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Note on Meyers-Serrin's theorem

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Abstract. We generalize the Meyers Serrin's theorem proving that Sobolev function can be approximated by smooth functions with the same behavior at the boundary. Then we apply this to the boundary value problems.

For the notational convention we shall recall the definition of Sobolev space. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We define

$$W^{m,p}(\Omega) = \{f \in \mathscr{D}'(\Omega) : D^\alpha f \in L^p(\Omega) \quad \text{for} \quad |\alpha| \le m\}$$

where m is a nonnegative integer and $1 \le p \le \infty$.

$W^{m,p(\Omega)}$ is a Banach space when endowed with the norm

$$\|f\|_{m,p,\Omega} = \sum_{|\alpha| \le m} \|D^\alpha f\|_{L^p(\Omega)}.$$

The completion of the set $C_0^\infty(\Omega)$ in the $\|\cdot\|_{m,p,\Omega}$ norm is denoted by $W_0^{m,p}(\Omega)$.

If we replace L^p by L^p_{loc} then we obtain the definition of $W^{m,p}_{\mathrm{loc}}$. One can prove (see e.g. [3], thm. 1.2.2) that

$$W^{m,p}_{\mathrm{loc}}(\Omega) = \{f \in \mathscr{D}'(\Omega) : D^\alpha f \in L^p_{\mathrm{loc}}(\Omega) \quad \text{for } |\alpha| = m\}.$$

The following theorem generalizes the classical Meyers-Serrin's theorem.

Theorem 1 *If $f \in W^{m,p}_{\mathrm{loc}}(\Omega)$ where $1 \le p < \infty$ then to every $\varepsilon > 0$ there exists $g \in C^\infty(\Omega)$ such that*
1) $f - g \in W_0^{m,p}(\Omega)$
2) $\|f - g\|_{m,p,\Omega} < \varepsilon$.

Remark For the classical Meyers-Serrin's theorem see e.g. [1], [3], [4].

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Proof. Let $\{B_k\}_{k=1}^{\infty}$ be a locally finite covering of Ω by balls with subordinated partition of unity $\{\varphi_k\}_{k=1}^{\infty}$ such that the family $\{2B_k\}_{k=1}^{\infty}$ also forms a locally finite covering of Ω (by $2B_k$ we denote a ball with the same center as B_k and twice enlarged radius).

Let $\varepsilon > 0$ be taken at will. The function $f\varphi_k$ (which belongs to $W_0^{m,p}(\Omega)$) can be approximated by a smooth function with compact support included in $2B_k$ (standard approximation by convolution). Hence there exists the function $g_k \in C_0^{\infty}(2B_k)$ such that

$$\|f\varphi_k - g_k\|_{m,p,\Omega} \le \varepsilon/2^k.$$

Hence the series $\sum(f\varphi_k - g_k)$ is convergent in $W_0^{m,p}(\Omega)$, but we have also a pointwise convergence $\sum(f\varphi_k - g_k) = f - g$ where $g = \sum g_k \in C^{\infty}(\Omega)$, so $f - g \in W_0^{m,p}(\Omega)$ and

$$\|f - g\|_{m,p,\Omega} \le \sum \|f\varphi_k - g_k\|_{m,p,\Omega} \le \varepsilon. \quad \blacksquare$$

Sometimes the boundary value problem is stated in the form:

Given $v \in W^{m,p}(\Omega)$. Find $u \in W^{m,p}(\Omega)$ such that

$$\begin{cases} P(u) = 0 \\ u - v \in W_0^{m,p}(\Omega) \end{cases}$$

It follows from the above theorem (but not from the classical Meyers-Serrin's theorem) that the boundary condition v can be replaced by a smooth function $v' \in C^{\infty}(\Omega)$. Moreover we can assume that

$$\|v'\|_{m,p,\Omega} \le (1+\varepsilon)\|v\|_{m,p,\Omega}$$

(because v' can be taken arbitrary close to v).

Theorem 2 *If $1 < p < \infty$ and $\Omega \in C^{m,1}$, then each element of $\prod_{l=0}^{m-1} W^{m-l-1/p,p}(\partial\Omega)$ is a trace of a smooth function from $W^{m,p}(\Omega)$. (For definition of $C^{m,1}$ domains and $\prod_{l=0}^{m-1} W^{m-l-1/p,p}(\partial\Omega)$ see [2].)*

Proof. This is a direct consequence of the above theorem, fact that each element of $\prod_{l=0}^{m-1} W^{m-l-1/p,p}(\partial\Omega)$ is a trace of a function from $W^{m,p}(\Omega)$ ([2] thm. 6.10.3(ii)) and fact that functions from $W_0^{m,p}(\Omega)$ have trace equal to zero. $\blacksquare$

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