the space $BD(\Omega)$ is strictly larger than $BV(\Omega)^n$, there are functions $u \in BD(\Omega)$ such for certain $i$, $j$, the distributional derivative $\partial u_i/\partial x_j$ is not a measure. However, it was conjectured few years ago that functions with bounded deformation are approximately differentiable almost everywhere (see Section 3 for the definition of approximate differentiability). The only known result in this direction was the one due to Bellettini, Coscia and Dal Maso, [4, Theorem 8.2] stating that the function with the bounded deformation has approximate symmetric differential a.e. This result was, however, easy, since by the definition we know "a lot" about symmetric part of the gradient $\frac{1}{2}(\nabla u + (\nabla u)^T)$. The problem is to investigate the properties of remaining, skew-symmetric part $\frac{1}{2}(\nabla u - (\nabla u)^T)$.

In the paper we give the affirmative answer to the above conjecture. While the paper was in preparation, Ambrosio, Coscia and Dal Maso, [2], obtained another proof of this conjecture. In fact, we prove a more general result. Namely, instead of assuming that $\nabla u + (\nabla u)^T$ is a measure, we assume that $\{P_i u\}_{i=1}^m$ are measures, where $P_i$ are certain partial differential operators with constant coefficients, see Theorem 5 and Corollary 1.

**Notation.** Symbol $W^{m,p}(\Omega)$ will denote the usual Sobolev space of functions whose distributional derivatives of order less then or equal to $m$, belong to $L^p(\Omega)$.

If $F \subset \Omega$ is a Borel set then the measure $\mu\lfloor F$ is defined by $(\mu\lfloor F)(A) = \mu(A \cap F)$. The Lebesgue measure of $A$ will be simply denoted by $|A|$.

If $u_k, u \in L^1_{loc}(\mathbb{R}^n)$ then we say that $u_k$ converges to $u$ in the sense of distributions if for every $\psi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} u_k(x)\psi(x)\,dx \to \int_{\mathbb{R}^n} u(x)\psi(x)\,dx.$$

We denote such a convergence by writing $u_k \to u$ in $\mathcal{D}'$.

By mollifier we mean a function $\varphi_\varepsilon$ defined as $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, where $\varphi \in C_0^\infty(\mathbb{R}^n)$, is a fixed function with $\varphi \geq 0$ and $\int_{\mathbb{R}^n}\varphi(x)\,dx = 1$. The symbol $\varphi_\varepsilon$ will always stand for a mollifier. By $\langle A, B\rangle$ we will denote the scalar product of vectors in $\mathbb{R}^n$. In the paper $C$ will denote a general constant which may change even in a single string of estimates. We will write $u \approx v$ to express that there are two positive constants $C_1$ and $C_2$ such that $C_1 u \leq v \leq C_2 u$ .

**Acknowledgements.** The author wishes to thank Giovanni Alberti for bringing the problem considered here to his attention.

## 2. Integral representation

In this section we recall Smith integral representation formula, [22], for $C_0^\infty(\mathbb{R}^n)$ functions and we show that this formula holds also for $BD(\mathbb{R}^n)$ functions with compact support. First, we start with an elementary case of Smith's formula, which is, however, sufficient for the applications to $BD$ functions.

Let $u = (u_1, u_2, \ldots, u_n) \in C_0^\infty(\mathbb{R}^n)^n$. For $k = 1, 2, \ldots, n$, we have the well known integral formula, see [17, Theorem 1.1.10/2],

$$u_k = \frac{2}{n\omega_n}\sum_{1\leq i\leq j\leq n}\frac{\partial^2 u_k}{\partial x_i \partial x_j} * K_{ij}, \tag{1}$$

where $K_{ij}(x) = x_i x_j/|x|^n$ and $\omega_n$ denotes volume of the unit ball. Note that

$$\frac{\partial \varepsilon_{jk}}{\partial x_i} - \frac{\partial \varepsilon_{ij}}{\partial x_k} + \frac{\partial \varepsilon_{ki}}{\partial x_j} = \frac{\partial^2 u_k}{\partial x_i \partial x_j}.$$

Placing this identity in (1) and integrating by parts we obtain

$$u_k = \frac{2}{n\omega_n}\sum_{1\leq i\leq j\leq n}\left(\varepsilon_{jk} * \frac{\partial K_{ij}}{\partial x_i} - \varepsilon_{ij} * \frac{\partial K_{ij}}{\partial x_k} + \varepsilon_{ki} * \frac{\partial K_{ij}}{\partial x_j}\right). \tag{2}$$

Thus we obtained an explicit integral formula to represent $u$ in terms of $\{\varepsilon_{ij}\}$. Note that the assumption about the compactness of the supprot of $u$ was essential, since (2) does not hold for a general $u \in C^\infty$ as $\varepsilon_{ij}$ vanishes on a certain class of polynomials of order 1. Now we show that (2) holds also for $BD(\mathbb{R}^n)$ functions with compact support.

THEOREM 1 *If $u \in BD(\mathbb{R}^n)$ has compact support, then formula (2) holds a.e.*

*Proof.* Since $|\partial K_{ij}/\partial x_k| \leq C|x|^{1-n}$, the theorem follows immediately from Lemma 1 and Lemma 2 below.

LEMMA 1 *Let $K \in L^1_{\rm loc}(\mathbb{R}^n)$. If $\mu$ is a signed Borel measure on $\mathbb{R}^n$ with compact support and finite total variation, then $K * \mu \in L^1_{\rm loc}(\mathbb{R}^n)$.*

LEMMA 2 *Assume that $|K(z)| \leq C|z|^{a-n}$ with certain constants $C > 0$ and $a > 0$. Let $\mu$ be a signed Borel measure on $\mathbb{R}^n$ with compact support and finite total variation. If $\mu_\varepsilon = \mu * \varphi_\varepsilon$, then $K * \mu_\varepsilon \to K * \mu$ in $\mathcal{D}'$.*

Lemma 1 follows from Fubini's theorem. According to Lemma 1, both $K * \mu_\varepsilon$ and $K * \mu$ are locally integrable functions, so we can ask about the convergence in the sense of distributions. The proof of Lemma 2 follows from the Fubini theorem and from the Dominated Convergence Theorem. The growth condition for the kernel $K$ in Lemma 2 leads to the uniform estimate, independent of $0 < \varepsilon \leq 1$,

$$(|K| * \varphi_\varepsilon)(z) \leq C(|z|^{a-n} + 1),$$

see [6, Lemma 2], which allows us to apply Dominated Convergence Theorem. We leave the details to the reader. The proof of Theorem 1 is complete.

Formula (2) is strictly related to the so called Korn's inequality, see [11], [22], [19], [5], [21, Theorem 12.20], [14] and references therein.

Now we state a more general version of Smith's representation formula. For the simplicity sake we do not pursue to state the result in its most general form.

Let $P_j = (P_{j1}, \ldots, P_{jM})$, $j = 1, \ldots, N$ be linear homogeneous partial differential operators of order $m \geq 1$, with constant coefficients, acting on vector functions

$$u = (u_1, \ldots, u_M) \quad \text{and} \quad P_j u = \sum_{k=1}^{M} P_{jk} u_k.$$

Homogeneity of order $m$ means $P_{jk} = \sum_{|\alpha|=m} a_\alpha D^\alpha$. By $p_{jk}(\xi)$ we will denote the characteristic polynomial of $P_{jk}$. The following result is due to Smith, [22].

THEOREM 2 *If for every $\xi \in \mathbb{C}^n \setminus \{0\}$, the matrix $\{p_{jk}(\xi)\}$ has rank $M$, then there exist $K_{ij} \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $K_{ij}(x) = |x|^{m-n} K_{ij}(x/|x|)$ when $x \neq 0$, such that for $u = (u_1, \ldots, u_M) \in C_0^\infty(\mathbb{R}^n)^M$, we have*

$$u_i = \sum_{j=1}^{N} K_{ij} * P_j u \tag{3}$$

Formula (2) is a particular case of formula (3). Indeed

$$\varepsilon_{ij} = \sum_{k=1}^{n} P_{(ij),k} u_k,$$

where

$$P_{(ij),k} = \frac{1}{2}\left(\delta_{ki}\frac{\partial}{\partial x_j} + \delta_{kj}\frac{\partial}{\partial x_i}\right).$$

Thus $M = n$, $N = n^2$ and $m = 1$. Here $\varepsilon_{ij}$ plays a role of $P_j$. It is also easy to check, that the rank of suitable matrix equals $n$.

Now the counterpart of Theorem 1 reads as follows.

THEOREM 3 *Assume that $P_j$, $K_{ij}$, $M$, $N$ and $m$ are as in Theorem 2. If $u \in L^1(\mathbb{R}^n)^M$ has compact support and $P_j u$, $j = 1, \ldots, N$ are measures with bounded total variation, then formula (3) holds a.e.*

Proof is that same as that for Theorem 1.

In many applications it is important to have integral representations in domains, rather than those for compactly supported functions. For the extension of Smith's theorem to domains, see the paper of Kalamajska [13]. It is also possible to extend Theorem 3 to domains, but we will not go into details.

## 3. Approximate differentiability

Let $u$ be a real valued function defined on a measurable subset $E \subset \mathbb{R}^n$. We say that $L = (L_1, \ldots, L_n)$ is an *approximate total differential* (in short *a.t.d.* ) of $u$ at $x_0$ if for every $\varepsilon > 0$ the set

$$A_\varepsilon = \left\{ x \in E \setminus \{x_0\} \ : \ \frac{|u(x) - u(x_0) - L(x - x_0)|}{|x - x_0|} < \varepsilon \right\}$$

has $x_0$ as a density point. If this is the case then $x_0$ is a density point of $E$ and $L$ is uniquely determined.

We recall that $x \in \mathbb{R}^n$ is a density point of a measurable set $A \subset \mathbb{R}^n$ if $\lim_{r \to 0} |A \cap B(x,r)|/|B(x,r)| = 1$.

When we say that $u$ is differentiable in a point $x_0$ we will mean the classical definition.

If a function $u : E \to \mathbb{R}$ has the following "Lusin type" property: for every $\varepsilon > 0$ there exists a locally Lipschitz function $h : \mathbb{R}^n \to \mathbb{R}$ such that $|\{x \in E : u(x) \neq h(x)\}| < \varepsilon$, then $u$ has a.t.d. almost everywhere in $E$. This is an elementary consequence of the a.e. differentiability of Lipschitz functions ($u$ has a.t.d. in $x$ if $x$ is a density point of the set $\{u = h\}$ and $h$ is differentiable at $x$).

Since every Lipschitz function can be extended from any subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ as a Lipschitz function, [10, 2.10.4], the above remark leads to the following

LEMMA 3 *Let $E \subset \mathbb{R}^n$ be a measurable subset and $u, I : E \to \mathbb{R}$ measurable functions. If $|u(x) - u(y)| \leq |x - y|(I(x) + I(y))$ a.e. in $E$, than $u$ has a.t.d. almost everywhere in $E$.*

*Remark.* The inequality in the lemma holds a.e. in the following sense: there is a set $F \subset E$, $|F| = 0$ such that the inequality holds for all $x, y \in E \setminus F$.

The above mentioned Lusin type property is not only sufficient but also necessary for a.e. existence of a.t.d. The necessity is a difficult part. This is due to Whitney, [26].

THEOREM 4 *Let $E \subset \mathbb{R}^n$ be a measurable set and $u : E \to \mathbb{R}$ a measurable function. Then the following two conditions are equivalent.*

1. *$u$ is approximately totally differentiable a.e. in $E$.*

2. *For each $\varepsilon > 0$ there exists a locally Lipschitz function $h : \mathbb{R}^n \to \mathbb{R}$ such that $|\{x \in E : \ u(x) \neq h(x)\}| < \varepsilon$.*

We will not use this theorem in the sequel. For more striking result, see the original paper of Whitney [26].

Now we can formulate the main result.

THEOREM 5 *Assume that $P_j$ are operators of order $m \geq 1$, as in Theorem 2 and $\Omega \subset \mathbb{R}^n$ is an open set. If $u \in W^{m-1,1}_{\text{loc}}(\Omega)^M$ has the property that distributional derivatives $P_j u$, $j = 1, \ldots, N$ are measures with bounded total variation, then all the functions $D^\alpha u$ for $|\alpha| = m - 1$ have a.t.d. almost everywhere in $\Omega$.*

*Remark.* The assumption $u \in W^{m-1,1}_{\text{loc}}(\Omega)^M$ is superfluous. Indeed, if we assume only that $u \in \mathcal{D}'(\mathbb{R}^n)^M$, then representation formula (3) which holds for $C_0^\infty$ allows us to apply a version of Deny and Lions' argument (cf. [13, Corollary 2], [24, Theorem 2.1], [9], [17, Theorem 1.1.2]). This implies that $u$ which is a priori a distribution, already belongs to $W^{m-1,1}_{\text{loc}}(\Omega)$. We skip details because it is standard and we will not use it in the sequel.

COROLLARY 1 *$u \in BD(\Omega)$ has a.t.d. almost everywhere.*

Since $P_j(\varphi u)$ are measures for $\varphi \in C_0^\infty(\Omega)$, Theorem 5 follows immediately from Theorem 3, Lemma 7 and Theorem 6. Note that the fact $u \in W^{m-1,1}_{\text{loc}}(\Omega)^M$ is employed in the proof that $P_j(\varphi u)$ are measures.

## 4. Calderón, Marcinkiewicz and Zygmund

In this section we recall some definitions and results related to Calderón and Zygmund's theory of singular integrals.

Let $\mu$ be a signed Borel measure on $\mathbb{R}^n$ with finite total variation. We define the maximal function of $\mu$ as

$$M\mu(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{|B(x,r)|}.$$

The following lemma is well known.

LEMMA 4 *If the measure $\mu$ is as above then for every $t > 0$*

$$|\{x \in \mathbb{R}^n : M\mu(x) > t\}| \leq Ct^{-1}\|\mu\|.$$

*The constant $C$ depends on $n$ only.*

The proof when $\mu$ is an absolutely continuous measure is given in [23, p. 5], however, the same argument works in the general case (cf. [23, p. 77]).

LEMMA 5 *If $\mu$ is as above then for every $t > 0$, the measure $\mu\lfloor\{M\mu \le t\}$ is absolutely continuous.*

*Proof.* We need to prove that for $E \subset \{M\mu \le t\}$ with $|E| = 0$ there is $|\mu|(E) = 0$. For $\varepsilon > 0$ let $E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$, where $x_i \in E$ and $\sum_{i=1}^{\infty} |B(x_i, r_i)| < \varepsilon$. Then

$$|\mu|(E) \le \sum_{i=1}^{\infty} |\mu|(B(x_i, r_i)) \le \sum_{i=1}^{\infty} t|B(x_i, r_i)| < t\varepsilon.$$

The lemma follows by letting $\varepsilon \to 0$.

For every $t > 0$ we define a Calderón–Zygmund decomposition of $\mu$ as follows.

Let $E_t = \{M\mu \le t\}$. The set $\Omega_t = \mathbb{R}^n \setminus E_t$ is open. Let $\Omega_t = \bigcup_{i=1}^{\infty} Q_i$ be a decomposition into Whitney cubes (i.e., $Q_i$ are closed cubes with pairwise disjoint interiors and such that $\operatorname{diam} Q_i$ is comparable to $\operatorname{dist}(Q_i, E_t)$, see [23, p. 16] for more details). By Calderón–Zygmund decomposition of $\mu$ we mean $\mu = g + \mu^b$, where

$$g = \mu\lfloor E_t + \sum_{i=1}^{\infty} \left(\frac{\mu(Q_i)}{|Q_i|}\right) \chi_{Q_i}; \qquad \mu^b = \sum_{i=1}^{\infty} \mu_i^b$$

and $\mu_i^b = (\mu - \mu(Q_i)/|Q_i|)\lfloor Q_i$, i.e., $\mu_i^b(A) = \mu(A \cap Q_i) - |A \cap Q_i|\mu(Q_i)/|Q_i|$. By lemma 5, $\mu\lfloor E_t$ and hence $g$ are identified with integrable functions. The Calderón–Zygmund decomposition depends on $t$, but for the simplicity of notation we do not put $t$ as a subscript. The letters "g" and "b" correspond to "good" and "bad" part of $\mu$. It is well known that

1. $|\Omega_t| \le Ct^{-1}\|\mu\|$,

2. $|\mu|(Q_i)/|Q_i| \le Ct$.

The constants $C$ depend on $n$ only. Inequality 1. is a reformulation of Lemma 4. Inequality 2. follows from the fact that $\operatorname{diam} Q_i$ is comparable to $\operatorname{dist}(Q_i, E_t)$ and from the definition of $E_t$, see [23, p. 19] for details. It follows from "differentiation theorem", [27, 1.3.9], that $|\mu\lfloor E_t| \le t$, ($\mu\lfloor E_t$ is a function) and hence $|g| \le Ct$. Thus $g \in L^1 \cap L^\infty$.

If $F \subset \mathbb{R}^n$ is a closed set then we define *Marcinkiewicz's integral associated to $F$* as

$$I_*(x) = \int_{\mathbb{R}^n} \frac{\delta(z)}{|x - z|^{n+1}}\, dz,$$

where $\delta(z) = \operatorname{dist}(z, F)$. Obviously $I_*(x) = \infty$ for $x \in \mathbb{R}^n \setminus F$. The following result is well known, see [23, pp. 14–15].

LEMMA 6 *Let $F \subset \mathbb{R}^n$ be a closed set such that $|\mathbb{R}^n \setminus F| < \infty$. Then $I_*(x) < \infty$ for almost every $x \in F$. Moreover*

$$\int_F I_*(x)\, dx \le C|\mathbb{R}^n \setminus F|.$$

*The constant $C$ depends on $n$ only.*

This lemma follows easily from Fubini's theorem.

## 5. Differentiability properties of convolution

LEMMA 7 *Let $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $K(x) = |x|^{m-n}K(x/|x|)$, $m \ge 1$ and let $\mu$ be a signed Borel measure on $\mathbb{R}^n$ with compact support and finite total variation. Then $K * \mu \in W^{m-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and*

$$D^\alpha(K * \mu) = (D^\alpha K) * \mu,$$

*for $|\alpha| \le m - 1$.*

This is a classical formula for differentiation of distributions, combined with Lemma 1. If $m = 1$, then Lemma 7 states only that $K * \mu \in L^1_{\mathrm{loc}}$.

It is natural to ask what we can say about derivatives $D^\alpha(K * \mu)$ when $|\alpha| = m$. Assume for a moment that instead of $\mu$ we have a function $g \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ (with compact support as well). The case of general measure $\mu$ will be treated later (Theorem 6).

If we try to compute the derivative in question, formally, using the formula "$D^\alpha(K*g) = (D^\alpha K)*g$", then we arrive into troubles: kernel $D^\alpha K$, $|\alpha| = m$ has a nonintegrable singularity (of order $n$), so the formula makes no sense. Mikhlin proved, however, that if we interpret the convolution with $D^\alpha K$ as a singular integral, then $D^\alpha(K * g) = (D^\alpha K) * g + cg$ with surprising appearance of the term $cg$, where $c$ is a constant (depending on $K$). Roughly speaking, the reason why $cg$ appears is the following: in the definition of singular integral we cut the kernel $D^\alpha K$ near origin, this causes the appearance of $\delta$ distribution and hence that of the term $cg$.

Now the direct application of celebrated theorem of Calderón and Zygmund on boundedness of singular integrals in $L^p$ readily establishes the following result of Mikhlin, [18, Theorem 1.29].

LEMMA 8 *If $K$ is as in Lemma 7 and $g \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ has compact support, then $K * g \in W^{m,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and $D^\alpha(K * g) \in L^p(\mathbb{R}^n)$ for $|\alpha| = m$.*

If we replace, however, $g$ in Lemma 8 by $g \in L^1$ or by a measure $\mu$, then the distributional derivative $D^\alpha(K * \mu)$, $|\alpha| = m$ does not need to be a measure (otherwise Theorem 1 would imply $BD = BV$). We can only prove the existence of derivatives in the approximate sense. More precisely for every $|\alpha| = m-1$, the function $D^\alpha(K*\mu)$ has a.t.d. almost everywhere. This is the main technical tool in the proof of Theorem 5. Since, according to Lemma 7, $D^\alpha(K*\mu) = (D^\alpha K)*\mu$, $D^\alpha K(x) = |x|^{1-n} D^\alpha K(x/|x|)$, when $|\alpha| = m-1$, the problem reduces to the case $m = 1$.

THEOREM 6 *Let $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $K(x) = |x|^{1-n}K(x/|x|)$ when $x \neq 0$ and let $\mu$ be a signed Borel measure on $\mathbb{R}^n$ with finite total variation. Then the function $K*\mu$ has approximate total differential almost everywhere.*

*Remark.* We do not assume that the support of $\mu$ is compact, however, for our applications it would suffice to assume it.

This theorem is essentially due to Calderón and Zygmund, [8, Remark, p. 129]. Namely Calderón and Zygmund sketched the proof in the particular case $K(x) = |x|^{1-n}$. The general case goes along the same line. There are, however, two reasons for which we include all the details here. The first reason is that the paper of Calderón and Zygmund contains a very short sketch only; the second reason is that we realized that this result can be used to solve some questions which arose in fields of calculus of variations where singular integrals did not appear so far.

*Proof of Theorem 6.* Given $t > 0$, let $\mu = g + \mu^b$ be a Calderón–Zygmund decomposition of $\mu$. We will use the notation from the Section 4. Note that $|\mathbb{R}^n \setminus E_t| \to 0$ as $t \to \infty$, so it suffices to prove that $K*\mu$ has a.t.d. almost everywhere in $E_t$, for every $t > 0$. We have $K*\mu = K*g + K*\mu^b$. Since $g \in L^1 \cap L^\infty$, then $g \in L^p$ for every $1 < p < \infty$, and hence $K*g$ is differentiable a.e. in $\mathbb{R}^n$ as every $W^{1,p}_{\mathrm{loc}}$ function for $p > n$ is differentiable a.e., [7]. Thus it remains to prove that the function $K*\mu^b$ has a.t.d. almost everywhere in $E_t$. Let $I_*$ be the integral of Marcinkiewicz associated to $E_t$. Since $I_* < \infty$ a.e. in $E_t$ (Lemma 6), the desired property of $K*\mu^b$ follows immediately from Lemma 3 and the lemma below.

LEMMA 9 *The inequality $|K*\mu^b(y) - K*\mu^b(x)| \leq C|y-x|(I_*(x) + I_*(y))$ holds for almost all $x, y \in E_t$, with $C$ depending on $n$ only.*

*Remarks.* 1) See the remark following Lemma 3. 2) Lemma 9 can be interpreted in terms of generalized Sobolev spaces introduced by the author in [12]. Namely the inequality of Lemma 9 implies $K*\mu^b \in W^{1,1}(E_t, |\cdot|, H^n)$. We will not use this interpretation in the sequel.

*Proof of Lemma 9.* Note that

$$|K*\mu^b(y) - K*\mu^b(x)| \leq \sum_{i=1}^{\infty} |K*\mu_i^b(y) - K*\mu_i^b(x)|$$

a.e., so it remains to prove that

$$\frac{|K*\mu_i^b(y) - K*\mu_i^b(x)|}{|y-x|} \leq C\left(\int_{Q_i} \frac{\delta(z)}{|x-z|^{n+1}}\,dz + \int_{Q_i} \frac{\delta(z)}{|y-z|^{n+1}}\,dz\right)$$

for $i = 1, 2, \ldots$ Fix $i \in \mathbb{N}$.

By $z^i$ we will denote the center of the cube $Q_i$. In what follows $x$ and $y$ will always belong to $E_t$. The heart of the matter is to estimate the expression $\heartsuit = K * \mu_i^b(y) - K * \mu_i^b(x)$ which I dedicate to my sweetheart Joanna. We need to consider three cases.

**Case 1:** $|x - y| < \operatorname{diam} Q_i$; For $z \in Q_i$, there is a variable point $w(z) \in \overline{xy}$ such that

$$\begin{aligned}\heartsuit &= \int_{Q_i} (K(y-z) - K(x-z))\, d\mu_i^b(z) \\ &= \int_{Q_i} \langle \nabla K(w(z) - z), y - x\rangle\, d\mu_i^b(z) \\ &= \int_{Q_i} \langle \nabla K(w(z) - z) - \nabla K(x - z^i), y - x\rangle\, d\mu_i^b(z)\end{aligned}$$

In the last step we employed the fact $\int_{Q_i} d\mu_i^b = 0$. Now using the property $|\mu_i^b|(Q_i) \leq Ct|Q_i|$ we get

$$|\heartsuit| \leq C|y - x||Q_i| \sup_{z\in Q_i} |\nabla K(w(z) - z) - \nabla K(x - z^i)|.$$

(C depends on $t$.) For a certain point $v(z)$ belonging to the segment joining $x$ with $w(z) + z^i - z$

$$\begin{aligned}|\nabla K(w(z) - z) - \nabla K(x - z^i)| &\leq |\nabla^2 K(v(z) - z^i)||w(z) + z^i - z - x| \\ &\leq C|x - z^i|^{-(n+1)} \operatorname{diam} Q_i.\end{aligned}$$

Thus

$$|\heartsuit| \leq C|x - y| \frac{\operatorname{diam} Q_i}{|x - z^i|^{n+1}} |Q_i| \leq C|x - y| \int_{Q_i} \frac{\delta(z)}{|x - z|^{n+1}}\, dz.$$

The last inequality follows from the observation that $\delta(z) \approx \operatorname{diam} Q_i$ for $z \in Q_i$ and $|x - z^i| \approx |x - z|$ for $z \in Q_i$.

**Case 2:** $|x - y| \geq 10^{-1} \operatorname{dist}(\{x, y\}, Q_i)$;

$$|\heartsuit| \leq \int_{Q_i} |K(y - z) - K(y - z^i)|\, d|\mu_i^b|(z) + \int_{Q_i} |K(x - z) - K(x - z^i)|\, d|\mu_i^b|(z).$$

We employed here the fact $\int_{Q_i} d\mu_i^b = 0$. Now for certain $s(z) \in \overline{zz^i}$

$$\begin{aligned}|K(x - z) - K(x - z^i)| &\leq |\nabla K(x - s(z))||z - z^i| \\ &\leq C|x - s(z)|^{-n} \operatorname{diam} Q_i \leq C|x - z^i|^{-n} \operatorname{diam} Q_i.\end{aligned}$$

We obtain a similar estimate with $x$ replaced by $y$. Thus

$$\begin{aligned}|\heartsuit| &\leq C\left(|x - z^i| \frac{\operatorname{diam} Q_i}{|x - z^i|^{n+1}} |Q_i| + |y - z^i| \frac{\operatorname{diam} Q_i}{|y - z^i|^{n+1}} |Q_i|\right) \\ &\leq C\left(|x - z^i| \int_{Q_i} \frac{\delta(z)}{|x - z|^{n+1}}\, dz + |y - z^i| \int_{Q_i} \frac{\delta(z)}{|y - z|^{n+1}}\, dz\right).\end{aligned}$$

Now the estimates $|x - z^i|$, $|y - z^i| < C|x - y|$ lead to the desired inequality.

**Case 3:** $\operatorname{diam} Q_i \leq |x-y| \leq 10^{-1}\operatorname{dist}(\{x,y\},Q_i)$; There exist points $w_{x,z}, w_{y,z} \in \overline{zz^i}$ such that

$$\begin{aligned} |\heartsuit| &= \left|\int_{Q_i} (K(y-z)-K(y-z^i))\, d\mu_i^b(z) - \int_{Q_i} (K(x-z)-K(x-z^i))\, d\mu_i^b(z)\right| \\ &= \left|\int_{Q_i} \langle \nabla K(y-w_{y,z}), z-z^i\rangle\, d\mu_i^b(z) - \int_{Q_i} \langle \nabla K(x-w_{x,z}), z-z^i\rangle\, d\mu_i^b(z)\right| \\ &= \left|\int_{Q_i} \langle \nabla K(y-w_{y,z}) - \nabla K(x-w_{x,z}), z-z^i\rangle\, d\mu_i^b(z)\right| \\ &\leq C|x-z^i|^{-(n+1)}|x-y|(\operatorname{diam} Q_i)|Q_i|. \end{aligned}$$

Hence

$$|\heartsuit| \leq C|x-y|\frac{\operatorname{diam} Q_i}{|x-z^i|^{n+1}}|Q_i| \leq C|x-y|\int_{Q_i}\frac{\delta(z)}{|x-z|^{n+1}}\,dz.$$

The proof for Lemma 9 and hence that for Theorem 6 is complete.

# References


[1] Ambrosio, L., Braides, A.: Energies in SBV and variational models in fracture mechanics. Preprint SISSA 124/95/M (October 1995)

[2] Ambrosio, L., Coscia, A., Dal Maso, G.: Fine properties of functions with bounded deformation, in preparation

[3] Anzolletti, G., Giaquinta, M.: Existence of the displacement field for an elastostatic body subject to Henky's law and von Mises yield condition. Manuscripta Math. **32**, 101–136 (1989)

[4] Bellettini, G., Coscia, A., Dal Maso, G.: Special functions of bounded deformation. Preprint SISSA 76/95/M (June 1995)

[5] Besov, O. V., Il'in, V. P. and Nikol'skiĭ.: Integral representations and imbedding theorems, John Wiley& Sons, 1978

[6] Bojarski, B., Hajłasz, P.: Pointwise inequalities for Sobolev functions and some applications. Studia Math. **106**, 77–92 (1993)

[7] Calderón, A. P.: On the differentiability of absolutely continuous functions. Riv. Mat. Univ. Parma **2**, 203–213 (1951)

[8] Calderón, A. P., Zygmund, A.: On the existence of certain singular integrals. Acta Math. **88** (1952), 85–139. Reprinted in Selected Papers of Antoni Zygmund. vol. 3 (Hulanicki, A., Wojtaszczyk, P., Żelazko, W. eds.) Mathematics and Its Applications, vol. 41/3, Kluwer Academic Publishers, 1989

[9] Deny, J., Lions, J. L.: Les espace du type de Beppo Levi. Ann. Inst. Fourier, **5**, 305–370 (1954)

[10] Federer, H.: Geometric measure theory. Springer-Verlag, 1969

[11] Friedrichs, K. O.: On the boundary-value problems of the theory of elasticity and Korn's inequality, Ann. Math. **48**, 441-471 (1947)

[12] Hajłasz, P.: Sobolev spaces on an arbitrary metric space. Potential Analysis, to appear

[13] Kałamajska, A.: Pointwise multiplicative inequalities and Nirenberg type estimates in weighted Sobolev spaces, Studia Math. **108**, 275–290 (1994)

[14] Kałamajska, A.: Coercive inequalities on weighed Sobolev spaces, Colloq. Math. **66**, 309–318 (1994)

[15] Kohn, R.: New estimates for deformations in terms of their strains. Ph. D. Thesis, Princeton University, 1979

[16] Matthies, H., Strang, G. and Christiansen, E.: The saddle point of a differential program, in "Energy methods in finite element analysis," volume dedicated to Professor Fraeijs de Veubeke; Glowinski, R., Rodin, E., Zienkiewicz, O.C., eds. New York: John Wiley 1979

[17] Maz'ya, V. G.: Sobolev spaces. Springer-Verlag, 1985

[18] Mikhlin, S. G.: Multidimensional singular integrals and integral equations. Pergamon press, 1965

[19] Mosolov, P. P. and Myasnikov, V. P.: A proof of Korn's inequality, Dokl. Acad. Nauk SSSR **201**, 36-39 (1971), English transl. Soviet. Math. Dokl. **12** (1971)

[20] Ornstein, D.: A non-inequality for differential operators in the $L^1$ norm, Arch. Rat. Mech. Anal. **11**, 40–49 (1968)

[21] Reshetnyak, Yu, G.: Space mappings with bounded distorsion. Translation of Mathematical Monographs, vol. 73, American Mathematical Society, Providence 1989

[22] Smith, K. T.: Formulas to represent functions by their derivatives, Math. Ann. **188**, 53–77 (1970)

[23] Stein, E.: Singular integrals and differentiability properties of functions. Princeton Univ. Press, 1970

[24] Temam, R. and Strang G.: Functions of bounded deformation, Arch. Rat. Mech. Anal. **75**, 7–21 (1980)

[25] Temam, R. and Strang G.: Duality and relaxation in the variational problems of plasticity. J. de Mecanique **19**, 493–527 (1980)

[26] Whitney, H.: On totally differentiable and smooth functions, Pacific J. Math. **1**, 143-159 (1951)

[27] Ziemer, W.: Weakly differentiable functions. Graduate Texts in Mathematics 120, Springer-Verlag, 1989