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Equivalent Statement of the Poincaré Conjecture (*).

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Abstract. - *We find an equivalent statement of the Poincaré Conjecture in an analytical form involving the notion of the Sobolev mappings between manifolds.*

1. - Introduction.

We start with recalling basic notions from the Sobolev mappings theory. More information can be found in papers given in references.

Let M^m be a smooth compact Riemannian m-dimensional manifold. We define the Sobolev space $W^{1,p}(M^m)$ as a completion of $C^\infty(M^m)$ in the norm

$$\|f\|_{1,p} = \left(\int_{M^m} (|f|^p + |\nabla f|^p)\, dx \right)^{1/p}.$$

Now let M^m and N^n be two smooth compact Riemannian manifolds (M^n may have a boundary but N^n not), and additionally let's assume that N^n is embedded in the Euclidean space $\mathbb{R}^k$. Then we define

$$W^{1,p}(M^m, N^n) = \{ f \in W^{1,p}(M^m, \mathbb{R}^k) \colon\ f(x) \in N^n \ \text{a.e. } x \in M^m \}$$

where

$$W^{1,p}(M^m, \mathbb{R}^k) = \{ (f_1, \dots, f_k) \colon\ f_i \in W^{1,p}(M^m),\ i = 1, 2, \dots, k \}.$$

One can prove that in fact the space $W^{1,p}(M^m, N^n)$ is independent of choice of metrics on M^m and N^n and of embedding of N^n.

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REMARK. – Analogously one can define the space $W^{r,p}(M^m, N^n)$ for $r > 1$. This space is also independent of choice of metrics and of embedding of N^n (see [5]).

THEOREM 1 ([6], [7]). – *If* $p \geq m$ *then* $C^\infty(M^m, N^n)$ *is dense in* $W^{1,p}(M^m, N^n)$. ■

In the case $p < m$ in general this statement doesn't hold (see Lemma 1 and Theorem 4 below).

THEOREM 2 ([8]). – *If* $f \in W^{1,m}(M^m, N^n)$ *then there exists* $\varepsilon > 0$ *such that if* g_1, $g_2 \in C^\infty(M^m, N^n)$, $\|f - g_i\|_{1,m} < \varepsilon$, $i = 1, 2$ *then* g_1 *and* g_2 *are homotopic.* ■

The following lemma was proved in [2] in a special case $M^m = B^n$. It is also an easier part of a very difficult theorem of Bethuel ([1], [4], Theorem 4 below). The proof presented below extends the method given in [2].

LEMMA 1. – *If* $\pi_{[p]}(N^n) \neq 0$ *and* $p < m$ *then smooth mappings are not dense in* $W^{1,p}(M^m, N^n)$.

(π_k denotes the k-th. homotopy group and $[p]$ is the greatest integer not greater than p.)

PROOF. – It is not difficult to find a smooth mapping $f\colon B^{[p]+1} \to S^{[p]}$ with two singular points of degree $+1$ and -1 such that near the boundary, f is a mapping into a point. Now we can construct the mapping $g\colon B^{[p]+1} \times S^{m-[p]-1} \to S^{[p]}$ putting $g(b, s) = f(b)$. And in the end we can embed the torus $B^{[p]+1} \times S^{m-[p]-1}$ into the manifold M^m and extended the mapping g on the completion of this torus sending it into the corresponding point.

Let's assume that $\varphi\colon S^{[p]} \to N^n$ is a smooth representative of a nontrivial element in $\pi_{[p]}(N^n)$. We will prove that the mapping $\varphi \circ g \in W^{1,p}(M^m, N^n)$ cannot be approximated by smooth mappings. For if not there would exist a sequence $f_k \in C^\infty(M^m, N^n)$ such that $f_k \to \varphi \circ g$ in a $W^{1,p}$ norm. Then the Fubini theorem implies that there exists a small sphere of the type $S^{[p]}$ around the singularity $+1$ (in one film $B^{[p]+1} \subset B^{[p]+1} \times S^{m-[p]-1} \subset M^m$) such that after taking a subsequence of f_k (denoted also by f_k) $f_{k|S^{[p]}} \to \varphi \circ g|_{S^{[p]}}$ in $W^{1,p}$ norm. Mappings $f_{k|S^{[p]}}\colon S^{[p]} \to N^n$ are homotopically trivial but $\varphi \circ g|_{S^{[p]}}$ is a representative of a nontrivial element in $\pi_{[p]}(N^n)$. This contradicts Theorem 2. ■

In the sequel we will use also the following lemma:

LEMMA 2. – *If a mapping* $f \in W^{1,p}(M^m, N^n)$ *can be approximated by Lipschitz mappings then it can be approximated by smooth ones.*

PROOF. – Lipschitz mapping can be approximated by a composition of a standard convolution approximation with the nearest point projection from the tubular neighborhood onto N^n. For the proof of the continuity of such composition in $W^{1,p}$ norm see e.g. [2] p. 63, [5], [7]. (By the convolution approximation we mean the convolution approximation of the components of the given mapping. This gives the mapping with values in $\mathbb{R}^k$, but near N^n as the given mapping is continuous.)

2. – Poincaré conjecture.

We restrict our attention (in the rest of the paper) to the category of differentiable manifolds, but fortunately in the most important case $n = 3$ all manifolds admit such structure. Let's start with recalling the Poincaré conjecture in its classical statement.

P.C. (first form). *The n-dimensional manifold is homeomorphic with the sphere S^n if and only if it has the same homotopical type.*

The following lemma can be readily derived from Hurewicz and Whitehad's theorems.

LEMMA 3. – *The n-dimensional manifold has the same homotopical type as the sphere S^n if and only if it is compact, $(n-1)$-connected, without boundary.* ■.

This lemma leads to the another equivalent statement of the Poincaré conjecture.

P.C. (second form). *Compact, n-dimensional manifold without boundary is homeomorphic with the sphere S^n if and only if it is $(n-1)$-connected.*

In the sequel we will use the following lemma.

LEMMA 4. – *If K^n is a compact, contractible (hence with boundary) submanifold of $\mathbb{R}^k$ then there exists a Lipschiltz retraction from $\mathbb{R}^k$ onto K^n.*

PROOF. – Proof of this lemma is more or less standard, so we will sketch it only. Let's divide $\mathbb{R}^k$ into a family of small cubes using the lattice points. Now we can construct this retraction as follows: Near the manifold we define it as the nearest point projection. On the rest of the space we define our mapping by extending it on skeletons. The extension of the mapping to the higher dimensional skeletons is possible by vanishing of the homotopy groups of the manifold K^n. ■

3. – Main result.

The aim of this paper is to prove the following

THEOREM 3. – *The following conjecture is equivalent to the Poincaré conjecture*:

CONJECTURE. – *Compact, connected manifold M^n, without boundary is homeomorphic with the sphere S^n if and only if $C^\infty(M^n, M^n)$ is dense in $W^{1,p}(M^n, M^n)$ for all $1 \leqslant p < \infty$.*

PROOF. – According to the second statement of the P.C. it is enough to prove that the manifold M^n is $(n-1)$-connected if and only if $C^\infty(M^n, M^n)$ is dense in $W^{1,p}(M^n, M^n)$ for all $1 \leqslant p < \infty$.

Let's assume that $\pi_k(M^n) \neq 0$ for some $1 \leqslant k \leqslant n-1$ then as follows from Lemma 1, $C^\infty(M^n, M^n)$ is not dense in $W^{1,k}(M^n, M^n)$.

To prove the opposite implication, we note that as follows from Theorem 1, $C^\infty(M^n, M^n)$ is dense in $W^{1,p}(M^n, M^n)$ when $p \geqslant n$. Hence, assume that $p < n$ and that M^n is embedded in $\mathbb{R}^k$. Let $f \in W^{1,p}(M^n, M^n)$. We have to prove that f can be approximated by smooth mappings.

LEMMA 5. – *If M^n is a compact, $(n-1)$-connected manifold without boundary then for every $x \in M^n$, $M^n \setminus \{x\}$ is contractible.*

PROOF. – The case $n = 2$ is obvious. Let's assume that $n \geqslant 3$. It is enough to prove that $\pi_k(M^n \setminus \{x\}) = 0$ for $k = 1, 2, \ldots,$ but by the dimension argument $\pi_1(M^n \setminus \{x\}) = \ldots = \pi_{n-2}(M^n \setminus \{x\}) = 0$, hence by Hurewicz theorems (simple and reverse) it is enough to prove that $H_{n-1}(M^n \setminus \{x\}) = 0$. $M^n \setminus \{x\}$ has a homotopy type of $M^n \setminus B^n$ where B^n is a small ball, hence:

$$H_{n-1}(M^n \setminus \{x\}) = H_{n-1}(M^n \setminus B^n) = (\textit{Lefschetz duality}) = H^1(M^n \setminus B^n, \partial B^n) =$$
$$= (\textit{excision}) = H^1(M^n, B^n) = H^1(M^n / B^n) = H^1(M^n) = 0 . \qquad \blacksquare$$

(In the following reasoning we will use some informal notations.)

If $\varepsilon > 0$ is a sufficiently small then the ball $B^n(x, \varepsilon)$ lying on the manifold M^n can be identified with the ball in euclidean space (using the system of normal coordinates), so we can introduce the linear structure on $B^n(x, \varepsilon)$ with x as a zero.

There exists the family of homotopies (for sufficiently small ε)

$$H^{x,\varepsilon}\colon (M^n \setminus B^n(x, \varepsilon)) \times [0, 1] \to M^n \setminus B^n(x, \varepsilon),$$

$$H^{x,\varepsilon}(\cdot, 0) = id, \qquad H^{x,\varepsilon}(\cdot, 1) = * .$$

(We restrict our attention to these x whose distance from the fixed point $*$ is greater than 2ε.)

Let $\Pi_{x,\varepsilon}: B^n(x,\varepsilon)\setminus\{x\} \to S^{n-1}(x,\varepsilon)$ denotes the radial projection. Let $\varphi \in C_0^\infty(B^n(1))$ be such that $0 \leq \varphi(x) \leq 1$ and $\varphi|_{B^n(1/2)} = 1$. Now we can define the family of retractions

$$P^{x,\varepsilon}: M^n \to M^n \setminus B^n(x,\varepsilon),$$

$$P^{x,\varepsilon}|_{M^n\setminus B^n(x,\varepsilon)} = id_{M^n\setminus B^n(x,\varepsilon)},$$

by the formula

$$P^{x,\varepsilon}(y) = \begin{cases} H^{x,\varepsilon}(\Pi_{x,\varepsilon}(y), \varphi(y/\varepsilon)) & \text{when } y \in B^n(x,\varepsilon)\setminus\{x\}, \\ * & \text{when } y = x, \\ y & \text{when } y \in M^n\setminus B^n(x,\varepsilon). \end{cases}$$

We can choose the family $H^{x,\varepsilon}$ «uniformly» and hence we obtain that $P^{x,\varepsilon}$ is a Lipschitz retraction with the Lipschitz constant bounded by $L\varepsilon^{-1}$ where the constant L does not depend on x. Now we can finish our proof following some ideas of [2] Th. 1. There exists a constant C such that the maximal number $k(\varepsilon)$ of disjoint balls with radius ε contained in M^n is not less than $C\varepsilon^{-n}$.

Let $\{B^n(x_i,\varepsilon)\}_{i=1}^{k(\varepsilon)}$ be such a disjointed family of balls. We have

$$\int\limits_{\bigcup_{i=1}^{k(\varepsilon)} f^{-1}(B^n(x_i,\varepsilon))} (|f|^p + |\nabla f|^p)\,dx = \sum_{i=1}^{k(\varepsilon)} \int\limits_{f^{-1}(B^n(x_i,\varepsilon))} (|f|^p + |\nabla f|^p)\,dx \leq \|f\|_{1,p}^p.$$

Hence there exists $j \in \{1, \dots, k(\varepsilon)\}$ such that

$$\int\limits_{f^{-1}(B^n(x_j,\varepsilon))} (|f|^p + |\nabla f|^p)\,dx \leq \frac{1}{k(\varepsilon)}\|f\|_{1,p}^p \leq \frac{1}{C}\varepsilon^n\|f\|_{1,p}^p.$$

The mapping f can be approximated in the Sobolev norm by mappings $P^{x_j,\varepsilon}\circ f$. Indeed. We can assume that $|f^{-1}(B^n(x_j,\varepsilon))|$ tends to zero with ε (Lebesgue's measure), hence we have an approximation in L^p-norm. Bearing absolute continuity of the integral in mind it is enough to prove that

$$\int\limits_{f^{-1}(B^n(x_j,\varepsilon))} |\nabla(P^{x_j,\varepsilon}\circ f)|^p\,dx \to 0 \quad \text{when } \varepsilon \to 0.$$

We have

$$\int\limits_{f^{-1}(B^n(x_j,\varepsilon))} |\nabla(P^{x_j,\varepsilon}\circ f)|^p\,dx \leq L^p\varepsilon^{-p} \int\limits_{f^{-1}(B^n(x_j,\varepsilon))} |\nabla f|^p\,dx \leq L^p\varepsilon^{-p}C^{-1}\varepsilon^n\|f\|_{1,p}^p \to 0$$

when $\varepsilon \to 0$. Now it remains to prove that the mapping $P^{x,\varepsilon}\circ f$ can be approximated by smooth mappings.

Note that there exists a smooth mapping $\pi: \mathbb{R}^k \to \mathbb{R}^k$ which is an extension of the retraction from the tubular neighborhood onto M^n. Composing the convolution approximation of $P^{x,\varepsilon} \circ f$ with mapping π we obtain the smooth approximation g_δ of $P^{x,\varepsilon} \circ f$ (compare with the proof of Lemma 2) with the following properties

1) $g_\delta \to P^{x,\varepsilon} \circ f$ in $W^{1,p}$ norm and a.e. when $\delta \to 0$.
2) If $E_\delta = \{x \in M^n \colon g_\delta(x) \in M^n \setminus B^n(x, \varepsilon/2)\}$ then $|M^n \setminus E_\delta| \xrightarrow[\delta \to 0]{} 0$.

Unfortunately in general not all values of the mapping g_δ belong to M^n. To obtain proper approximation by Lipschitz mappings (compare with Lemma 2) it is enough to compose g_δ with the retraction described in Lemma 4. Namely the manifold $M^n \setminus B^n(x, \varepsilon/2)$ is contractible (Lemma 5) hence there exists a Lipschitz retraction $T\colon \mathbb{R}^k \to M^n \setminus B^n(x, \varepsilon/2)$ (Lemma 4). We denote its Lipschitz constant by K. Now it suffices to prove that $T \circ g_\delta$ tends to $P^{x,\varepsilon} \circ f$ in $W^{1,p}$-norm. $T \circ g_\delta$ tends to $P^{x,\varepsilon} \circ f$ a.e. and hence we have L^p-convergence. Now it suffices to prove that

$$I_\delta = \int_{M^n} |\nabla(T \circ g_\delta - P^{x,\varepsilon} \circ f)|^p \, dx \to 0 \quad \text{when } \delta \to 0 .$$

We have

$$I_\delta = \int_{E_\delta} + \int_{M^n \setminus E_\delta} = I_{1,\delta} + I_{2,\delta},$$

$$I_{1,\delta} = \int_{E_\delta} |\nabla(g_\delta - P^{x,\varepsilon} \circ f)|^p \, dx \to 0 \quad \text{when } \delta \to 0 .$$

(We used the fact that $\nabla(T \circ g_\delta)(x) = \nabla g_\delta(x)$ for a.e. $x \in E_\delta$. It is a simply consequence of the fact that $T \circ g_\delta(x) = g_\delta(x)$ when $x \in E_\delta$)

$$I_{2,\delta} \leqslant 2^{p-1} K^p \int_{M^n \setminus E_\delta} |\nabla g_\delta|^p \, dx + 2^{p-1} \int_{M^n \setminus E_\delta} |\nabla(P^{x,\varepsilon} \circ f)|^p \, dx .$$

Second integral converges to zero because $|M^n \setminus E_\delta| \to 0$. First integral also converges to zero because

$$\int_{M^n \setminus E_\delta} |\nabla g_\delta|^p \, dx \leqslant 2^{p-1} \left(\int_{M^n \setminus E_\delta} |\nabla(g_\delta - P^{x,\varepsilon} \circ f)|^p \, dx + \int_{M^n \setminus E_\delta} |\nabla(P^{x,\varepsilon} \circ f)|^p \, dx \right).$$

This completes the proof. ■

We should note that our main result follows also from the following very difficult theorem of Bethuel [1].

Theorem 4. – *If $p < m$ then $C^\infty(M^m, N^n)$ is dense in $W^{1,p}(M^m, N^n)$ if and only if* $\pi_{[p]}(N^n) = 0$.

Although Theorem 3 follows also from Theorem 4, the proof presented above is much simpler than that of Theorem 4. Moreover modification of the above method leads to new results concerning the sequentially weakly density of smooth mappings in Sobolev spaces of mappings—see [4].

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