curl and div

For a vectorfield $F = \langle P, Q, R \rangle$ we define

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} =$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

<u>Theorem</u> $\operatorname{curl}(\nabla f) = \vec{0}$.

Indeed

$$\operatorname{curl} \nabla f = \nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle$$

$$= \langle 0, 0, 0 \rangle.$$

Thus if $\vec{F}$ is conservative, i.e. $\vec{F} = \nabla f$, then

$$\operatorname{curl} \vec{F} = \operatorname{curl} \nabla f = 0.$$

It turns out that this property characterizes conservative vector fields in the following sense

Theorem A vector field $\vec{F}$ in $\mathbb{R}^3$ is conservative if and only if
$$\operatorname{curl} \vec{F} = 0.$$

For a vector field $\vec{F} = \langle P, Q, R \rangle$ we also define

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$
$$= \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}$$

Theorem $\operatorname{div} \operatorname{curl} \vec{F} = 0$

Indeed,
$$\operatorname{div} \operatorname{curl} \vec{F} = \operatorname{div} \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$
$$= \cancel{\frac{\partial^2 R}{\partial x \partial y}} - \cancel{\frac{\partial^2 Q}{\partial x \partial z}} + \cancel{\frac{\partial^2 P}{\partial y \partial z}} - \cancel{\frac{\partial^2 R}{\partial y \partial x}} + \cancel{\frac{\partial^2 Q}{\partial z \partial x}} - \cancel{\frac{\partial^2 P}{\partial z \partial y}}$$
$$= 0.$$

Example Show that the vector field $\vec{F} = \langle xz, xyz, -y^2 \rangle$ cannot be represented as curl of another vector field, i.e. there is no vector field $\vec{G}$ such that $\operatorname{curl} \vec{G} = \vec{F}$.

Suppose that such a vector field $\vec{G}$ exists

$$\operatorname{curl} \vec{G} = \vec{F}.$$

Then

$$\operatorname{div} \vec{F} = \operatorname{div} \operatorname{curl} \vec{G} = 0,$$

but on the other hand

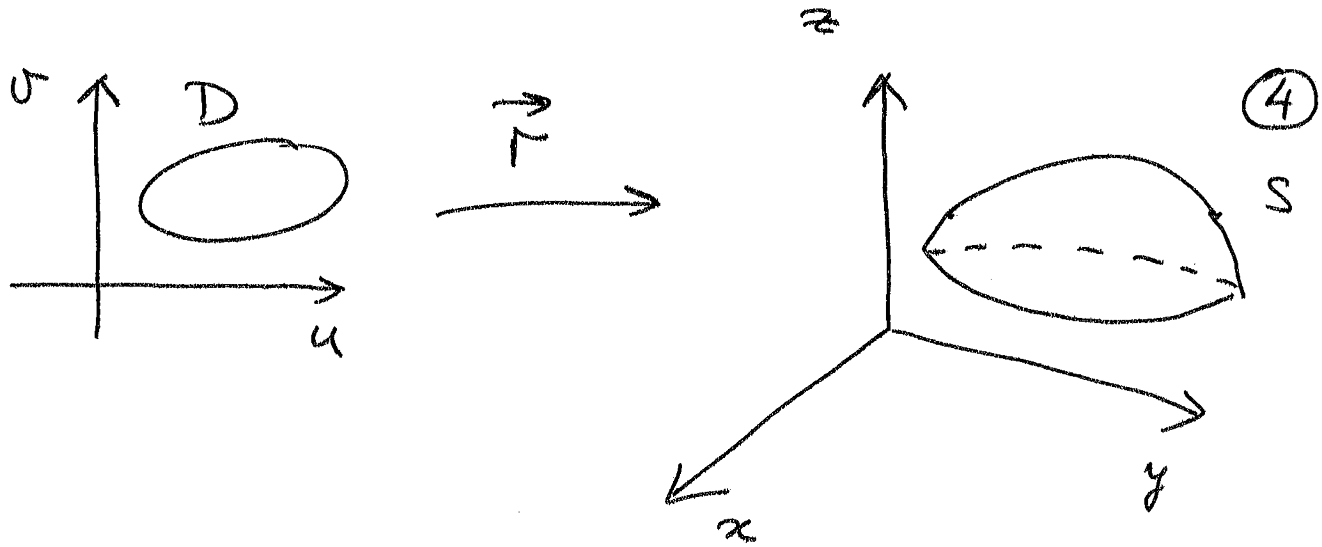
$$\operatorname{div} \vec{F} = z + xz \neq 0.$$

Parametric surfaces

Suppose that

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

is a vector valued function defined on a planar domain D in the uv plane.



We assign points $\vec{r}(u,v)$ in $\mathbb{R}^3$ to points (u,v) in $\mathbb{R}^2$. In general the image of $\vec{r}(u,v)$ will be a 2-dimensional surface S in $\mathbb{R}^3$. It consists of points

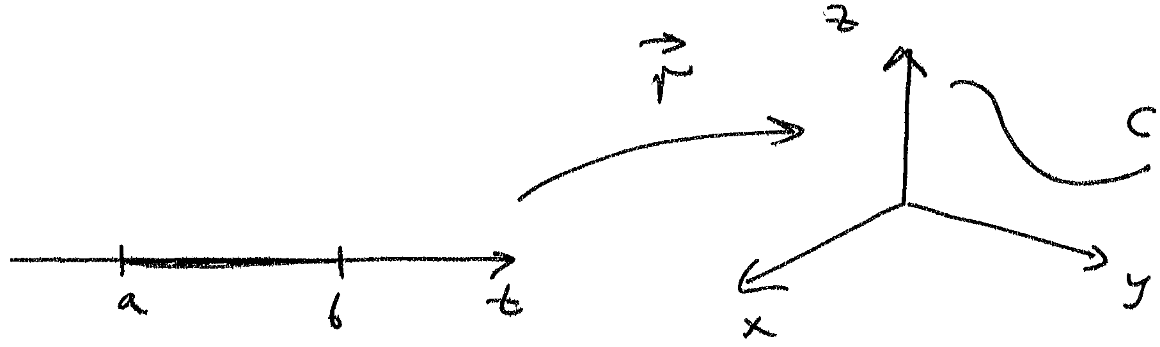
$$S = \left\{ (x,y,z) \middle| \begin{array}{l} x = x(u,v),\ y = y(u,v),\ z = z(u,v) \\ \text{for some } (u,v) \in D \end{array} \right\}.$$

S is called a <u>parametric surface</u> and

$$x = x(u,v),\quad y = y(u,v),\quad z = z(u,v)$$

are called <u>parametric equations of S.</u>

Let us compare this definition with the definition of a parametric curve in $\mathbb{R}^3$



In the case of a parametric curve, $\vec{r}$ is defined on a one dimensional segment, so its image is a one dimensional curve. The definition of a parametric surface is pretty similar, but now $\vec{r}$ is defined on a two dimensional domain, so its image is a 2-dimensional surface. Let us start with examples.

<u>Example</u> The graph of a function

$$z = f(x,y), \quad (x,y) \in D \text{ in } \mathbb{R}^2$$

is clearly a surface. It consists of points

$$\langle x, y, f(x,y) \rangle \text{ for } (x,y) \in D.$$

Hence the graph can be represented as a parametric surface

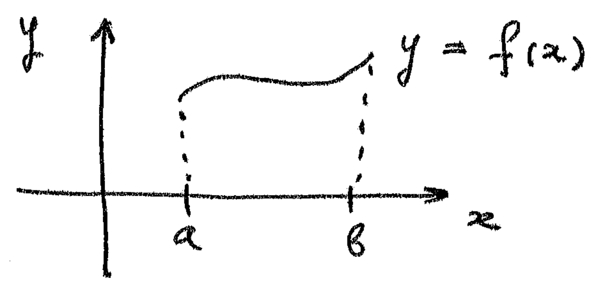
$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle, \quad (x,y) \in D.$$

In a general definition of a parametric surface we used variables (u,v), but there is no reason why we shouldn't be allowed to use variables (x, y) instead of (u, v).

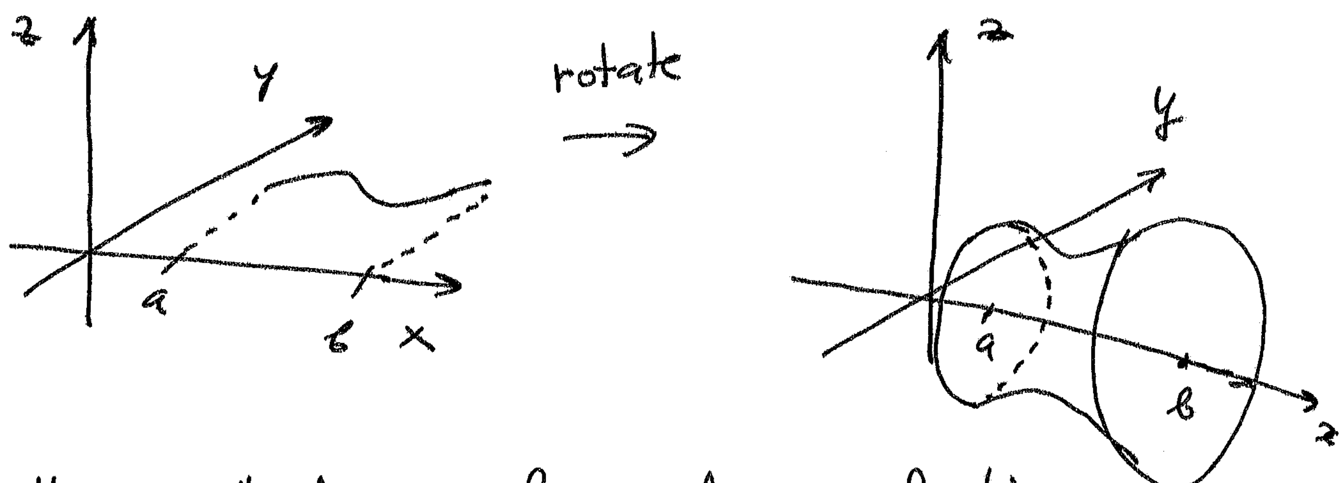
<u>Example</u> Consider the graph of a function

$$y = f(x), \quad x \in [a, b]$$

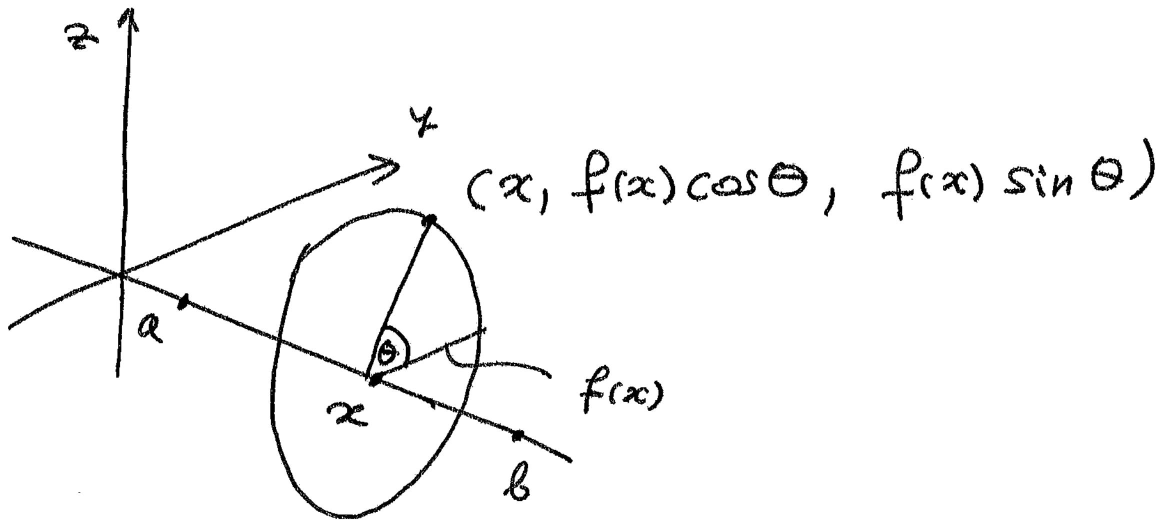
and assume that $f > 0$.



If we rotate the graph about the x-axis in the xyz space we will obtain a surface



It is called a surface of revolution.
Now we will show how to represent the surface of revolution as a parametric surface.



Fix $a \le x \le b$. The point $(x, f(x))$ in the xy plane, i.e. a point on the graph of f will rotate along the circle centered at $(x, 0, 0)$ of radius $r = f(x)$ in the plane parallel to the ~~x~~ yz plane. Such a circle can be parametrized by

$$\theta \longmapsto \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle.$$

We obtain our surface of revolution if we take all such circles for $a \le x \le b$. Hence this surface can be parametrized by

$$\vec{r}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$$

$$a \le x \le b, \; 0 \le \theta \le 2\pi.$$

Example Use cylindrical coordinates to find a parametrization of the surface
$$z = 3 + x^2 + y^2.$$

Solution $z = 3 + x^2 + y^2 = 3 + r^2$

In the cylindrical coordinates
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 3 + r^2 \end{cases}$$

Hence
$$\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, 3 + r^2 \rangle$$
$$r \geq 0, \quad 0 \leq \theta \leq 2\pi$$
is a parametrization of this surface.

Remark this is a surface of a graph of a function and hence it has a parametrization
$$\vec{\omega}(x,y) = \langle x, y, 3 + x^2 + y^2 \rangle.$$
$$-\infty < x < \infty, -\infty < y < \infty.$$

This is just a different parametrization of the same surface. In general a surface has infinitely many different parametrizations.

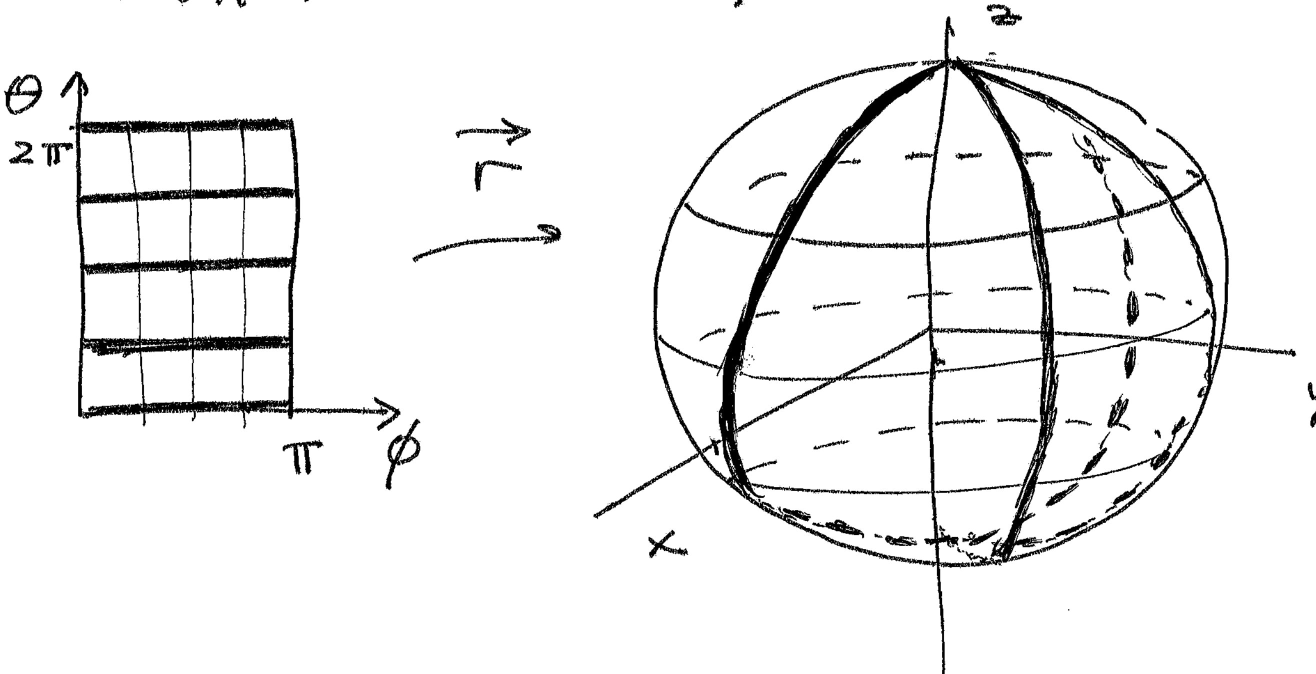
Example Find a parametrization of the sphere
$$x^2 + y^2 + z^2 = a^2$$

Solution We will use the spherical coordinates to parametrize the sphere. Note that $\rho = a$ is fixed so

$$\begin{cases} z = a\cos\phi \\ x = a\sin\phi\cos\theta \\ y = a\sin\phi\sin\theta \end{cases} \qquad \begin{array}{l} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{array}$$

and the parametrization is

$$\vec{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi \rangle.$$



This picture is terrible, but try to understand what are the images of thin and thick lines parallel to the θ and and ϕ axes. You will see that we just created a map of the Earth.

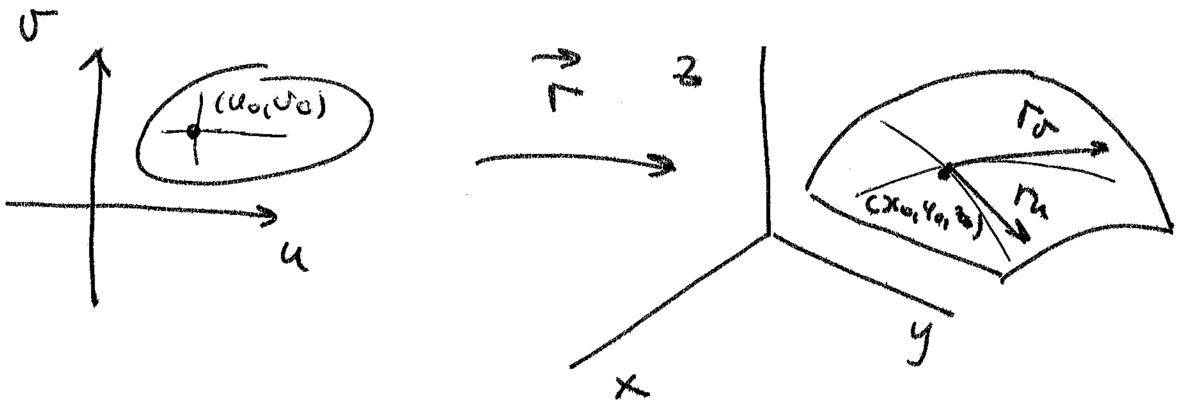
Tangent planes

If $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ is a parametric surface, then

$$\vec{r_u}(u_0, v_0) = \langle x_u(u_0, v_0), y_u(u_0, v_0), z_u(u_0, v_0) \rangle$$

$$\vec{r_v}(u_0, v_0) = \langle x_v(u_0, v_0), y_v(u_0, v_0), z_v(u_0, v_0) \rangle$$

are tangent vectors to the surface at the point $(x_0, y_0, z_0) = \vec{r}(u_0, v_0)$



The normal vector is

$$\vec{r_u} \times \vec{r_v}(u_0, v_0)$$

Once we know the normal vector, say

$$\vec{r_u} \times \vec{r_v}(u_0, v_0) = \langle a, b, c \rangle$$

we can write the equation of the tangent plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

However to do this we must know that

$$\vec{r_u} \times \vec{r_v}\,(u_0, v_0) \neq \vec{0}$$

otherwise the tangent plane may not exist. This justifies the definition:

We say that a parametric surface is smooth if

$$\vec{r_u} \times \vec{r_v} \neq \vec{0} \quad \text{for all } (u,v) \in D$$

This guarantees the existence of the normal vector and hence the tangent plane at every point of the surface.

Example Find the tangent plane to the parametric surface

$$x = u^2,\ y = v^2,\ z = u + 2v \text{ at } (1,1,3)$$

Solution $\vec{r_u} = \langle 2u, 0, 1 \rangle$, $\vec{r_v} = \langle 0, 2v, 2 \rangle$

$$\vec{r_u} \times \vec{r_v} = \langle -2v, -4u, 4uv \rangle.$$

The surface is not smooth at

$\vec{r}(0,0) = \langle 0, 0, 0 \rangle$ only

Since $\vec{r}(1,1) = \langle 1, 1, 3 \rangle$,

the tangent plane at $(1,1,3)$ has

the normal vector

$$\vec{r}_u \times \vec{r}_v\,(1,1) = \langle -2, -4, 4 \rangle$$

and hene the equation of the tangent plane is

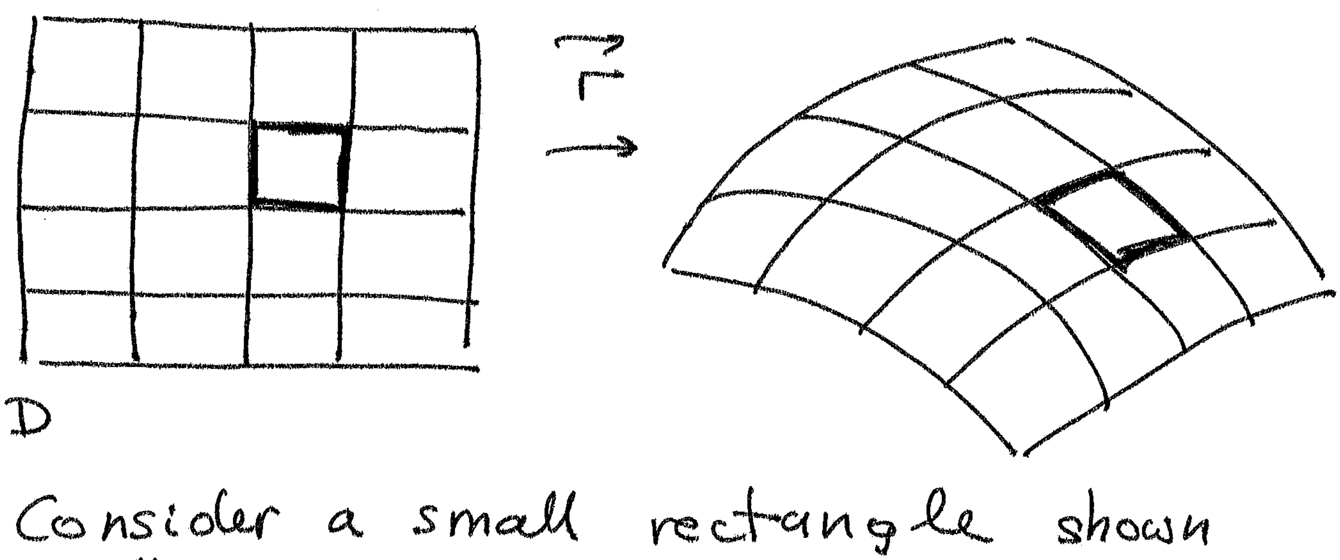
$$-2(x-1) - 4(y-1) + 4(z-3) = 0.$$

Surface area

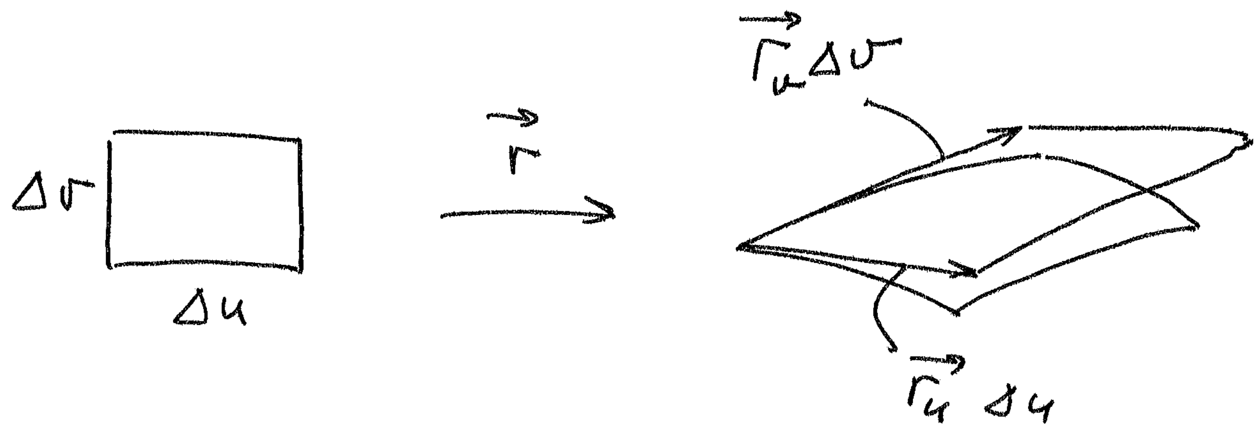
Consider a parametric surface

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$
$$(u,v) \in D$$

Assume for simplicity that D is a rectangle. Do the partition of the rectangle into small rectangles $\Delta u \times \Delta v$



Consider a small rectangle shown on the picture



If the rectangle $\Delta u \times \Delta v$ is rey small, the curred rectangle in the image of $\vec{r}$ is almost flat and hence its area is very well approximated by the tangent paralleogram with sides

$$\vec{r}_u \Delta u \quad \text{and} \quad \vec{r}_v \Delta v$$

whose area is

$$|(\vec{r}_u \Delta u) \times (\vec{r}_v \Delta v)| = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

If we add the areas of all curred rectangls we will obtain the area of the entire surface which is well approximated by the Riemann sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} |\vec{r}_u(u_i, v_i) \times \vec{r}_v(u_i v_i)| \Delta u_i \Delta v_i$$

This is, however the Riemann sum of the double integral

$$\iint_D |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

Thus we provided a heuristic argument for the following fact

The area of the parametric surface S, $\vec{r}(u,v)$, $(u,v) \in D$ equals

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

This is true for any domain D, not necessarily a rectangle.

<u>Example</u> Use the spherical coordinates parametization of $x^2+y^2+z^2=a^2$ to find the surface area of the sphere of radius a.

<u>Solution</u> We have

$$\vec{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, \, a\sin\phi\sin\theta, \, a\cos\phi \rangle$$

$$D = \{ (\phi,\theta) \mid 0 \le \phi \le \pi, \; 0 \le \theta \le 2\pi \}$$

Easy calculation shows that

$r_\phi = \langle a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi \rangle$

$r_\theta = \langle -a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0 \rangle$

$r_\phi \times r_\theta = \langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\sin\phi\cos\phi \rangle$

$|r_\phi \times r_\theta| = \sqrt{a^4\sin^4\phi\,\underline{\cos^2\theta} + a^4\sin^4\phi\,\underline{\sin^2\theta} + a^4\sin^2\phi\cos^2\phi}$

$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi}$

$= \sqrt{a^4\sin^2\phi\,\underline{\sin^2\phi} + a^4\sin^2\phi\,\underline{\cos^2\phi}}$

$= \sqrt{a^4\sin^2\phi} = a^2\sin\phi$

Note that $\sin\phi \geq 0$ because $0 \leq \phi \leq \pi$

Thus

$$A = \iint_D |r_\phi \times r_\theta|\, d\phi\, d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta = a^2 \underbrace{\int_0^{2\pi} d\theta}_{2\pi}\ \underbrace{\int_0^{\pi}\sin\phi\, d\phi}_{2} = \boxed{4\pi a^2}$$

<u>Example</u> The graph of

$$z = f(x,y), \quad (x,y) \in D$$

has a parametrization

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle, \quad (x,y) \in D.$$

$$\vec{r}_x = \langle 1, 0, f_x \rangle, \quad \vec{r}_y = \langle 0, 1, f_y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + f_x^2 + f_y^2}$$

Hence the area of the graph of f equals

$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$$

Recall that the surface area of the parametric surface $\vec{r}(u,v)$ is

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv.$$

Suppose now that a function $f(x,y,z)$ is defined at all points (x,y,z) of the surface S. Then we define the integral of f on S as follows

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv.$$

This is a natural definition, because if $f = 1$, we will obtain the surface area of S

$$\iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v|\,du\,dv = A(S).$$

Compare this definition with the definition of the integral along a parametric curve

$$A(S) = \iint_D |r_u \times r_v|\,du\,dv \qquad \ell(C) = \int_a^b |r'(t)|\,dt$$

$$\iint_S f\,dS = \iint_D f(\vec{r}(u,v))\,|r_u \times r_v|\,du\,dv \qquad \int_C f\,ds = \int_a^b f(\vec{r}(t))\,|r'(t)|\,dt$$

There is a clear analogy which should help you memorize the formulas.

<u>Example</u> If the surface S is the graph of $z = g(x,y)$, $(x,y) \in D$, then

$$\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$$

is a parametrization of S and

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + g_x^2 + g_y^2}$$

Hence for a function $f(x,y,z)$ defined on the graph of $z = g(x,y)$ we have

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\, dx\, dy}$$

In order to remember this formula you need just to remember that

$$dS = \sqrt{1+g_x^2+g_y^2}\, dx\, dy .$$

Change of variables

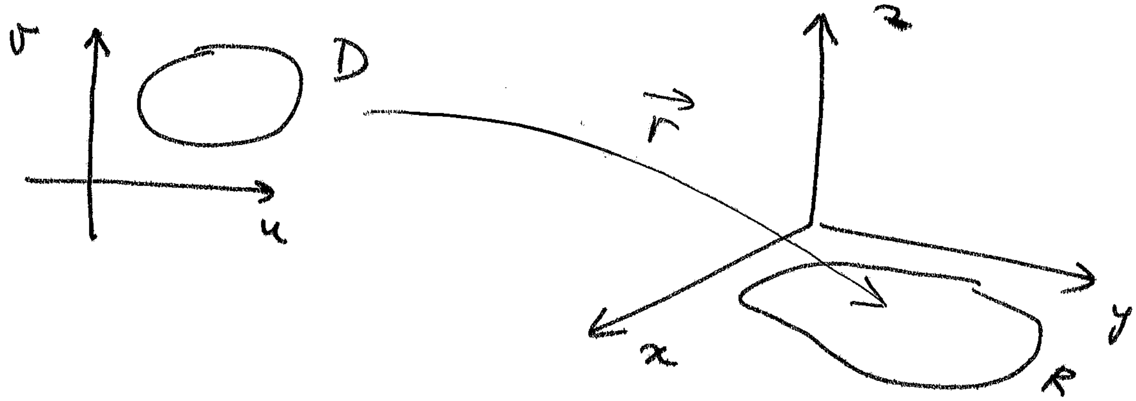
Consider a mapping

$$\vec{r}(u,v) = \langle x(u,v), y(u,v) \rangle$$

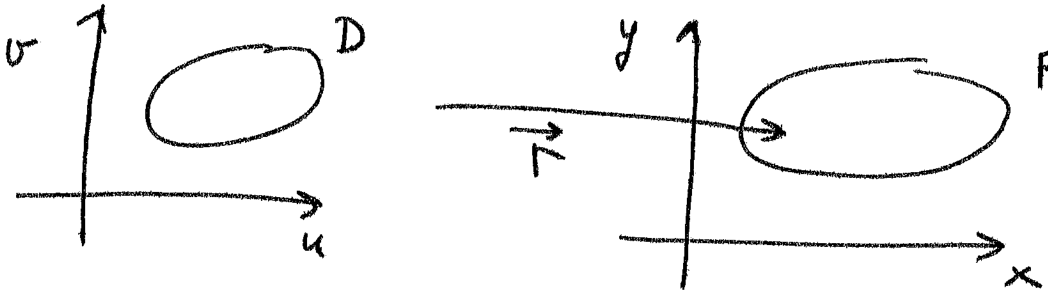
We can regard it as a parametric surface in $\mathbb{R}^3$ by writing

$\vec{r}(u,v) = \langle x(u,v), y(u,v), 0 \rangle.$

Thus this is a flat surface which is contained in the coordinate xy plane



or if we neglect the z-axis and just look at the xy-plane



We assume that the function $\vec{r}(u,v)$ is one-to-one and that $|r_u \times r_v| \neq 0$, because we want it to be a smooth parametric surface. According to the formula for the integration on surfaces

$$(*) \quad \iint_R f(x,y)\,dx\,dy = \iint_D f(x(u,v), y(u,v))\,|r_u \times r_v|\,du\,dv$$

$$r_u = \langle x_u, y_u, 0 \rangle$$

$$r_v = \langle x_v, y_v, 0 \rangle$$

$$r_u \times r_v = \langle 0, 0, x_u y_v - x_v y_u \rangle$$

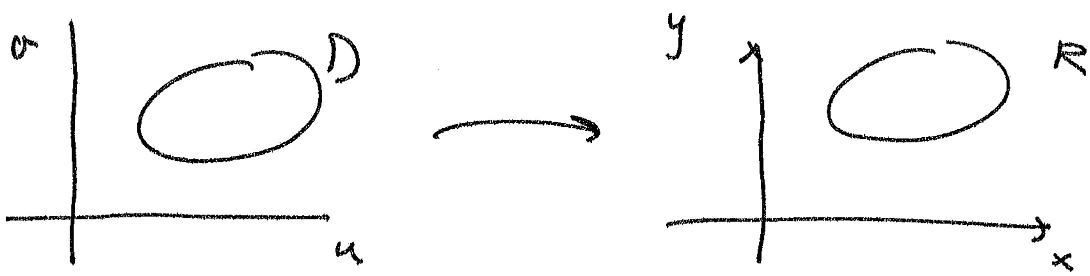
$$|r_u \times r_v| = |x_u y_v - x_v y_u|$$

We introduce notation

$$\frac{\partial(x,y)}{\partial(u,v)} = \underbrace{\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}}_{\text{determinant}} = x_u y_v - x_v y_u$$

and we call it the Jacobian of the transformation

$$x = x(u,v), \quad y = y(u,v)$$



since $\quad |r_u \times r_v| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$

the formula (*) can be written as

$$\boxed{\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_D f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv}$$

which is a general formula for the change of variables in the integral.

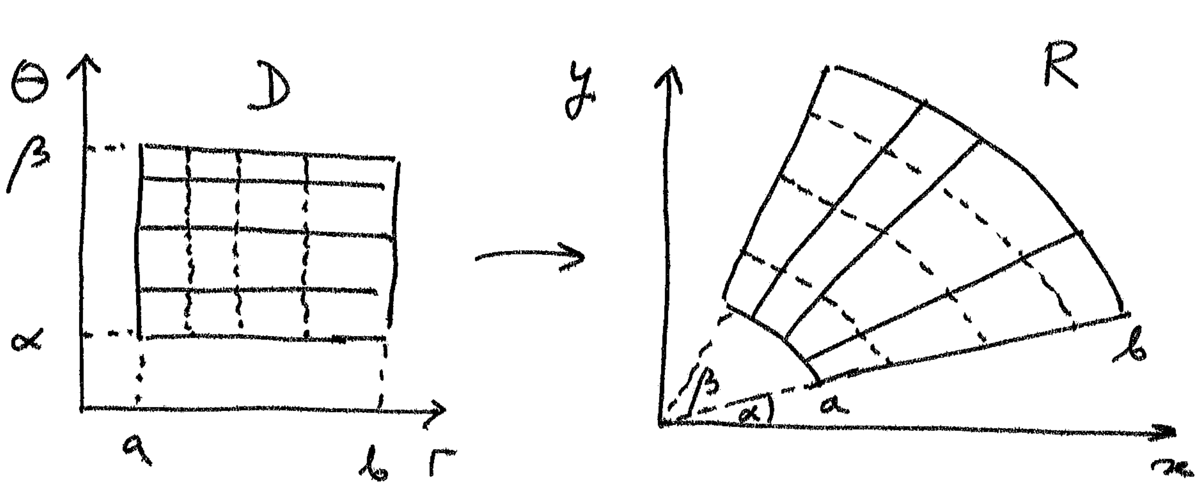
<u>Example</u> The formula for the integration in polar coordinates is a special case of this formula

$$x = x(r,\theta), \quad y = y(r,\theta),$$

i.e.

$$x = r\cos\theta, \quad y = r\sin\theta$$

is actually a transformation: to each point (r,θ) in the $r\theta$ plane we associate a point (x,y) in the xy plane. For example:



This picture shows also how the lines in the $r\theta$ plane are transformed into lines and circular arcs in the xy plane.

We have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

Hence

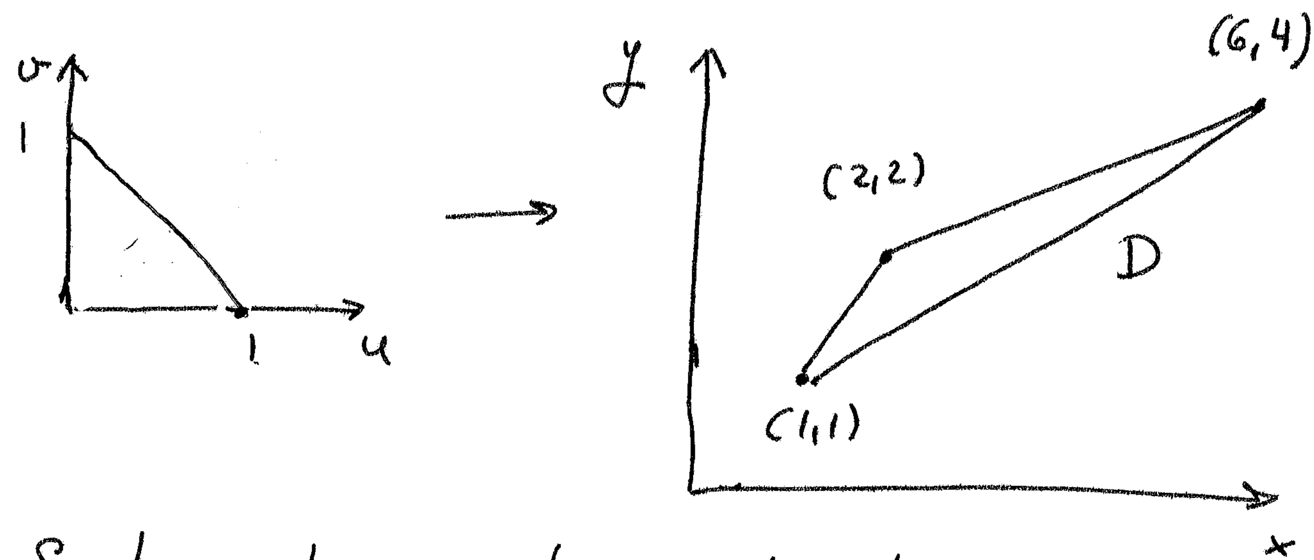
$$\iint_R f(x,y) = \iint_D f(x(r,\theta), y(r,\theta)) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\, d\theta$$

$$= \int_\alpha^\beta \int_a^b f(x\cos\theta, y\sin\theta)\, r\, dr\, d\theta .$$

Example Find the double integral

$\iint_D x^2-y^2\,dx\,dy$, where D is the triangle with vertices $(1,1)$, $(2,2)$, $(6,4)$.

Solution We are looking for a linear change of coordinates that sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(6,4)$ and $(0,1)$ to $(2,2)$



Such a change of coordinates will be of the form

$$x = Au + Bv + C \qquad y = Du + Ev + F$$

To find coefficients A, B, C, D, E, F we need to solve equations

$$(1,1) = (A\cdot 0 + B\cdot 0 + C,\ D\cdot 0 + E\cdot 0 + F)$$
$$(6,4) = (A\cdot 1 + B\cdot 0 + C,\ D\cdot 1 + E\cdot 0 + F)$$
$$(2,2) = (A\cdot 0 + B\cdot 1 + C,\ D\cdot 0 + E\cdot 1 + F)$$

i.e.

$$(1,1) = (C, F)$$
$$(6,4) = (A+C, D+F)$$
$$(2,2) = (B+C, E+F).$$

Hence

$$C = 1, \quad F = 1$$
$$A = 6 - C = 5 \qquad D = 4 - F = 3$$
$$B = 2 - C = 1 \qquad E = 2 - 1 = 1$$

i.e.

$$x = 5u + v + 1$$
$$y = 3u + v + 1.$$

We have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 5 & 1 \\ 3 & 1 \end{vmatrix} = 2$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = 2.$$

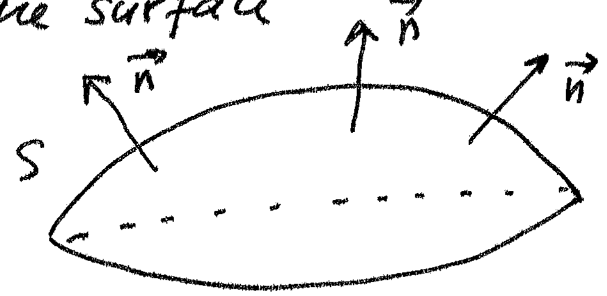
Thus the change of variables formula yields

$$\iint_D x^2-y^2\,dx\,dy = \iint \left(x(u,v)^2 - y(u,v)^2\right)\cdot 2\,du\,dv$$

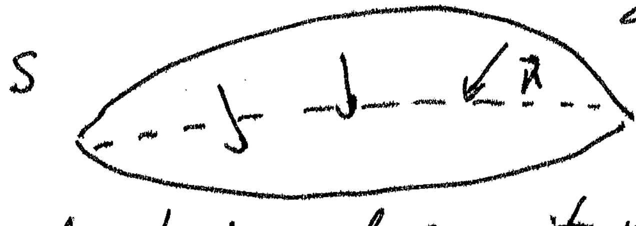
$$= \int_0^1 \int_0^{1-u} \left((5u+v+1)^2 - (3u+v+1)^2\right)\cdot 2\,dv\,du = \frac{13}{3}.$$

Oriented surfaces

In order to integrate a vector field on a surface S we will need to choose a vector field of unit normal vectors $\vec{n}$ on the surface



We can choose such a vector field in two different ways



Here is another choice for the same surface

A choice of a unit normal vector field

on S is called an orientation of the surface. Thus a surface has two different orientations. (There are however surfaces such as the Möbius band that have no orientation because we cannot choose a normal vector field to be continuous on the entire surface)

A surface with a choosen normal vector field is called an oriented surface.

If S is a closed surface, i.e. if it is the whole boundary of a 3D solid, we choose the outward normal vector field and we call it a positive orientation. Thus if in a problem no orientation of a closed surface is mentioned, it is assumed that the surface is equipped with the outward normal vector field. However, in some problems it may be stated that the orientation is inward.

<u>Example</u> On the sphere $x^2+y^2+z^2=r^2$ of radius r, the radius is orthogonal to the sphere, but it has length r. Thus at the point (x,y,z) the outward unit normal vector is

$$\vec{n} = \frac{\langle x,y,z\rangle}{r}.$$

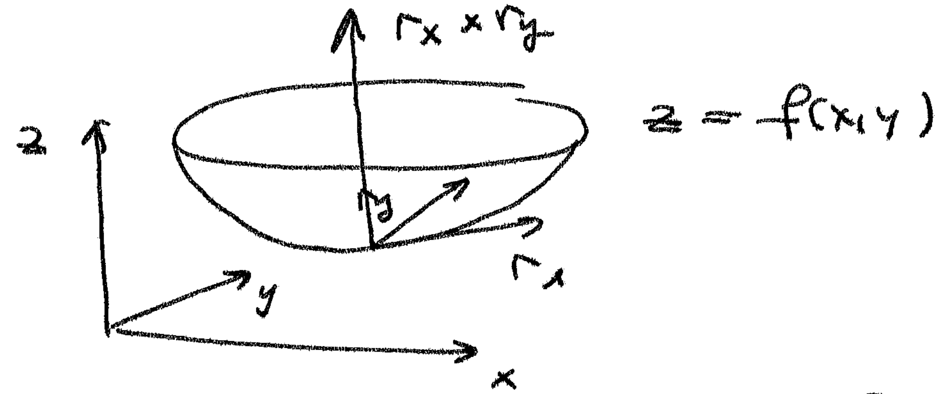
If S is the graph of $z = f(x,y)$, then

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

is a parametrization of S and

$$\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle$$

is a vector orthogonal to the surface that points in the upward direction, i.e. in the positive direction of the z-axis



To be more precise the vector $\vec{r}_x \times \vec{r}_y$ is not necessarily parallel to the z-axis, but its $\vec{k}$ component (equal 1) is positive and this is what we mean when we say that it points in the positive direction of the z-axis. The vector $\vec{r}_x \times \vec{r}_y$ is not necessarily of the unit length, so the upward normal vector field is

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{|r_x \times r_y|} = \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+f_x^2+f_y^2}}.$$

Usually the graph of a function is equipped with the upward orientation, but you may expect problems that explicitely assume downward orientation which is

$$\vec{n} = -\frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+f_x^2+f_y^2}} = \frac{\langle f_x, f_y, -1\rangle}{\sqrt{1+f_x^2+f_y^2}}.$$

If S is a parametric surface with a parametrization $\vec{r}(u,v)$, then

$$\vec{n} = \frac{\vec{r_u}\times\vec{r_v}}{|r_u \times r_v|}$$

is a unit normal vector field. However, we need to make sure that this normal vectorfield is consistent with the orientation of the surface.

<u>Example</u> Represent the outward orientation of the sphere $x^2+y^2+z^2=a^2$ is spherical coordinates.

<u>Solution</u> $\vec{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$

$\vec{r_\phi}\times\vec{r_\theta} = \langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi\rangle$

$$\vec{n} = \frac{\vec{r}_\phi \times \vec{r}_\theta}{|r_\theta \times r_\theta|} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$$

(for the calculations see page 15). We could, however, get this answer immediately with the following argument:

The outward unit normal vector at the point (x, y, z) on the sphere of radius a is

$$\vec{n} = \frac{\langle x, y, z \rangle}{a} \qquad \text{(see p. 26)}$$

If $(x, y, z) = \vec{r}(\phi, \theta)$, then

$$\vec{n} = \frac{\langle x, y, z \rangle}{a} = \frac{\vec{r}(\phi, \theta)}{a} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle.$$

The Flux

If $\vec{F}(x, y, z)$ is a vector field and S is a surface equipped with an orientation $\vec{n}$ (unit normal vector field on S) then we define the flux of $\vec{F}$ accross S by

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS.$$

This is how we integrate vector fields

on oriented surfaces. Note that $\vec{F}\cdot\vec{n}$ is a function, not a vector field and the integral

$$\iint_S \vec{F}\cdot\vec{n}\, dS$$

is just an integral of a function on the surface S, the integral that we discussed on page 17.

Recall that if f is a function on a parametric surface, then

$$\iint_S f\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, du\, dv .$$

In our situation $f = \vec{F}\cdot\vec{n}$ and

$$\vec{n} = \frac{r_u \times r_v}{|r_u \times r_v|} .$$ Hence

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D \underbrace{\vec{F}(\vec{r}(u,v))\cdot \frac{\vec{r}_u \times \vec{r}_v}{|r_u \times r_v|}}_{f}\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\, du\, dv .$$

Thus you need to remember that

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv}$$

Example Find the flux of $\vec{F} = \langle z, y, x\rangle$ accross the unit sphere $x^2+y^2+z^2=1$

Solution 1 The spherical parametrization of the sphere is

$$\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi\rangle$$

$$\vec{F}(\vec{r}(\phi,\theta)) = \langle \underbrace{\cos\phi}_{z},\ \underbrace{\sin\phi\sin\theta}_{y},\ \underbrace{\sin\phi\cos\theta}_{x}\rangle$$

$$\vec{r}_\phi\times\vec{r}_\theta = \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi\rangle$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\phi,\theta))\cdot(\vec{r}_\phi\times\vec{r}_\theta)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\theta\cos\phi\right)d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta\right) d\phi\, d\theta$$

$$= \underbrace{\int_0^{2\pi}\cos\theta\, d\theta}_{0} \int_0^{\pi} 2\sin^2\phi\cos\phi\, d\phi + \int_0^{\pi}\sin^3\phi\, d\phi \int_0^{2\pi}\sin^2\theta\, d\theta = \heartsuit$$

$$\int_0^{2\pi}\sin^2\theta\, d\theta = \int_0^{2\pi}\cos^2\theta\, d\theta, \quad \int_0^{2\pi}\sin^2\theta\, d\theta = \frac{1}{2}\int_0^{2\pi}\underbrace{\sin^2\theta + \cos^2\theta}_{1}\, d\theta = \pi$$

$$\heartsuit = \pi\int_0^{\pi}\sin^3\phi\, d\phi = \pi\int_0^{\pi}\sin\phi\left(1-\cos^2\phi\right) d\phi$$

$$= \pi\left(\underbrace{\int_0^{\pi}\sin\phi\, d\phi}_{2} - \int_0^{\pi}\sin\phi\cos^2\phi\, d\phi\right) =$$

$$= \pi\left(2 - \left.\frac{\cos^3\phi}{-3}\right|_0^{\pi}\right) = \pi\left(2 - \left(\frac{-1}{-3} - \frac{1}{-3}\right)\right)$$

$$= \pi\left(2 - \frac{2}{3}\right) = \boxed{\frac{4\pi}{3}}$$

Solution II This solution is short, but tricky. We know that on the unit sphere the outward unit normal vector field is $\vec{n} = \langle x, y, z \rangle$. Hence

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \langle z, y, x \rangle \cdot \langle x, y, z \rangle\, dS$$

$$= \iint_S 2xz + y^2\, dS. = \heartsuit$$

If we split S into upper S_+ and lower S_- hemisphere, then

$$\iint_{S_+} xz\, dS = -\iint_{S_-} xz\, dS$$

because z will change sign, but x will remain the same. Hence

$$\iint_S 2xz\, dS = 2\left(\iint_{S_+} xz\, dS + \iint_{S_-} xz\, dS \right) = 0.$$

Also using a symmetry argument

$$\iint_S y^2\, dS = \frac{1}{3} \iint_S \underbrace{x^2 + y^2 + z^2}_{1}\, dS = \frac{1}{3} \underbrace{4\pi}_{\text{area of } S}.$$

Hence

$$\heartsuit = \boxed{\frac{4\pi}{3}}$$

Later we will see one more solution based on the divergence theorem. (pp. 39-40).

Now we will show how to find the flux of $\vec{F}(x,y,z)$ accross the surface of the graph of

$$z = g(x,y), \quad (x,y) \in D.$$

We assume that the graph is oriented upward.

We have

$$\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y)\, dx\,dy$$

$$= \iint_D \vec{F}(x,y,g(x,y)) \cdot \langle -g_x, -g_y, 1 \rangle \, dx\,dy$$

$$= \iint_D -Pg_x - Q g_y + R \, dx\,dy.$$

The easiest way to remember this formula is to memorize that for a graph we have

$$\boxed{\vec{dS} = \langle -g_x, -g_y, 1 \rangle \, dx\,dy}$$

Then we have

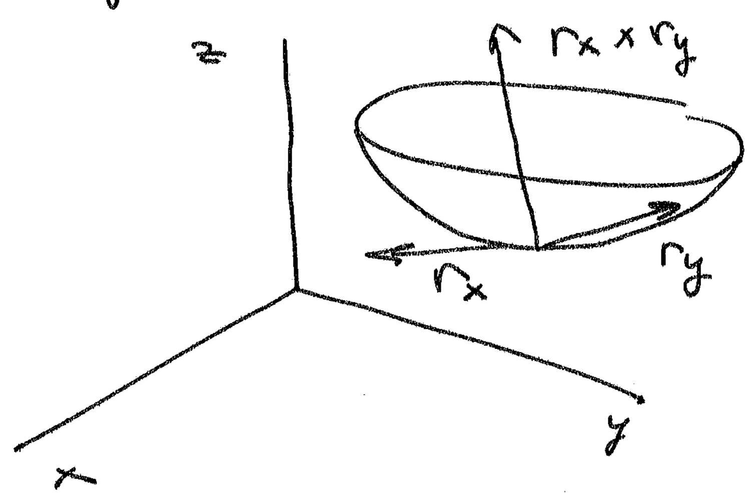
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle \, dx\,dy$$

$$= \iint_D -Pg_x - Qg_y + R \;\; dx\,dy \, .$$

Recall that the vector

$$\vec{r_x} \times \vec{r_y} = \langle -g_x, -g_y, 1 \rangle$$

has upward orientation



Suppose now that we have a surface

$$y = g(x, z) \, .$$

The difference is that z is replaced by y and y is replaced by z. Hence by analogy we should have

$$d\vec{S} = \langle -g_x, 1, -g_z \rangle \, dx\, dz \qquad (*)$$

Let us check it carefully.

$$\vec{r}(x,z) = \langle x, g(x,z), z \rangle$$

is a parametrization.

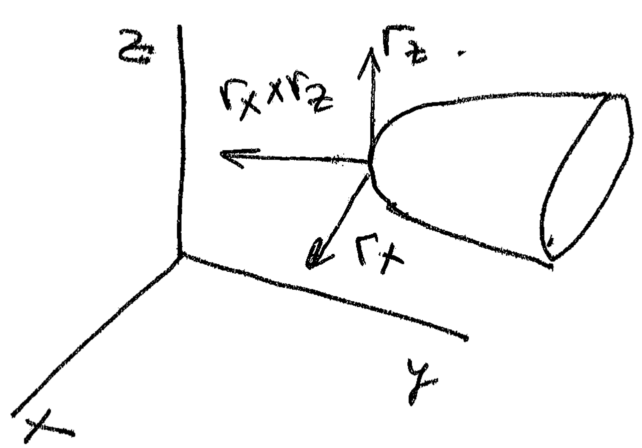
$$\vec{r}_x = \langle 1, g_x, 0 \rangle$$
$$\vec{r}_z = \langle 0, g_z, 1 \rangle$$

and

$$\vec{r}_x \times r_z = \begin{vmatrix} i & j & k \\ 1 & g_x & 0 \\ 0 & g_z & 1 \end{vmatrix} = \langle g_x, -1, g_z \rangle.$$

That looks wrong, because the sign is different than in $(*)$. Where is a mistake?

Let us look at the picture



The problem is that the vector $r_x \times r_z$ has the downward orientation with respect to the y-axis. To obtain a vector with the upward orientation (and this is what we want) we need to take

$$\vec{r}_z \times \vec{r}_x = \langle -g_x, 1, -g_z \rangle$$

and indeed, the formula

$$\boxed{d\vec{S} = \langle -g_x, 1, -g_z \rangle \, dx\, dz}$$

is correct

Exercise Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = y\vec{j} - z\vec{k}$ and S is the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ oriented upward (i.e. in the positive direction of the y-axis).

Solution The paraboloid is the graph of

$$g(x,z) = x^2 + z^2$$

over the unit disc

$$D = \{ (x,z) \mid x^2 + z^2 \le 1 \}.$$

As we already explained

$$d\vec{S} = \langle -g_x, 1, -g_z \rangle \, dx\,dz$$

and hence

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle 0, y, -z\rangle \cdot \langle -2x, 1, -2z\rangle \, dx\,dz$$

$$= \iint_D \underbrace{y}_{x^2+z^2} + 2z^2 \, dx\,dz = \iint_D x^2 + 3z^2 \, dx\,dz = \heartsuit$$

In polar coordinates

$$x^2 + 3z^2 = \underbrace{x^2 + z^2}_{r^2} + 2z^2 = r^2 + 2r^2\sin^2\theta$$

$$\heartsuit = \int_0^{2\pi}\int_0^1 (r^2 + 2r^2\sin^2\theta)\, r\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 (1+2\sin^2\theta)\, r^3\,dr\,d\theta =$$

$$= \frac{1}{4}\int_0^{2\pi} 1 + 2\sin^2\theta \, d\theta = \clubsuit$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} \underbrace{\sin^2\theta + \cos^2\theta}_{1} \, d\theta = \pi$$

$$\clubsuit = \frac{1}{4}\left(2\pi + 2\cdot\pi\right) = \pi.$$

Later we will see another solution based on the divergence theorem.

The Divergence Theorem

Theorem Let E be a solid with piecewise smooth boundary that has positive (outward) orientation. Let $\vec{F}$ be a vector field defined in a domain that contains E. Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_E \operatorname{div} \vec{F}\, dV$$

The divergence theorem tells us how to compute the flux accross a closed surface using tripple integrals.

Example Find the flux of $\vec{F} = \langle z, y, x\rangle$ accross the unit sphere $x^2+y^2+z^2=1$.

On pages 31–33 we have already seen two different solutions to this problem. Now we will see the third one

Solution III $\quad \operatorname{div} \vec{F} = \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} = 1$

The unit sphere is the boundary of the unit ball B. Hence

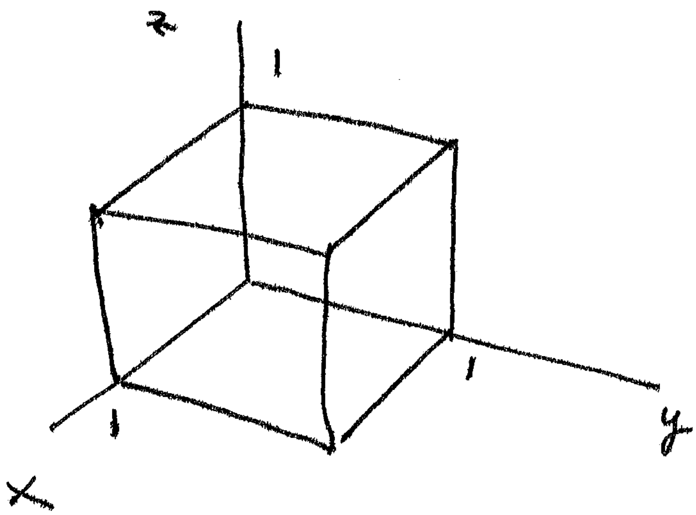
$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_B \operatorname{div} \vec{F} \, dV = \iiint_B dV = \frac{4\pi}{3}$$

(formula for the volume of the unit ball).

Example Find the flux of

$\vec{F} = (3x + 2yz)\vec{i} + (2x - y + z)\vec{j} + (x - 3y + 2z)\vec{k}$

accross the surface of the unit cube in the first octant



<u>Solution</u> By the divergence theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV = \diamondsuit$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(3x+2yz) + \frac{\partial}{\partial y}(2x-y+z) + \frac{\partial}{\partial z}(x-3y+2z)$$

$$= 3 - 1 + 2 = 4.$$

$$\diamondsuit = \iiint_E 4 \, dV = 4.$$

<u>Example</u> Find the flux of the vector field from the previous example accross all five sides of the cube except the top one

<u>Solution</u> The flux equals the flux accross the boundary of the whole cube which is 4 by the divergence theorem (see above) minus the flux accross the top surface of the cube. We will compute the flux accross the top side using a direct parametrization.

$\vec{r}(x,y) = \langle x, y, 1\rangle$, $0 \le x \le 1$, $0 \le y \le 1$

is a parametrization of the top side.

The normal vector is $\vec{n} = \vec{k}$ so $d\vec{S} = \vec{k}\,dx\,dy$

$$\iint_{\text{Top}} \vec{F}\cdot d\vec{S} = \int_0^1\!\!\int_0^1 \vec{F}\cdot\vec{k}\,dx\,dy =$$

$$\int_0^1\!\!\int_0^1 (x - 3y + \underset{z=1}{\underset{\uparrow}{2}})\,dx\,dy = 1$$

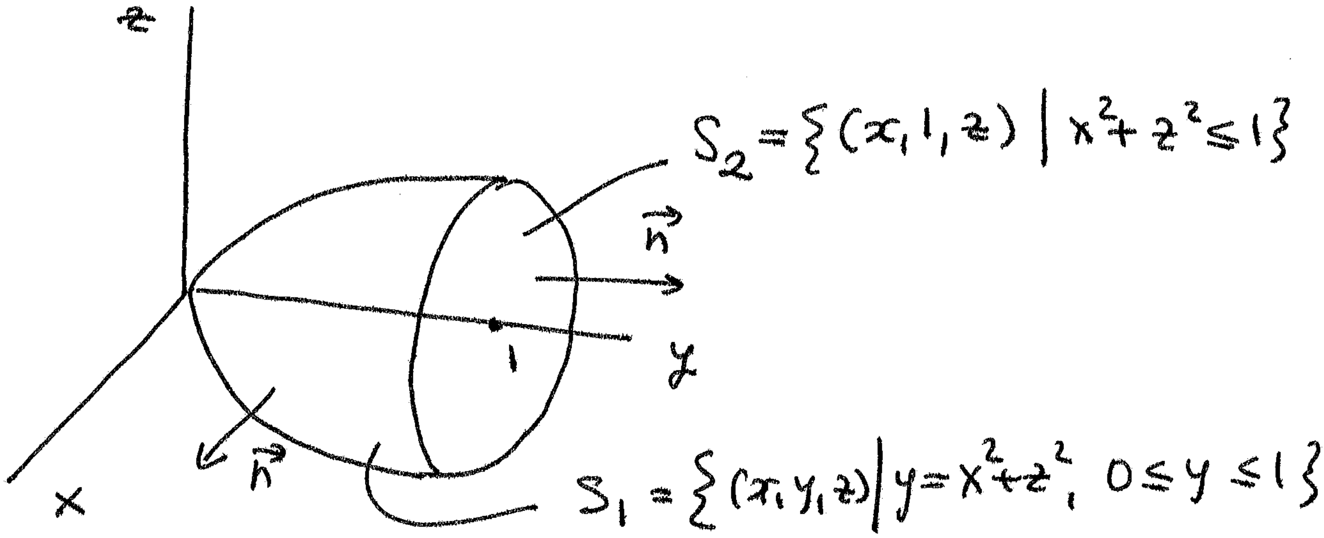
Hence the flux accross the five sides of the cube equals $4 - 1 = 3$.

<u>Exercise</u> Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = y\vec{j} - z\vec{k}$ and S is the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ oriented upward (i.e. in the positive direction of the y-axis)

We have already seen a solution by a direct parametrization of the paraboloid (pp. 37–38), but now we will use the divergence theorem.

Solution 11 We would like to apply the divergence theorem, but the problem is that the surface is not closed. However, we can close the surface by adding the disc:

$$\{ (x,1,z) \mid x^2+z^2 \leq 1 \}$$



The solid E has the boundary $S_1 + S_2$. Hence

$$\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} = 0$$

because $\operatorname{div} \vec{F} = 1 - 1 = 0$.

[Before trying to use the divergence theorem you should compute $\operatorname{div} \vec{F}$ to see if it is an easy expression whose tripple integral would be easy to compute]

Thus $-\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot d\vec{S}$

The surface $S_1 + S_2$ has an outward orientation, i.e. the normal vector to S_2 is $\vec{n} = \vec{j}$. Hence

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{x^2+z^2\le 1} \langle 0, \underbrace{y}_{1}, -z\rangle \cdot \vec{j}\, dx\,dz$$

$$= \iint_{x^2+z^2\le 1} 1\, dx\,dz = \pi \quad \text{(the area of the unit disc)}.$$

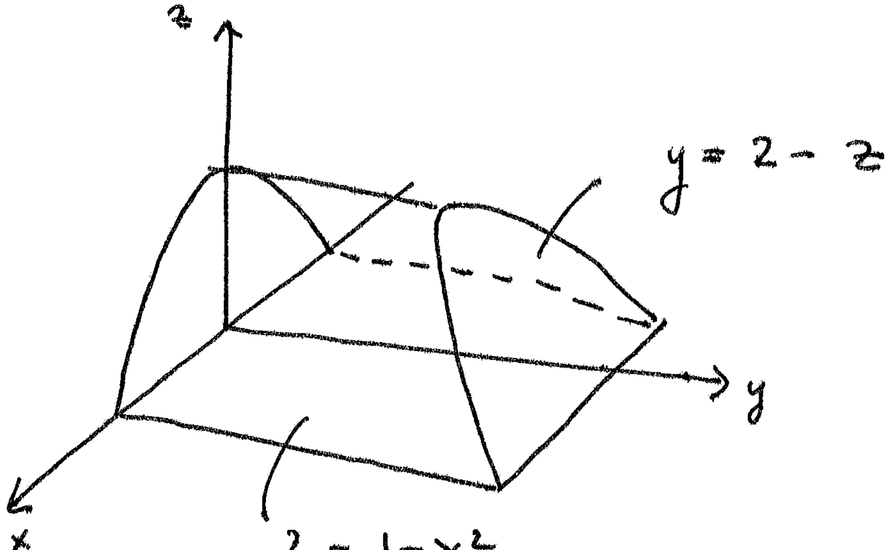
The outward normal vector to S_1 is oriented downward - in the negative direction of the y-axis. In the original problem the surface S of the paraboloid was oriented upward. Hence we have to change the sign

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot d\vec{S} = \pi.$$

Example Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where

$$\vec{F}(x,y,z) = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y + z = 2$.



Solution. We will apply the divergence theorem

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}\sin(xy)$$

$$= y + 2y = 3y$$

The region is

$$E = \{(x,y,z) \mid -1 \le x \le 1,\ 0 \le z \le 1 - x^2;\ 0 \le y \le 2 - z\}$$

Hence

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 3y\, dV$$

$$= 3 \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} y\, dy\, dz\, dx = \frac{184}{35}.$$

Example Let E denote the portion of the solid sphere of radius R in the first octant, and let

$$\vec{F} = (2x+y)\vec{i} + y^2 \vec{j} + \cos(xy)\vec{k}$$

Find the flux of $\vec{F}$ accross the boundary of E

Solution E is oriented by the outward normal vector field

$$\operatorname{div} \vec{F} = 2 + 2y$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} = \iiint_E (2+2y)\, dV$$

$$= 2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R (1+\rho \sin\theta \sin\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \frac{1}{3}\pi R^3 + \frac{1}{8}\pi R^4$$

(we split the integral as two integrals).

<u>Exercise</u> Suppose that S is the boundary of E as in the divergence theorem. Prove that if functions f, g have continuous partial derivatives, then

$$\iint_S (f\nabla g - g\nabla f)\cdot d\vec{S} = \iiint_E (f\Delta g - g\Delta f)\,dV$$

<u>Proof</u> $f\nabla g - g\nabla f$ is a vector field, so the divergence theorem yields

$$\iint_S (f\nabla g - g\nabla f)\cdot d\vec{S} = \iiint_E \operatorname{div}(f\nabla g - g\nabla f)$$

$$\operatorname{div}(f\nabla g - g\nabla f) = \operatorname{div}\langle f g_x - g f_x,\ f g_y - g f_y,\ f g_z - g f_z\rangle$$

$$= (f g_x - g f_x)_x + (f g_y - g f_y)_y + (f g_z - g f_z)_z$$

$$= \cancel{f_x g_x} + f g_{xx} - \cancel{g_x f_x} - g f_{xx}$$

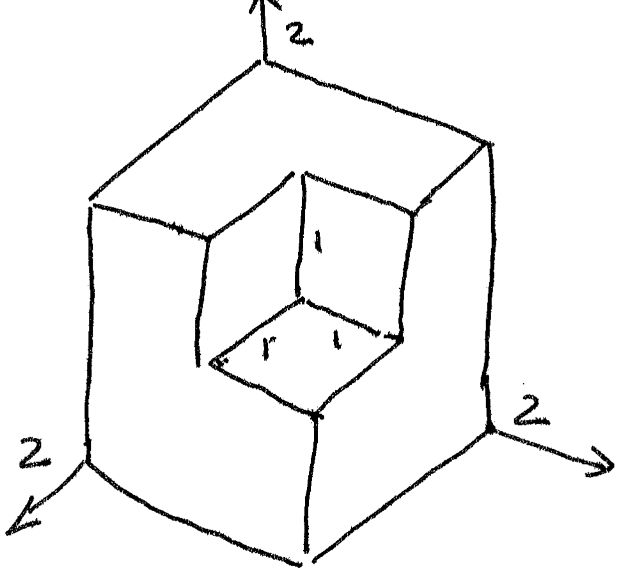
$$+ \cancel{f_y g_y} + f g_{yy} - \cancel{g_y f_y} - g f_{yy}$$

$$+ \cancel{f_z g_z} + f g_{zz} - \cancel{g_z f_z} - g f_{zz}$$

$$= f\,(g_{xx} + g_{yy} + g_{zz}) - g\,(f_{xx} + f_{yy} + f_{zz})$$

$$= f\Delta g - g\Delta f.$$

Exercise Find $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle x, y, z \rangle$ and S is the surface shown on the picture oriented inward



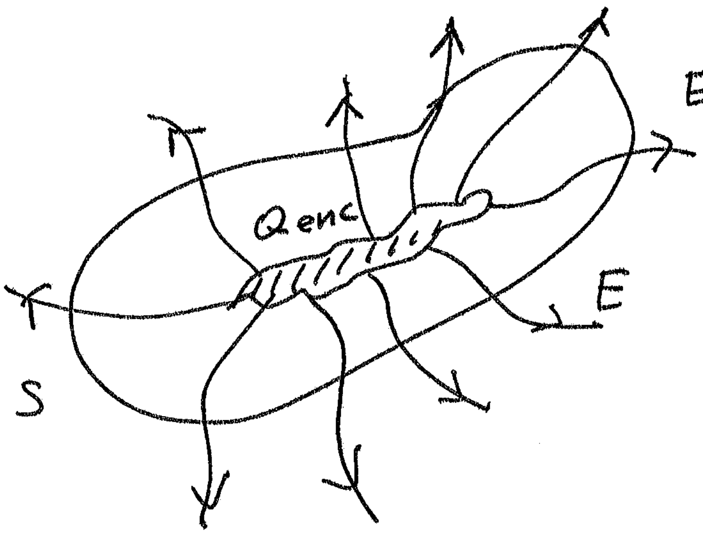
Solution $\iint_S \vec{F} \cdot d\vec{S} = -\iiint_E \operatorname{div} \vec{F} = -\iiint_E 3$

inward orientation

$= -3\,\mathrm{Vol}(E) = -3(8-1) = -21.$

The Gauss law

The flux of an electric field E through a closed surface multiplied by ε_0 equals the total charge Q_{enc} enclosed inside the surface.



$$\varepsilon_0 \iint_S \vec{E} \cdot d\vec{S} = Q_{enc} \quad (*)$$

If the enclosed charge is positive, the lines of the electric field go out of the surface and the flux is positive. If the charge is negative the lines of the electric field go into the surface and the flux is negative. It is consistent with the formula (*).

Suppose now that the density of charge in space is σ, so the total charge enclosed in the interior D of the surface S equals

$$Q_{enc} = \iiint_D \sigma \, dV$$

Hence the Gauss law can be written as

$$\varepsilon_0 \iint_S \vec{E} \cdot d\vec{S} = \iiint_D \sigma \, dV .$$

On the other hand the divergence theorem implies that

$$\varepsilon_0 \iint_S \vec{E} \cdot d\vec{S} = \varepsilon_0 \iiint_D \operatorname{div} \vec{E} \, dV$$

Hence on <u>any</u> domain

$$\varepsilon_0 \iiint_D \operatorname{div} \vec{E} \, dV = \iiint_D \sigma \, dV . \qquad (*)$$

If two continuous functions f, g have the property that on <u>any</u> domain D

$$\iiint_D f \, dV = \iiint_D g \, dV \qquad (*)$$

then $f = g$. Indeed, if $f \neq g$ somewhere, then $f - g > 0$ or $f - g < 0$ at some point and hence this inequality is also true in some small neighborhood D of that point

This implies that

$$\iiint_D f-g\,dV > 0 \quad \text{or} \quad \iiint_D f-g\,dV < 0$$

i.e.

$$\iiint_D f\,dV > \iiint_D g\,dV \quad \text{or} \quad \iiint_D f\,dV < \iiint_D g\,dV .$$

which contradicts (**).

Since the equality (*) is true on any region D we conclude that

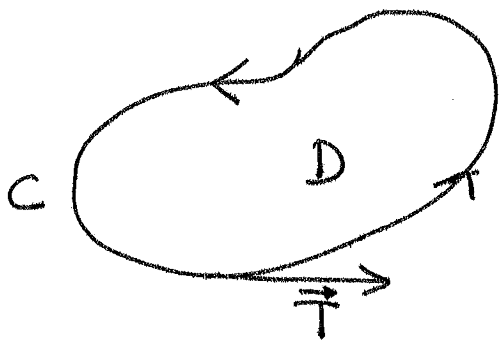
$$\boxed{\varepsilon_0 \operatorname{div} \vec{E} = \sigma}$$

This is an equivalent reformulation of the Gauss Law. In particular the electric field in a part of the space where there is no charge satisfies $\operatorname{div} \vec{E} = 0$.

The Green Theorem vs. the Divergence Theorem

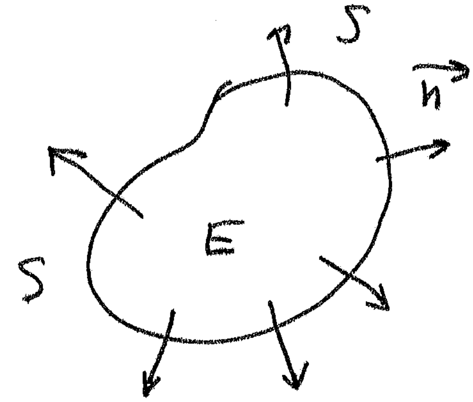
Green's theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

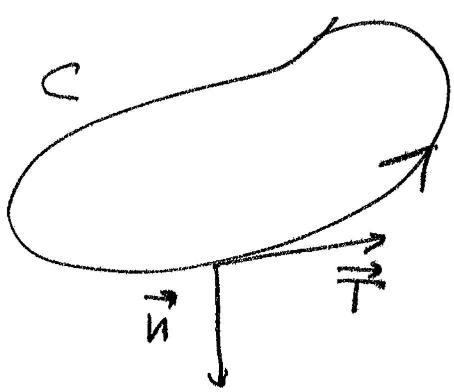


The divergence theorem:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_E \operatorname{div} \vec{F}\, dx\, dy\, dz$$

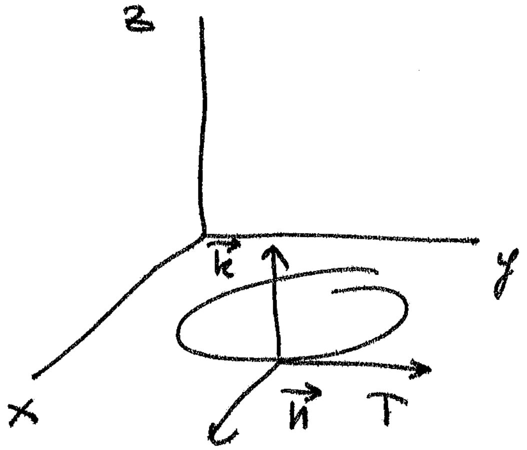


We would like to understand if Green's theorem can be interpreted as a two dimensional analogue of the divergence theorem. In both theorems we express an integral along the boundary in terms of an integral in the interior. However everything else seems different. In Green's theorem we take the dot product $\vec{F}\cdot\vec{T}$ with the unit tangent vector while in the divergence theorem we take the dot product $\vec{F}\cdot\vec{n}$ with the unit normal vector. Moreover the expression $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ looks more like curl and it is not equal to div $\vec{F}$. However, as we will see now the Green theorem can be seen as a two dimensional version of the divergence theorem, but we need a trick.



If $\vec{T}$ is a unit tangent vector oriented counter clockwise, then

$\vec{n} = \vec{T} \times \vec{k}$ is the unit normal outer vector. Suppose that $T = \langle a, b \rangle$. To compute its cross product with $\vec{k}$ we need to regard $\vec{T}$ as a vector in $\mathbb{R}^3$, i.e.



$$\vec{T} = \langle a, b, 0 \rangle.$$

We have

$$\vec{n} = \vec{T} \times \vec{k} = \langle a, b, 0 \rangle \times \langle 0, 0, 1 \rangle$$
$$= \langle b, -a, 0 \rangle.$$

Then if we forget about z-axis and we are back to the xy-plane,

$$\vec{n} = \langle b, -a \rangle.$$

We have

$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_C \langle P, Q \rangle \cdot \langle b, -a \rangle\, ds$$

$$= \int_C Pb - Qa\, ds = \int_C \underbrace{\langle -Q, P \rangle}_{\tilde{F}} \cdot \langle a, b \rangle\, ds$$

$$= \int_C \vec{\tilde{F}} \cdot \vec{T}\, ds \overset{\text{Green}}{=} \iint_D \frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y}\, dA$$

$$= \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA = \iint_D \operatorname{div} \vec{F}\, dA.$$

That means we reformulated the Green theorem as

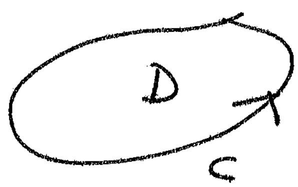
$$\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA$$

and this is exactly the divergence theorem.

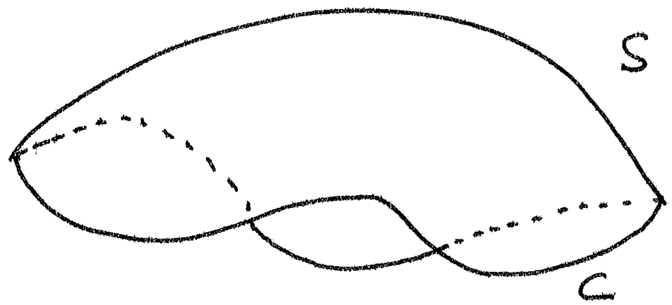
The Stokes Theorem

The Stokes theorem is another generalization of the Green theorem. Recall that

$$\int_C \vec{F} \cdot d\vec{r} \overset{\text{Green}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



Here D is a planar domain and C is its positively oriented boundary. The Stokes theorem generalizes the Green theorem to the situation in which C is a curve in $\mathbb{R}^3$ (not necessarily planar) and D is replaced by a surface S spanned on C, i.e. C is the boundary of S

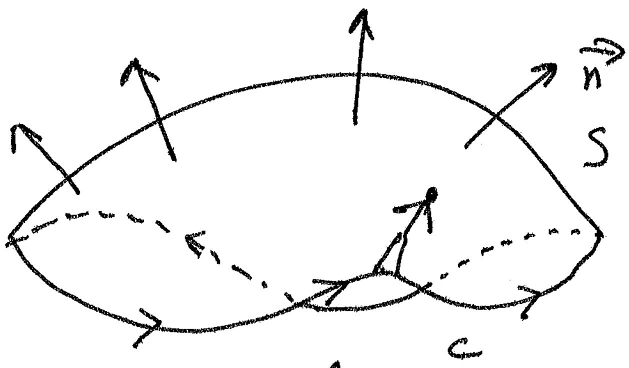


The Stokes theorem represents the integral $\int_C \vec{F} \cdot d\vec{r}$ as an integral of something over S

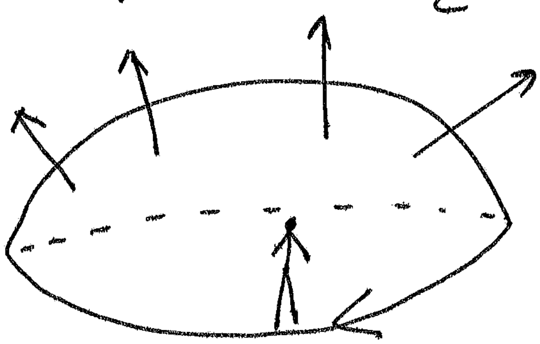
$$\int_C \vec{F} \cdot d\vec{r} = \iint_S ???$$

In the Green theorem the boundary C has positive orientation. Thus we need to discuss the orientation of the boundary C of a surface S.

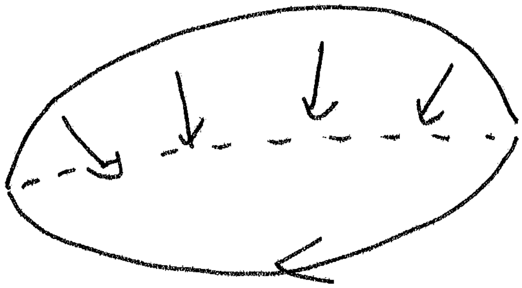
We assume that the surface S has an orientation, i.e. it is equipped with a unit normal vector field. Then the boundary is positively oriented if when we walk along the boundary, the domain is on the left. I hope pictures will explain what I mean by that



positively oriented boundary



negatively oriented boundary



positively oriented boundary.

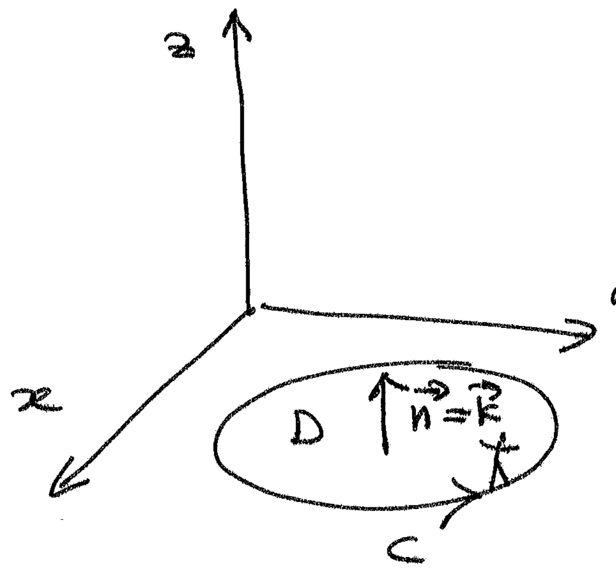
Now we can state the Stokes theorem.

<u>Theorem (Stokes)</u> S - oriented piecewise smooth surface. C - piecewise smooth boundary positively oriented. $\vec{F}(x,y,z)$ - vector field. Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F}\cdot d\vec{S} = \iint_S \operatorname{curl} \vec{F}\cdot \vec{n}\, dS$$

Let us check that the Green theorem is a special case of the Stokes theorem.

In the planar case



$S = D$

we take upward orientation of D, so $\vec{n}=\vec{k}$. Indeed, this orientation guarantees that if we walk counterclockwise, the domain is on the left. Now the Stokes theorem yields

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \operatorname{curl}\vec{F}\cdot \underbrace{\vec{n}}_{\vec{k}}\, dA$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} =$$

$$= \left\langle \star, \star, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

not important what

$$\operatorname{curl} \vec{F} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

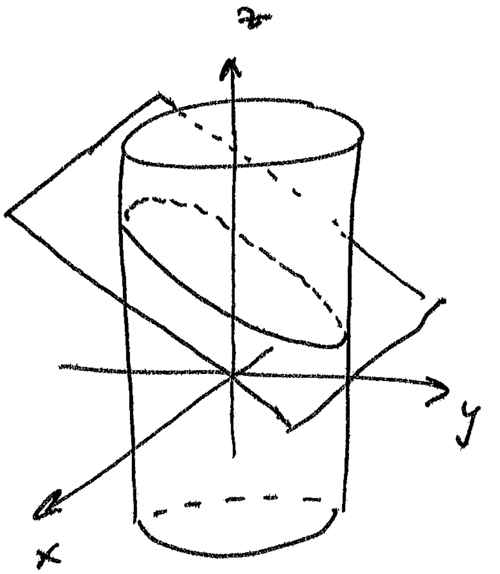
and hence

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

whitch is the Green theorem.

Now we will show some applications.

<u>Exercise</u> Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$, oriented counterclockwise as viewed from above.



In the intersection we obtain an ellipse. We will show two solutions, by a straightforward calculation and by an application of Stokes' theorem.

Solution I (straightforward calculation)

We need to parametrize the curve. The x, y components are on the unit circle $x^2+y^2=1$, so we can parametrize

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi.$$

This is a counterclockwise parametization when we look from above, just as we assumed in the problem. We find a formula for z from the equation $y+z=2$.

$$\sin t + z = 2$$
$$z = 2 - \sin t$$

Hence

$$\vec{r}(t) = \langle \cos t, \sin t, 2-\sin t \rangle, \quad 0 \le t \le 2\pi.$$

We have

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt =$$

$$\int_0^{2\pi} \langle \underbrace{-\sin^2 t}_{-y^2}, \underbrace{\cos t}_{x}, \underbrace{(2-\sin t)^2}_{z^2} \rangle \cdot \langle \underbrace{-\sin t}_{x'}, \underbrace{\cos t}_{y'}, \underbrace{-\cos t}_{z'} \rangle dt$$

$$= \int_0^{2\pi} \sin^3 t + \cos^2 t - \cos t\,(2-\sin t)^2\,dt = \pi.$$

We omitted the computations in the last step, because you should know how to do it (do it!), but it is clear that it is quite a lot of work.

Solution II (Stokes' theorem)

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y \rangle$$

Our surface is the ellipse. It is the graph of $z = 2-y$ over the unit disc. We have

$$\iint_S \underbrace{\text{curl}\,\vec{F}}_{\langle P, Q, R\rangle} \cdot d\vec{S} = \iint_D -Pg_x - Qg_y + R\, dA$$

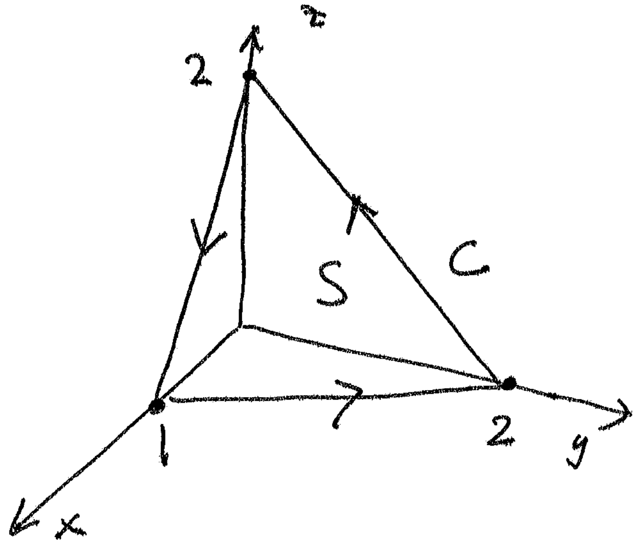
$$= \iint_D -0 \cdot g_x - 0 \cdot g_y + (1+2y)\, dx\,dy$$

$$= \iint_D (1+2y)\, dx\,dy \underset{\substack{\uparrow \\ \text{polar} \\ \text{coordinates}}}{=} \pi .$$

Here $g(x,y) = 2-y$, but we did not need to compute g_x, g_y because these derivatives were multiplied by 0.

<u>Example</u> Evaluate $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F} = xz\,\vec{i} + xy\,\vec{j} + 3xz\,\vec{k}$ and C is the boundary of the portion of the plane $2x+y+z=2$ in the first octant, parametrized counterclockwise as viewed from above.

Solution The portion of the plane in the first octant is a triangle whose vertices can be found by finding x, y, and z intercepts (63)

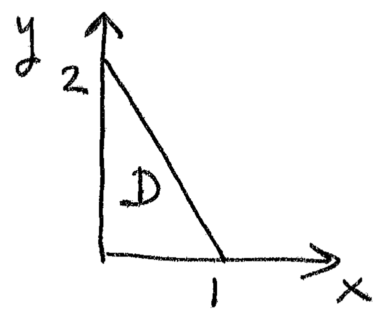


With this orientation of the boundary, the normal vector to the plane points up, i.e. in the positive direction of the z-axis.

This triangle is the graph of

$$z = g(x, y) = 2 - 2x - y$$

over the planar triangle



We have

$$d\vec{S} = \langle -g_x, -g_y, 1 \rangle \, dx\,dy = \langle 2, 1, 1 \rangle \, dx\,dy$$

and this is the "up" orientation.

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & 3xz \end{vmatrix}$$

$$= \langle 0,\ x-3z,\ y\rangle.$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} =$$

$$= \iint_D \langle 0, x-3z, y\rangle\cdot\langle 2,1,1\rangle\, dx\,dy =$$

$$= \iint_D (x-3z+y)\,dx\,dy = \iint_D (x-3\underbrace{(2-2x-y)}_{z}+y)\,dx\,dy$$

$$= \iint_D 7x+4y-6\,dx\,dy = \int_0^1\int_0^{2-2x} 7x+4y-6\,dy\,dx = -1.$$

$\square$

Again. Important formulas to remember

S – graph of $z = g(x,y)$, $(x,y)\in D$

$dS = \sqrt{1+g_x^2+g_y^2}\,dx\,dy$, $\quad d\vec{S} = \langle -g_x, -g_y, 1\rangle\,dx\,dy$

$$\iint_S f\,dS = \iint_D f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\,dx\,dy$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle P,Q,R\rangle\cdot\langle -g_x,-g_y,1\rangle\,dx\,dy$$

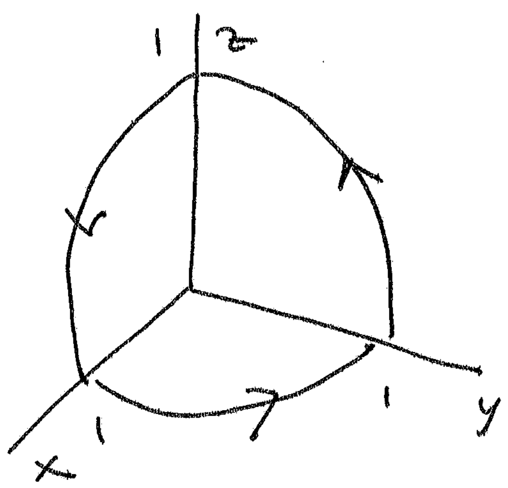
$$= \iint_D -Pg_x - Qg_y + R\,dx\,dy$$

Example Compute the integral

$\int_C \vec{F} \cdot d\vec{r}$, $F = \langle y, x, x^2+y^2 \rangle$,

where C is positively oriented boundary curve of the part of the unit sphere $x^2+y^2+z^2=1$ in the first octant.

Solution



The sphere has the outward orientation, so the positive orientation of the bounday is shown on the picture.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & x^2+y^2 \end{vmatrix} = \langle 2y, -2x, 0 \rangle$$

One could try to compute $\operatorname{curl} \vec{F} \cdot \vec{n}$ using spherical parametrization, but this would be a mistake.

Remember that on the unit sphere
$$\vec{n} = \langle x, y, z \rangle$$
(and $\vec{n} = \frac{\langle x, y, z\rangle}{r}$ on the sphere of radius r)
This often simplifies computations.

Thus
$$\operatorname{curl} \vec{F} \cdot \vec{n} = \langle 2y, -2x, 0\rangle \cdot \langle x, y, z\rangle$$
$$= 2yx - 2xy = 0.$$

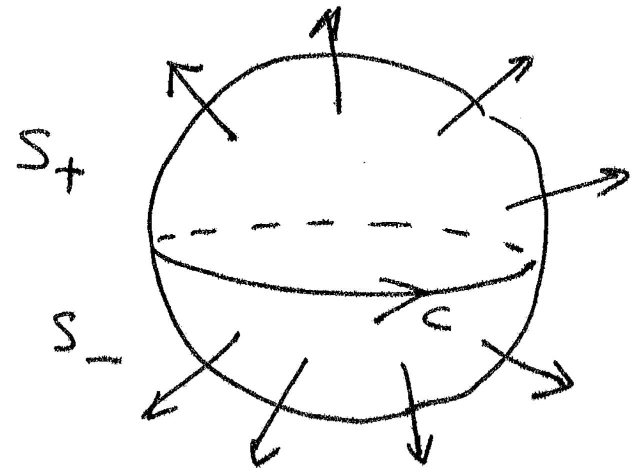
Hence
$$\int_C \vec{F} \cdot d\vec{r} = \iint_S 0 \, dS = 0.$$

Exercise Prove that if S is a sphere, then
$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = 0.$$

This problem looks surprising, because we do not know $\vec{F}$.

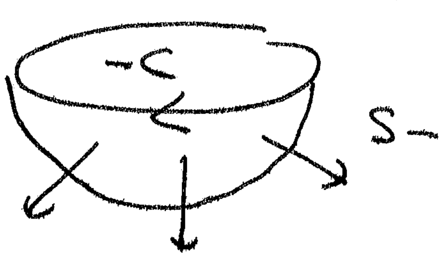
Proof I Recall that
$$\operatorname{div} \operatorname{curl} \vec{F} = 0 \quad \text{(page 2)}$$
Hence the divergence theorem yields
$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \operatorname{curl} \vec{F} = 0.$$

<u>Proof II</u> Let C be the equator on the sphere. Let S_+ and S_- be the upper and lower hemispheres. Choose the orientation of C as on the picture



The curve C has the positive orientation as the boundary of S_+

However it has the negative orientation as the boundary of S_- and the curve $-C$ has the positive orientation as the boundary of S_-



According to the Stokes theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_{S_+} \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$\int_{-C} \vec{F}\cdot d\vec{r} = \iint_{S_-} \operatorname{curl}\vec{F}\cdot d\vec{S}$$

Hence

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_+} \operatorname{curl}\vec{F}\cdot d\vec{S} + \iint_{S_-} \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$= \int_C \vec{F}\cdot d\vec{r} + \int_{-C} \vec{F}\cdot d\vec{r} = 0.$$

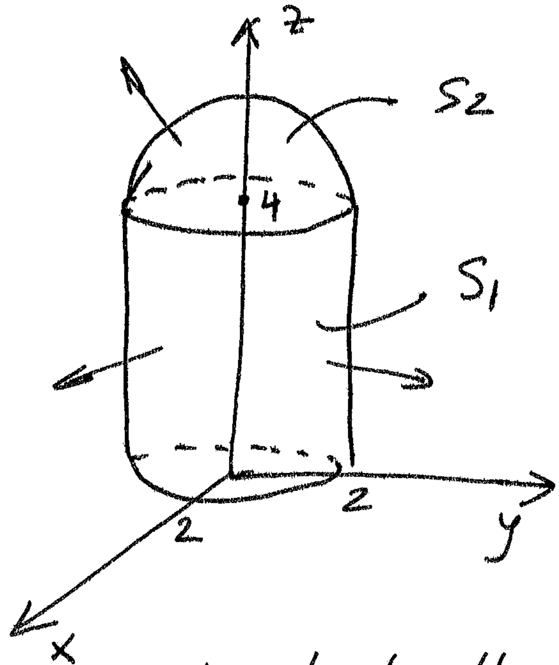
<u>Remark</u> The result remains true (with the same proofs) for any closed surface $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = 0$.

<u>Exercise</u> The surface S is obtained by taking the union of the cylinder

$$S_1 = \{(x,y,z) \mid 0 \le z \le 4,\ x^2+y^2 = 4\}$$

and the upper hemisphere of radius 2 centered at $(0,0,4)$

$$S_2 = \{(x,y,z) \mid x^2+y^2+(z-4)^2 = 4,\ z \geq 4\}$$



$S = S_1 + S_2$

The surface S is oriented away from the origin

Evaluate the integral $\iint_S (\text{curl}\vec{F}) \cdot d\vec{S}$, where
$$\vec{F} = \langle yx^2 + \cos(zx), e^{zy} - xy^2, e^{xy} \rangle.$$

Solution The boundary C of the surface $S = S_1 + S_2$ is the same as the boundary of the disc D of radius 2 in the xy-plane. The boundary is the circle C
$$x^2 + y^2 = 4.$$
The counterclockwise orientation of this circle corresponds to the orientation of S (away from the origin) and also to the orientation of D by the upward normal vector $\vec{n} = \vec{k}$. According to the Stokes thm.
$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}\,\vec{F} \cdot d\vec{S}$$
$$= \iint_D \text{curl}\,\vec{F} \cdot \vec{k}\, dx\,dy = ♡$$

Note that we applied the Stokes theorem twice.
We need to find the $\vec{k}$ component of curl $\vec{F}$ only

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yx^2+\cos(zx) & e^{zy}-xy^2 & e^{xy} \end{vmatrix}$$

$$= \langle \star, \star, -y^2-x^2 \rangle.$$

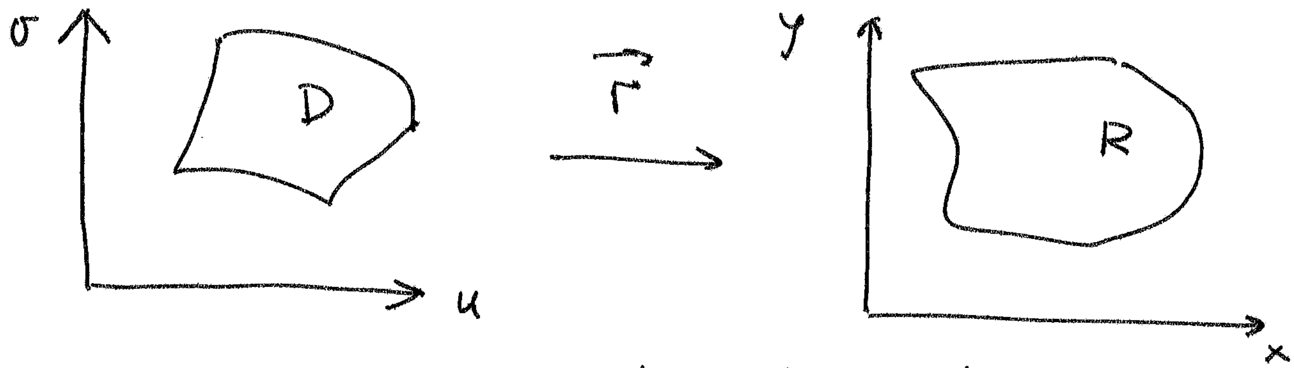
$$\heartsuit = \iint_D -y^2-x^2\,dx\,dy = -\int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta$$

$$= -2\pi \left.\frac{r^4}{4}\right|_0^2 = -2\pi\cdot\frac{16}{4} = \boxed{-8\pi}$$

Change of variables

Now we will discuss more examples for the application of the change of variables formula. Some examples have already been discussed on pages 21–25.

Recall that if $\vec{r}(u,v) = \langle x(u,v), y(u,v) \rangle$



is a one-to-one transformation such that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u \neq 0,$$

then

$$\iint_R f(x,y)\,dx\,dy = \iint_D f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

Exercise Use the transformation $u = x+2y$, $v = x-y$ to evaluate the integral

$$\int_0^{2/3} \int_y^{2-2y} (x+2y)\, e^{x-y}\, dx\, dy.$$

Solution We actually need to write x, y as functions of u, v, so we need to solve the equations

$$u = x+2y, \quad v = x-y$$

for x and y. We easily obtain

$$x = \frac{u+2v}{3}, \quad y = \frac{u-v}{3}$$

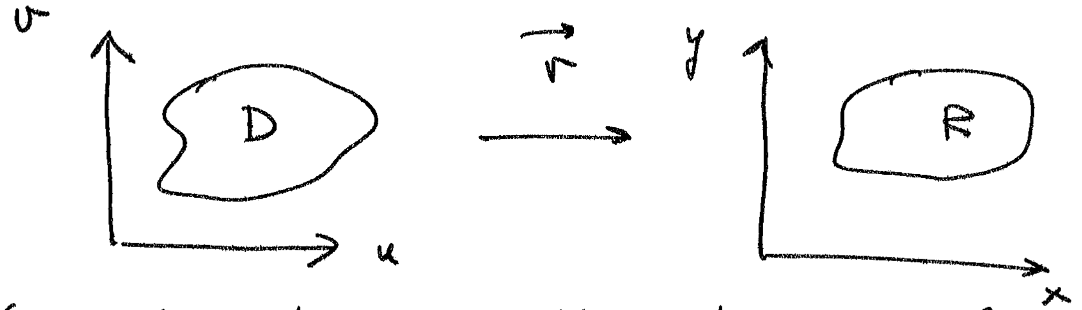
$$\vec{r}(u,v) = \Big(\underbrace{\frac{u+2v}{3}}_{x(u,v)}, \underbrace{\frac{u-v}{3}}_{y(u,v)} \Big).$$

$$f(x(u,v), y(u,v)) = (x+2y)\, e^{x-y} = u e^{v}.$$

Observe that the transformation was defined in a way to make the function simple.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{vmatrix} = -1/3$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{3}$$



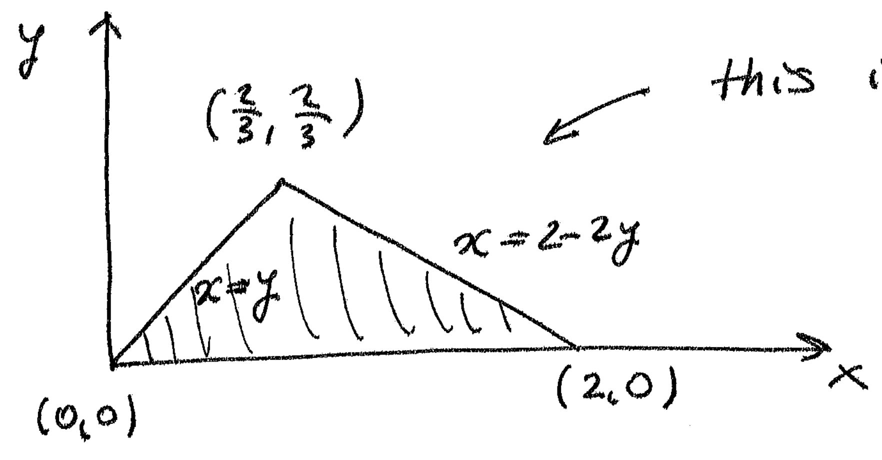
(we do not know the shapes of D and R yet, so we sketched some arbitrary domains)

$$\iint_R (x+2y)\, e^{x-y}\, dx\,dy = \iint_D u\, e^{v} \cdot \frac{1}{3}\, du\,dv.$$

We have to find what R and D are. The integral

$$\int_0^{2/3} \int_0^{2-2y} \ldots \, dx\,dy$$

is over the triangle



this is R

The domain D is also a triangle, because the transformation is linear and we just need to find vertices of D.

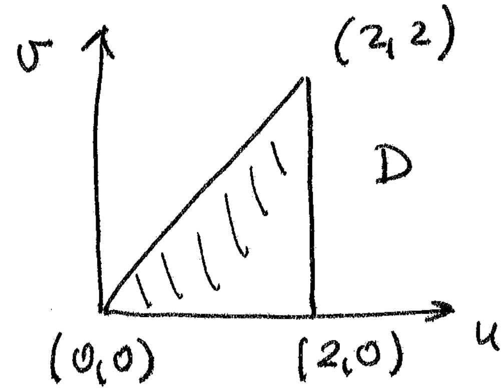
$$(x,y) \longmapsto (\underbrace{x+2y}_{u}, \underbrace{x-y}_{v})$$

$$(0,0) \longmapsto (0,0)$$

$$(2,0) \longmapsto (2,2)$$

$$\left(\tfrac{2}{3}, \tfrac{2}{3}\right) \longmapsto (2,0)$$

Hence



We have

$$\int_0^{2/3}\int_0^{2-2y} (x+2y)\, e^{x-y}\, dx\, dy = \iint_D u e^{v} \cdot \frac{1}{3}\, du\, dv$$

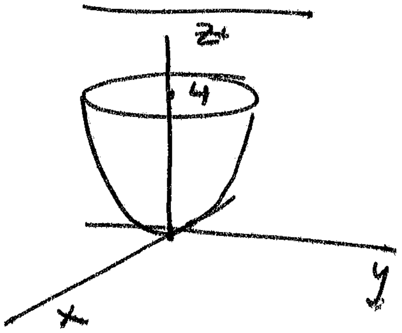
$$= \frac{1}{3}\int_0^2\int_0^u u e^{v}\, dv\, du = \frac{1}{3}\int_0^2 u e^{v}\Big|_0^u du$$

$$= \frac{1}{3}\int_0^2 u\,(e^u - 1)\,du =$$

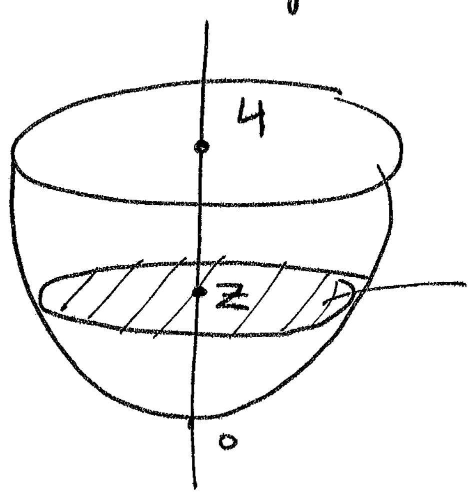
$$= \frac{1}{3}\left(ue^u - e^u - \frac{u^2}{2}\right)\Big|_0^2 = \frac{e^2-1}{3}$$

Exercise Find the tripple integral $\iiint_R \sqrt{z}\,dx\,dy\,dz$, where R is the region between the elliptic paraboloid $z = 2x^2+3y^2$ and the plane $z=4$.

Solution



The cross sections of R are ellipses. If we fix z, then we integrate with respect to x and y over



$2x^2+3y^2 \leq z$

Hence

$$\iiint_R \sqrt{z}\,dx\,dy\,dz = \int_0^4 \left(\iint_{2x^2+3y^2\le z} \sqrt{z}\,dx\,dy \right) dz$$

$$= \int_0^4 \sqrt{z} \left(\iint_{2x^2+3y^2\le z} dx\,dy \right) dz$$

Change of variables

$$2x^2 + 3y^2 \le z$$

$$(\underbrace{\sqrt{2}x}_{u})^2 + (\underbrace{\sqrt{3}y}_{v})^2 \le z$$

$$u^2 + v^2 \le z$$

turn the ellipse into a disc of radius $\sqrt{z}$.

$$u = \sqrt{2}x, \quad v = \sqrt{3}y$$

$$x = \frac{u}{\sqrt{2}}, \quad y = \frac{v}{\sqrt{3}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{vmatrix} = \frac{1}{\sqrt{6}}.$$

$$\iint_{2x^2+3y^2 \le z} dx\,dy = \iint_{\underbrace{u^2+v^2 \le z}_{\text{disc of radius } \sqrt{z}}} \frac{1}{\sqrt{6}}\, du\,dv$$

$$= \frac{1}{\sqrt{6}} \underbrace{\pi \left(\sqrt{z}\right)^2}_{\substack{\text{area of} \\ \text{the disc}}} = \frac{\pi z}{\sqrt{6}}.$$

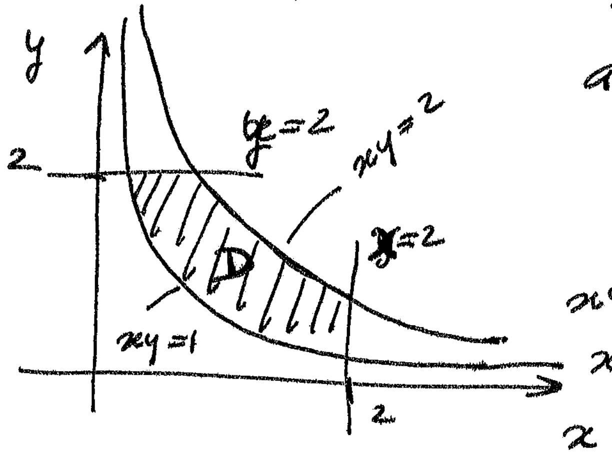
$$\int_0^4 \sqrt{z} \left(\iint_{2x^2+3y^2 \le z} \sqrt{z}\, dx\,dy \right) dz = \int_0^4 \sqrt{z} \cdot \frac{\pi z}{\sqrt{6}}\, dz$$

$$= \frac{\pi}{\sqrt{6}} \left. \frac{z^{5/2}}{5/2} \right|_0^4 = \frac{\pi \cdot 4^{5/2}}{\sqrt{6}} \cdot \frac{2}{5}$$

$$= \frac{\pi \cdot 32}{\sqrt{6}} \cdot \frac{2}{5} = \frac{64\pi}{5\sqrt{6}}.$$

__Exercise__ Find the double integral $\iint_D x^2 y^3\, dx\,dy$, where D is the region bounded by curves $xy=1$, $xy=2$, $x=2$, $y=2$ in the first quadrant using the change of variables $u = xy$, $v = y$.

Solution



$xy=1$ i.e. $y=\frac{1}{x}$

and

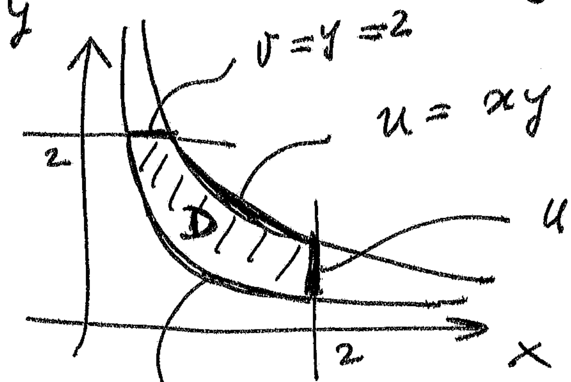
$xy=2$ i.e. $y=\frac{2}{x}$

are hyperbolas

$xy=2$

$xy=1$

We are integrating over the shaded region



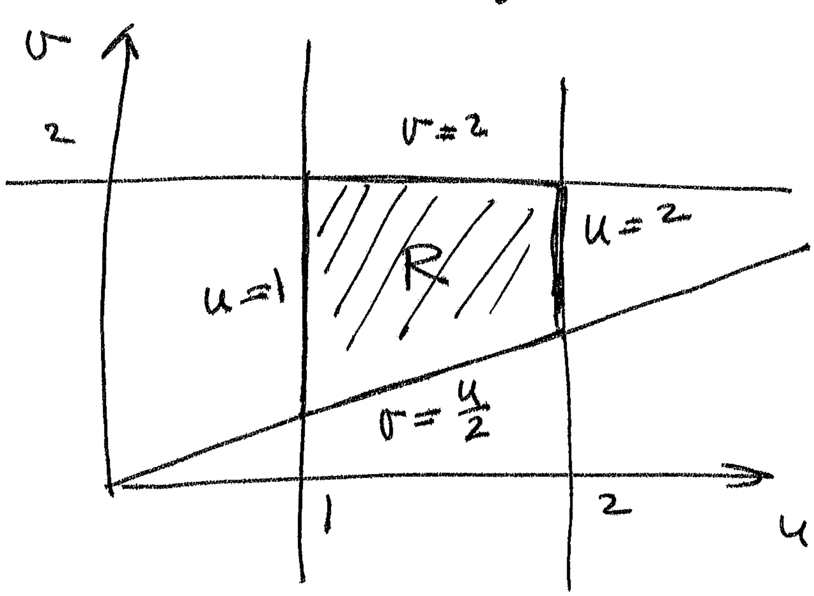
$v=y=2$

$u=xy=2$

$u=xy=2y=2v$, $v=\frac{u}{2}$

$u=xy=1$

Hence the given transformation maps the shaded region onto



$$x^2y^3 = (xy)^2 y = u^2 v$$

$$x = \frac{u}{y} = \frac{u}{v}, \quad y = v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v}, & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{v}$$

$$\iint_D x^2y^3 \, dx\,dy = \iint_R u^2 v \cdot \frac{1}{v} \, du\,dv$$

$$= \int_1^2 \int_{4/2}^{2} u^2 \, du = \frac{67}{24}. \qquad \square$$