

ON APPROXIMATE DIFFERENTIABILITY OF THE MAXIMAL FUNCTION

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ABSTRACT. We prove that if $f \in L^1(\mathbb{R}^n)$ is approximately differentiable a.e., then the Hardy-Littlewood maximal function $\mathcal{M}f$ is also approximately differentiable a.e. Moreover, if we only assume that $f \in L^1(\mathbb{R}^n)$, then any open set of \mathbb{R}^n contains a subset of positive measure such that $\mathcal{M}f$ is approximately differentiable on that set. On the other hand we present an example of $f \in L^1(\mathbb{R})$ such that $\mathcal{M}f$ is not approximately differentiable a.e.

1. INTRODUCTION

Juha Kinnunen [10] proved that the Hardy-Littlewood maximal function

$$\mathcal{M}f(x) = \sup_{r>0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y)| dy$$

is a bounded operator in the Sobolev space $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$. Recall that $W^{1,p}(\mathbb{R}^n)$ is the space of all functions $f \in L^p(\mathbb{R}^n)$ such that weak (distributional) partial derivatives $\partial f / \partial x_i$ also belong to $L^p(\mathbb{R}^n)$, and similarly for $W_{\text{loc}}^{1,p}(\mathbb{R}^n)$. Since the maximal function is not bounded in L^1 , there is no apparent reason to expect any kind of boundedness of the maximal function in $W^{1,1}(\mathbb{R}^n)$. However, Tanaka [25] proved that in the one dimensional case the noncentered maximal function of $f \in W^{1,1}(\mathbb{R})$ belongs locally to $W^{1,1}(\mathbb{R})$. Since that time it has been an open problem to extend Tanaka's result to the case of the Hardy-Littlewood maximal function and to find analogous results in the higher dimensional case; cf. [8, Question 1]. To the best of our knowledge there are no known higher dimensional results in the case $p = 1$, and even in the one dimensional case it is still not known whether the Hardy-Littlewood maximal function (i.e. the centered one) of $f \in W^{1,1}(\mathbb{R})$ belongs locally to $W^{1,1}(\mathbb{R})$; see, however, [2], [3]. The results proved in the paper are clearly motivated by this challenging problem.

Theorem 1. *If $f \in L^1(\mathbb{R}^n)$ is approximately differentiable a.e., then the maximal function $\mathcal{M}f$ is approximately differentiable a.e.*

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Since every function $f \in W^{1,1}(\mathbb{R}^n)$ is approximately differentiable a.e., the result implies a.e. approximate differentiability of $\mathcal{M}f$. This, in particular, implies (see Lemma 5) that $\mathcal{M}f$ coincides with a C^1 function off an open set of arbitrarily small measure. However, a.e. approximate differentiability of $\mathcal{M}f$ is much less than weak differentiability of $\mathcal{M}f$, which is still an open problem. On the other hand, the assumption about f in the theorem is much weaker than $f \in W^{1,1}$. In addition to this result, Theorem 2 provides a formula for the approximate derivative of $\mathcal{M}f$ when $f \in W^{1,1}$.

Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. It is easy to see (cf. [20]) that for a.e. $x \in \mathbb{R}^n$, either

$$(1) \quad \mathcal{M}f(x) = \fint_{B(x,r_x)} |f(y)| dy \quad \text{for some } r_x > 0$$

or

$$(2) \quad \mathcal{M}f(x) = |f(x)|.$$

Denote by E and P the sets of points in \mathbb{R}^n for which (1) and respectively (2) is satisfied. The following result is due to Luirio [20] when $p > 1$ and is new when $p = 1$. Our proof is new and simpler even in the case $p > 1$.

Theorem 2. *Let $f \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$. Then the weak derivative, when $p > 1$, and the approximate derivative, when $p = 1$, of the maximal function $\mathcal{M}f$ satisfy*

$$(3) \quad \begin{aligned} \nabla \mathcal{M}f(x) &= \fint_{B(x,r_x)} \nabla |f(y)| dy \quad \text{for a.e. } x \in E, \\ \nabla \mathcal{M}f(x) &= \nabla |f(x)| \quad \text{for a.e. } x \in P. \end{aligned}$$

Remark 3. If $x \in E$, then $r_x > 0$ is not necessarily uniquely defined and (3) holds for all such r_x .

In the next result we deal with differentiability properties of $\mathcal{M}f$ for any $f \in L^1(\mathbb{R}^n)$.

Theorem 4. *If $f \in L^1(\mathbb{R}^n)$, then any open set $\Omega \subset \mathbb{R}^n$ contains a subset $E \subset \Omega$ of positive Lebesgue measure such that $\mathcal{M}f$ is approximately differentiable a.e. in E .*

Again, Lemma 5 implies that for any open set $\Omega \subset \mathbb{R}^n$ there is a function $g \in C^1(\mathbb{R}^n)$ such that the set $\{x \in \Omega : f(x) = g(x)\}$ has positive measure.

In view of Theorem 4 it is natural to inquire whether for every $f \in L^1(\mathbb{R}^n)$ the maximal function $\mathcal{M}f$ is approximately differentiable a.e. Unfortunately the answer is in the negative, as an example presented at the end of the paper shows.

While the proofs of Theorems 1 and 2 are completely elementary, the proof of Theorem 4 requires some advanced potential theory.

Let us also mention that the result of Kinnunen [10] has been applied and generalized by many authors ([2], [3], [6], [7], [8], [11], [12], [14], [15], [16], [17], [18], [20], [21], [25]).

The notation used in the paper is pretty standard. The volume of the unit ball in \mathbb{R}^n is denoted by ω_n , and we use a barred integral to denote the integral average

$$\fint_{B(x,r)} f(y) dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy.$$

By C we will denote a generic positive constant whose actual value may change even in a single string of estimates.

2. APPROXIMATE DIFFERENTIABILITY

Let f be a real-valued function defined on a set $E \subset \mathbb{R}^n$. We say that f is *approximately differentiable* at $x_0 \in E$ if there is a vector $L = (L_1, \dots, L_n)$ such that for any $\varepsilon > 0$ the set

$$A_\varepsilon = \left\{ x : \frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} < \varepsilon \right\}$$

has x_0 as a density point. If this is the case, then x_0 is a density point of E and L is uniquely determined. The vector L is called the *approximate differential* of f at x_0 and is denoted by $\nabla f(x_0)$.

In what follows we will need the following theorem of Whitney [26], which provides several characterizations of a.e. approximate differentiability of a function. We state it as a lemma.

Lemma 5. *Let $f : E \rightarrow \mathbb{R}$ be measurable, $E \subset \mathbb{R}^n$. Then the following conditions are equivalent.*

- (a) f is approximately differentiable a.e.
- (b) For any $\varepsilon > 0$ there is a closed set $F \subset E$ and a locally Lipschitz function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f|_F = g|_F$ and $|E \setminus F| < \varepsilon$.
- (c) For any $\varepsilon > 0$ there is a closed set $F \subset E$ and a function $g \in C^1(\mathbb{R}^n)$ such that $f|_F = g|_F$ and $|E \setminus F| < \varepsilon$.

Remark 6. To illustrate the relevance of the maximal function in this part of the real analysis, let us mention a useful Lipschitz type estimate valid for Sobolev functions:

$$(4) \quad |f(x) - f(y)| \leq C|x - y|(\mathcal{M}|\nabla f|(x) + \mathcal{M}|\nabla f|(y)) \quad \text{a.e.};$$

see [1], [4], [5]. As an almost immediate consequence of (4) one obtains a well known result that each $f \in W_{\text{loc}}^{1,1}(\mathbb{R}^n)$ is approximately differentiable a.e.; cf. [22].

Investigating the positive and negative parts of a function separately, one can easily prove

Lemma 7. *A measurable function $f : E \rightarrow \mathbb{R}$ is a.e. approximately differentiable if and only if $|f|$ is a.e. approximately differentiable.*

3. PROOF OF THEOREM 1

We consider a restricted version of the maximal function

$$\mathcal{M}^\varepsilon f(x) = \sup_{r \geq \varepsilon} \int_{B(x,r)} |f(y)| dy.$$

Lemma 8. *If $f \in L^1(\mathbb{R}^n)$, then*

$$|\mathcal{M}^\varepsilon f(x) - \mathcal{M}^\varepsilon f(y)| \leq \frac{n}{\varepsilon} |x - y| (\mathcal{M}^\varepsilon f(x) + \mathcal{M}^\varepsilon f(y)) \leq \frac{2n}{\omega_n \varepsilon^{n+1}} \|f\|_1 |x - y|.$$

Proof. The second inequality of the lemma is obvious because

$$\mathcal{M}^\varepsilon f(x) \leq \frac{1}{\omega_n \varepsilon^n} \int_{\mathbb{R}^n} |f(y)| dy = \frac{1}{\omega_n \varepsilon^n} \|f\|_1.$$

Thus we are left with the proof of the first inequality. For $a, r > 0$ the function $\varphi(r) = r/(r+a)$ is increasing, and hence, applying Bernoulli's inequality, we have for $r \geq \varepsilon$,

$$\left(\frac{r}{r+|x-y|} \right)^n \geq \left(\frac{\varepsilon}{\varepsilon+|x-y|} \right)^n \geq 1 - n \frac{|x-y|/\varepsilon}{1+|x-y|/\varepsilon} \geq 1 - \frac{n}{\varepsilon} |x-y|.$$

Fix $x, y \in \mathbb{R}^n$. Then for any $r \geq \varepsilon$ we have $B(y, r) \subset B(x, r+|x-y|)$ and hence

$$\mathcal{M}^\varepsilon f(x) \geq \left(\frac{r}{r+|x-y|} \right)^n \int_{B(y,r)} |f| \geq \left(1 - \frac{n}{\varepsilon} |x-y| \right) \int_{B(y,r)} |f|.$$

Passing to the supremum over $r \geq \varepsilon$ we obtain

$$\mathcal{M}^\varepsilon f(x) \geq \left(1 - \frac{n}{\varepsilon} |x-y| \right) \mathcal{M}^\varepsilon f(y).$$

Since the inequality is also true if we replace x by y and y by x , one easily concludes the first inequality from the lemma. \square

Lemma 9. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\{x : \mathcal{M}f(x) > |f(x)|\} = Z \cup \bigcup_{k=1}^{\infty} E_k,$$

where $|Z| = 0$ and $\mathcal{M}f|_{E_k}$ is Lipschitz continuous for $k = 1, 2, \dots$. In particular $\mathcal{M}f$ is a.e. approximately differentiable in the set $\{x : \mathcal{M}f(x) > |f(x)|\}$.

Proof. Let Z be the set of points that are not Lebesgue points of $|f|$. Clearly $|Z| = 0$. Assume that $x \in \mathbb{R}^n \setminus Z$ and $\mathcal{M}f(x) > |f(x)|$. Let $r_i > 0$ be a sequence such that

$$\int_{B(x_i, r_i)} |f| \rightarrow \mathcal{M}f(x).$$

The sequence r_i is bounded (because $\mathcal{M}f(x) > 0$ and $f \in L^1$), and hence we can select a subsequence (still denoted by r_i) such that $r_i \rightarrow r$. Clearly $r > 0$ as otherwise we would have $\mathcal{M}f(x) = |f(x)|$. Thus

$$\mathcal{M}f(x) = \int_{B(x,r)} |f| \quad \text{for some } r > 0.$$

This easily implies that

$$\{x : \mathcal{M}f(x) > |f(x)|\} \subset Z \cup \bigcup_{k=1}^{\infty} \{x : \mathcal{M}f(x) = \mathcal{M}^{1/k}f(x)\}.$$

Since the function $\mathcal{M}^{1/k}f$ is Lipschitz continuous by Lemma 8, the first part of the result follows. The second part is a direct consequence of Lemma 5. \square

Now we can complete the proof of Theorem 1. Let $f \in L^1(\mathbb{R}^n)$ be approximately differentiable a.e. Then also $|f|$ is approximately differentiable a.e. (Lemma 7). According to Lemma 9,

$$\mathbb{R}^n = \{x : \mathcal{M}f(x) = |f(x)|\} \cup Z \cup \bigcup_{k=1}^{\infty} E_k,$$

where $|Z| = 0$ and $\mathcal{M}f|_{E_k}$ is Lipschitz continuous. Since $\mathcal{M}f|_{E_k}$ is approximately differentiable a.e. and $\mathcal{M}f = |f|$ is approximately differentiable a.e. in the set $\{x : \mathcal{M}f(x) = |f(x)|\}$, the theorem follows. \square

4. PROOF OF THEOREM 2

Since $\mathcal{M}f(x) = |f(x)|$ in P , clearly $\nabla \mathcal{M}f(x) = \nabla |f(x)|$ a.e. in P . Thus let $x \in E$ and $r_x > 0$ be such that equality (1) holds. Assume also that $\mathcal{M}f$ is approximately differentiable at x . Note that the function

$$\varphi(y) = \mathcal{M}f(y) - \int_{B(y, r_x)} |f(z)| dz = \mathcal{M}f(y) - \int_{B(0, r_x)} |f(y+z)| dz$$

is approximately differentiable at x and

$$\begin{aligned} \nabla \varphi(x) &= \nabla \mathcal{M}f(x) - \int_{B(0, r_x)} \nabla |f|(x+z) dz \\ &= \nabla \mathcal{M}f(x) - \int_{B(x, r_x)} \nabla |f(z)| dz. \end{aligned}$$

Indeed, $\mathcal{M}f$ is approximately differentiable at x , and since $f \in W^{1,p}$ we can differentiate in the second term under the sign of the integral. Note also that $\varphi \geq 0$ and $\varphi(x) = 0$, so φ attains a minimum at x , and hence its approximate derivative at x must be equal to 0, which is the claim we wanted to prove. \square

5. PROOF OF THEOREM 4

This proof requires some results from potential theory. We say that a locally integrable function $u : \Omega \rightarrow [0, \infty]$ defined on an open set $\Omega \subset \mathbb{R}^n$ is *superharmonic* if it is lower semicontinuous and

$$(5) \quad u(x) \geq \int_{B(x, r)} u(y) dy$$

whenever $B(x, r) \Subset \Omega$.

The following regularity result has been established in the setting of weak solutions of the p -Laplace equation in [19]. For the convenience of the reader, we include a short proof based only on a knowledge of classical potential theory.

Lemma 10. *If a locally integrable function $u : \Omega \rightarrow [0, \infty]$, $\Omega \subset \mathbb{R}^n$, is superharmonic, then $u \in W_{\text{loc}}^{1,p}(\Omega)$ for all $1 \leq p < n/(n-1)$. In particular u is a.e. approximately differentiable.*

Proof. Let $u : \Omega \rightarrow [0, \infty]$ be superharmonic and let $U \Subset \Omega$. According to the Riesz decomposition theorem [23], [9], u restricted to U can be represented as

$$(6) \quad u(x) = h(x) - \int_{\mathbb{R}^n} \Phi(x-y) d\mu(y), \quad \text{for } x \in U,$$

where h is harmonic, Φ is the fundamental solution to the Laplace equation and μ is a finite positive measure supported in U . It is easy to see that we can compute the weak first-order partial derivatives of u in U by differentiating the right-hand side of (6) under the sign of the integral

$$\frac{\partial u}{\partial x_i}(x) = \frac{\partial h}{\partial x_i}(x) - \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{(x_i - y_i)}{|x - y|^n} d\mu(y).$$

By Young's convolution inequality (cf. [24, II.1.1, p. 27]), the convolution is as integrable as the kernel. Since the measure μ has a bounded support and the function $x \mapsto \frac{x}{|x|^n}$ is clearly in $L^p_{\text{loc}}(\mathbb{R}^n)$, we deduce that $\nabla u \in L^p(U)$. \square

For an open set $\Omega \subset \mathbb{R}^n$ and $f \in L^1_{\text{loc}}(\Omega)$ we define a local maximal function

$$\mathcal{M}_\Omega f(x) = \sup \int_{B(x,r)} |f(y)| dy,$$

where the supremum is over all balls $B(x,r) \Subset \Omega$.

The following characterization of superharmonic functions will be very useful; see [13].

Lemma 11. *A locally integrable function $u : \Omega \rightarrow [0, \infty]$, $\Omega \subset \mathbb{R}^n$, is superharmonic if and only if*

$$\mathcal{M}_\Omega u(x) = u(x) \quad \text{for all } x \in \Omega.$$

Proof. If u is superharmonic, then taking the supremum over all balls in (5) gives $u(x) \geq \mathcal{M}_\Omega u(x)$ for all $x \in \Omega$. On the other hand, lower semicontinuity of u yields

$$\mathcal{M}_\Omega u(x) \geq \limsup_{r \rightarrow 0} \int_{B(x,r)} u(y) dy \geq \liminf_{y \rightarrow x} u(y) \geq u(x)$$

for all $x \in \Omega$. Hence $u(x) = \mathcal{M}_\Omega u(x)$ for all $x \in \Omega$.

Now suppose that $\mathcal{M}_\Omega u(x) = u(x)$ for all $x \in \Omega$. Since the maximal function is lower semicontinuous we conclude lower semicontinuity of u . The superharmonicity of u follows from the inequality

$$\int_{B(x,r)} u(y) dy \leq \mathcal{M}_\Omega u(x) = u(x),$$

which is satisfied on every ball $B(x,r) \Subset \Omega$. \square

Corollary 12. *If $f \in L^1(\mathbb{R}^n)$ and $|f(x)| = \mathcal{M}f(x)$ a.e. in an open set $\Omega \subset \mathbb{R}^n$, then we can redefine f on a set of measure zero in such a way that $|f|$ becomes superharmonic in Ω .*

Proof. It follows from the Lebesgue differentiation theorem that $|f(x)| \leq \mathcal{M}_\Omega |f|(x)$ a.e. in Ω . Hence

$$\mathcal{M}_\Omega |f|(x) \leq \mathcal{M}|f|(x) = \mathcal{M}f(x) = |f(x)| \leq \mathcal{M}_\Omega |f|(x)$$

a.e. in Ω and thus $|f(x)| = \mathcal{M}_\Omega |f|(x)$ a.e. in Ω . Now it is clear that we can modify f on a set of measure zero in such a way that

$$|f(x)| = \mathcal{M}_\Omega |f|(x) \quad \text{everywhere in } \Omega,$$

which makes the function $|f|$ superharmonic. \square

Now we can complete the proof of the theorem. Let $f \in L^1(\mathbb{R}^n)$ and let $\Omega \subset \mathbb{R}^n$ be open. According to Lemma 9, $\mathcal{M}f$ is a.e. approximately differentiable in the set

$$\{x \in \Omega : |f(x)| < \mathcal{M}f(x)\}.$$

If this set has positive measure, the theorem follows. If it has measure zero, then

$$|f(x)| = \mathcal{M}f(x) \quad \text{a.e. in } \Omega,$$

and hence $|f(x)|$ coincides a.e. with a superharmonic function in Ω ; see Corollary 12. Now Lemma 10 gives a.e. approximate differentiability of $|f|$ in Ω and hence that of f ; see Lemma 7. The proof is complete.

6. EXAMPLE

In this section we will construct a bounded integrable function $f \in L^1(\mathbb{R})$ such that the set of points where the maximal function $\mathcal{M}f$ is not approximately differentiable is of positive measure. In our construction $\mathcal{M}f$ will coincide with f on a contact set P of positive length, and f will not be approximately differentiable on P . This will imply the lack of approximate differentiability of $\mathcal{M}f$ at the Lebesgue points of P .

In the first step we will construct a bounded periodic function f with period 1 such that $\mathcal{M}f$ is not approximately differentiable a.e., and then it will be clear that also for $\tilde{f} = f\chi_{[0,1]} \in L^1(\mathbb{R})$ the maximal function is not approximately differentiable a.e.

6.1. Construction. We denote

$$r_k = 3^{-k(k+1)}, \quad \alpha_k = \exp(-9^{-k-2}).$$

For $k = 1, 2, \dots$, on the interval $[0, r_{k-1})$ we define

$$g_k(x) = \begin{cases} 1, & x \in [(i-1)r_k, ir_k), \quad i \in \{2, 4, \dots, 9^k - 1\}, \\ \alpha_k, & x \in [(i-1)r_k, ir_k), \quad i \in \{3, 5, \dots, 9^k - 2\}, \\ 0, & x \in [(i-1)r_k, ir_k), \quad i \in \{1, 9^k\}. \end{cases}$$

We extend g_k to \mathbb{R} periodically with the period r_{k-1} . Finally we set

$$f_0 = 1, \quad f_n = \prod_{k=1}^n g_k, \quad f = \lim_{n \rightarrow \infty} f_n.$$

Observe that the function g_k is constant on the intervals $[(i-1)r_k, ir_k)$, $i \in \mathbb{Z}$, and hence f_n is constant on the intervals $[(i-1)r_n, ir_n)$, $i \in \mathbb{Z}$.

6.2. Maximal function. We will now estimate the maximal function of f . We denote

$$P = \bigcap_k \{f_k > 0\}.$$

Let $x \in \mathbb{R}$ and $\rho > 0$. We consider the smallest $n \in \mathbb{N}$ such that $M_n(x, \rho) = (x - \rho, x + \rho) \cap r_n \mathbb{Z} \neq \emptyset$. In this situation $(x - \rho, x + \rho)$ is contained in one of the intervals $[(i-1)r_{n-1}, ir_{n-1})$, and hence f_{n-1} equals a constant β on $(x - \rho, x + \rho)$. Let us write $M = M_n(x, \rho)$. Now we will distinguish two cases.

Case 1. Let $\rho \leq r_{n+1}$. Then there is only one point $z \in M$. By reason of symmetry, we may assume that $x \geq z$. Then $x \in [z, z + r_{n+1})$. Since $g_{n+1} = 0$ on $[z, z + r_{n+1})$, $f_{n+1} = 0$ on that interval and hence $x \notin P$.

Case 2. Let $\rho > r_{n+1}$. We split $(x - \rho, x + \rho)$ into intervals $(x - \rho, x + \rho) \cap [(i-1)r_n, ir_n)$. For each interval I of the partition, with an endpoint $z \in M$, either $[z, z + r_{n+1}) \subset I$, $[z - r_{n+1}, z) \subset I$ or $I \subset (z - r_{n+1}, z + r_{n+1})$. Since $f_{n+1} = 0$ on $(z - r_{n+1}, z + r_{n+1})$ we have $(z - r_{n+1}, z + r_{n+1}) \cap P = \emptyset$. In each case

$$|I \cap P| \leq \left(1 - \frac{r_{n+1}}{r_n}\right) |I|.$$

Summing over I we obtain

$$|(x - \rho, x + \rho) \cap P| \leq 2\rho \left(1 - \frac{r_{n+1}}{r_n}\right).$$

It follows that

$$\int_{x-\rho}^{x+\rho} f \leq \frac{\beta}{2\rho} |(x-\rho, x+\rho) \cap P| \leq \beta \left(1 - \frac{r_{n+1}}{r_n}\right).$$

On the other hand, if $x \in P$, then

$$f(x) \geq \beta \alpha_n \alpha_{n+1} \dots$$

Since

$$1 - \frac{r_{n+1}}{r_n} = 1 - 9^{-n-1} \leq e^{-9^{-n-1}} < e^{-9^{-n-2} - 9^{-n-3} - 9^{-n-4} - \dots} = \alpha_n \alpha_{n+1} \dots,$$

we obtain

$$\mathcal{M}f(x) \leq f(x) \quad \text{on } P,$$

and hence $\mathcal{M}f(x) = f(x)$ a.e. in P .

6.3. The contact set. On the set P we have

$$f(x) \geq \beta_\infty := \alpha_1 \alpha_2 \alpha_3 \dots = \exp(-9^{-3} - 9^{-4} - 9^{-5} - \dots) > 0.$$

We will estimate the size of the set $P \cap [0, 1]$. We see that

$$\begin{aligned} |\{f_1 > 0\} \cap [0, 1]| &= 1 - 2r_1, \\ |\{f_2 > 0\} \cap [0, 1]| &= |\{f_1 > 0\} \cap [0, 1]| \left(1 - 2\frac{r_2}{r_1}\right) = (1 - 2r_1) \left(1 - 2\frac{r_2}{r_1}\right), \\ &\dots, \end{aligned}$$

so that

$$(7) \quad |P \cap [0, 1]| = (1 - 2r_1) \left(1 - 2\frac{r_2}{r_1}\right) \left(1 - 2\frac{r_3}{r_2}\right) \dots > 0$$

as

$$\sum_k \frac{r_{k+1}}{r_k} = \sum_k 9^{-k-1} < +\infty.$$

6.4. Differentiability. Let us consider $x \in P$, $k \in \mathbb{N}$ and an interval $[z, z + r_k]$ such that $z \in r_k \mathbb{Z}$ and

$$x \in [z, z + r_k] \subset \{f_k > 0\}.$$

This interval is contained in an interval $[(i-1)r_{k-1}, ir_{k-1}]$, where the function f_{k-1} has constant value $\beta \in (\beta_\infty, 1]$. Since $z \geq (i-1)r_{k-1} + r_k$ (otherwise $f_k(z) = 0$) we have that

$$[z - r_k, z + r_k] \subset [(i-1)r_{k-1}, ir_{k-1}]$$

and hence $f_{k-1} = \beta$ on $[z - r_k, z + r_k]$. There are three possibilities:

$$\begin{aligned} f_k &= \begin{cases} 0 & \text{on } [z - r_k, z) = I, \\ \beta & \text{on } [z, z + r_k) = J, \end{cases} \\ f_k &= \begin{cases} \beta & \text{on } [z - r_k, z) = J, \\ \alpha_k \beta & \text{on } [z, z + r_k) = I, \end{cases} \\ f_k &= \begin{cases} \alpha_k \beta & \text{on } [z - r_k, z) = I, \\ \beta & \text{on } [z, z + r_k) = J. \end{cases} \end{aligned}$$

In each case

$$f \leq \beta \alpha_k = \beta \exp(-9^{-k-2}) \quad \text{on } I,$$

whereas

$$f \geq \beta \alpha_{k+1} \alpha_{k+2} \alpha_{k+3} \dots = \beta \exp\left(-\frac{9^{-k-2}}{8}\right) \quad \text{on } J \cap P.$$

Hence

$$(8) \quad |f - f(x)| \geq \frac{1}{2} \beta \left(\exp\left(-\frac{9^{-k-2}}{8}\right) - \exp(-9^{-k-2}) \right)$$

on at least one of the sets I or $J \cap P$. Since the infinite product at (7) converges, for sufficiently large k we have

$$|J \cap P| = |J| \left(1 - 2\frac{r_{k+1}}{r_k}\right) \left(1 - 2\frac{r_{k+2}}{r_{k+1}}\right) \dots > \frac{1}{2} |J| = \frac{r_k}{2},$$

and hence inequality (8) is satisfied on a set $E_k \subset [z - r_k, z + r_k]$ of length $|E_k| > r_k/2$. To estimate the right hand side of (8), observe that $e^{-x} - e^{-y} \geq e^{-y}(e^{y-x} - 1) \geq (1-y)(y-x)$, $0 < x < y$, and thus

$$\exp\left(-\frac{9^{-k-2}}{8}\right) - \exp(-9^{-k-2}) \geq \frac{7}{8} \left(1 - 9^{-k-2}\right) 9^{-k-2} > 4 \cdot 9^{-k-3}.$$

Accordingly

$$(9) \quad \frac{|f(y) - f(x)|}{|y - x|} \geq \frac{4\beta 9^{-k-3}}{4r_k} = \beta 3^{k^2 - k - 6} \quad \text{for } y \in E_k.$$

Set $E_n^* = \bigcup_{k=n}^{\infty} E_k$. Since

$$\limsup_{h \rightarrow 0^+} \frac{|E_n^* \cap (x - h, x + h)|}{2h} \geq \limsup_{k \rightarrow \infty} \frac{|E_k|}{4r_k} \geq \frac{1}{8},$$

the approximate limit-superior of $\frac{|f(y) - f(x)|}{|y - x|}$ as $y \rightarrow x$ is at least $\beta 3^{n^2 - n - 6}$ for each n , and thus it is ∞ . Hence f cannot be approximately differentiable at x . If x is a density point of P , then also $\mathcal{M}f$ cannot be approximately differentiable at x .

6.5. An integrable function. Finally let $\tilde{f} = f\chi_{(0,1)}$. Then $\mathcal{M}\tilde{f}(x) \leq \mathcal{M}f(x) \leq f(x) = \tilde{f}(x)$ in $P \cap (0, 1)$ and hence $\mathcal{M}\tilde{f}(x) = \tilde{f}(x) = f(x)$ a.e. in $P \cap (0, 1)$. Since f is not approximately differentiable on $P \cap (0, 1)$, $\mathcal{M}f$ cannot be approximately differentiable a.e.

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