HARMONIC ANALYSIS

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1. Preliminaries

Let us first fix notation that will be used throughout the book. We will be working mostly in the Euclidean spaces and with the Lebesgue measure. The Lebesgue measure of a measurable set E will be denoted by $|E|$. The volume of a ball of radius r is $|B(0,r)| = \omega_n r^n$. Then it is well known that the $(n-1)$-dimensional measure of the sphere $S^{n-1}(0,r)$ equals $n\omega_n r^{n-1}$. The integral average over a set E of positive measure will be denoted by

$$⨍_E f(x)\, dx = \frac{1}{|E|}\int_E f(x)\, dx.$$

The Lebesgue measure on a surface will be denoted by $d\sigma$ so the integration in the spherical coordinates will take the form

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_{S^{n-1}} f(s\theta)\, d\sigma(\theta)\, ds.$$

Unless we will explicitly state otherwise, all functions and all function spaces will be complex valued.

A generic open set in $\mathbb{R}^n$ will usually be denoted by Ω and $C_0^\infty(\Omega)$ will stand for a space of smooth functions with compact support in Ω. The space of continuous functions vanishing at infinity will be denote by $C_0(\mathbb{R}^n)$, i.e. $f \in C_0(\mathbb{R}^n)$ if f is continuous and

$$\lim_{|x|\to\infty} f(x) = 0.$$

$C_0(\mathbb{R}^n)$ is a Banach space with respect to the supremum norm $\|\cdot\|_\infty$ and $C_0^\infty(\mathbb{R}^n)$ is a dense subset of $C_0(\mathbb{R}^n)$.

We will use the multiindex notation

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}, \quad x^\alpha = x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}$$

where $\alpha = (\alpha_1,\ldots,\alpha_n)$, $|\alpha| = \alpha_1+\ldots+\alpha_n$. Then the product rule takes the form

$$D^\alpha(fg) = \sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}D^\beta f D^\gamma g, \quad \text{where } \alpha! = \alpha_1!\cdot\ldots\cdot\alpha_n!.$$

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The characteristic function of a set E will be denoted by χ_E. By δ_a we will denote the *Dirac measure* centered at a i.e.,

$$\int_{\mathbb{R}^n} f(x)\, d\delta_a(x) = f(a).$$

By an *increasing* function we will mean a non-decreasing function and increasing functions that are actually increasing, will be called *strictly increasing*. Similar terminology applies to *decreasing* and *strictly decreasing* functions.

By $\operatorname{sgn} x$ we will denote the sign of the number x, i.e. $\operatorname{sgn} x = 1$ for positive x, $\operatorname{sgn} x = -1$ for negative x and $\operatorname{sgn} 0 = 0$.

Various constants will usually be denoted by C. By writing $C(n,m)$ we will indicate the the constant $C(n,m)$ depends on n and m *only*. We adopt the rule that a constant C may change its value within one string of estimates. This will allow us to minimize the use of indices.

2. The Maximal function

2.1. The maximal theorem.

Definition 2.1. For a locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ the *Hardy-Littlewood maximal function* is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

The operator $\mathcal{M}$ is not linear but it is subadditive. We say that an operator T from a space of measurable functions into a space of measurable functions is *subadditive* if

$$|T(f_1 + f_2)(x)| \le |Tf_1(x)| + |Tf_2(x)| \quad \text{a.e.}$$

and

$$|T(kf)(x)| = |k||Tf(x)| \quad \text{for } k \in \mathbb{C}.$$

The following integrability result, known also as the *maximal theorem*, plays a fundamental role in many areas of mathematical analysis.

Theorem 2.2 (Hardy-Littlewood-Wiener). *If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then $\mathcal{M}f < \infty$ a.e. Moreover*

(a) *For $f \in L^1(\mathbb{R}^n)$*

$$|\{x : \mathcal{M}f(x) > t\}| \le \frac{5^n}{t} \int_{\mathbb{R}^n} |f| \quad \textit{for all } t > 0. \tag{2.1}$$

(b) *If $f \in L^p(\mathbb{R}^n)$, $1 < p \le \infty$, then $\mathcal{M}f \in L^p(\mathbb{R}^n)$ and*

$$\|\mathcal{M}f\|_p \le 2 \cdot 5^{n/p} \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p \quad \textit{for } 1 < p < \infty,$$

$$\|\mathcal{M}f\|_\infty \le \|f\|_\infty .$$

Remark 2.3. Note that if $f \in L^1(\mathbb{R}^n)$ is a non-zero function, then $\mathcal{M}f \notin L^1(\mathbb{R}^n)$. Indeed, if $\lambda = \int_{B(0,R)} |f| > 0$, then for $|x| > R$, $B(0,R) \subset B(x, R + |x|)$ so

$$\mathcal{M}f(x) \ge \fint_{B(x,R+|x|)} |f| \ge \frac{\lambda}{\omega_n (R + |x|)^n}\,,$$

Since the function on the right hand side is not integrable on $\mathbb{R}^n$, the statement (b) of the theorem is not true for $p = 1$.

Remark 2.4. If $g \in L^1(\mathbb{R}^n)$, then the *Chebyschev inequality*

$$|\{x : |g(x)| > t\}| \leq \frac{1}{t}\int_{\mathbb{R}^n} |g| \quad \text{for } t > 0 \tag{2.2}$$

is easy to prove. Thus if an (not necessarily linear) operator T is bounded in L^1, $\|Tf\|_1 \leq C\|f\|_1$, it immediately follows from Chebyschev's inequality that

$$|\{x : |Tf(x)| > t\}| \leq \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|\, dx. \tag{2.3}$$

The maximal function $\mathcal{M}$ is *not* bounded in L^1, but it still satisfies the weaker estimate (2.3). For this reason estimate (2.1) is called the *weak type estimate.*

In the proof of Theorem 2.2 we will need the following two results.

Theorem 2.5 (Cavalieri's principle)**.** *If μ is a σ-finite measure on X and $\Phi : [0,\infty) \to [0,\infty)$ is increasing, absolutely continuous and $\Phi(0) = 0$, then*

$$\int_X \Phi(|f|)\, d\mu = \int_0^\infty \Phi'(t)\mu(\{|f| > t\})\, dt\,.$$

Proof. The result follows immediately from the equality

$$\int_X \Phi(|f(x)|)\, d\mu(x) = \int_X \int_0^{|f(x)|} \Phi'(t)\, dt\, d\mu(x)$$

and the Fubini theorem. □

Corollary 2.6. *If μ is a σ-finite measure on X and $0 < p < \infty$, then*

$$\int_X |f|^p\, d\mu = p\int_0^\infty t^{p-1}\mu(\{|f| > t\})\, dt\,.$$

The next result has many applications that go beyond the maximal theorem. Here and in what follows by $5B$ we denote the ball concentric with B and with five times the radius.

Theorem 2.7 ($5r$-covering lemma)**.** *Let $\mathcal{B}$ be a family of balls in a metric space such that $\sup\{\operatorname{diam} B : B \in \mathcal{B}\} < \infty$. Then there is a subfamily of pairwise disjoint balls $\mathcal{B}' \subset \mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{B}'} 5B\,.$$

More precisely, for every $B \in \mathcal{B}$ there is $B' \in \mathcal{B}'$ such that $B \cap B' \neq \emptyset$ and $B \subset 5B'$.

If the metric space is separable, then the family $\mathcal{B}'$ is countable (or finite) and we can arrange it as a (possibly finite) sequence $\mathcal{B}' = \{B_i\}_{i=1}^\infty$ so

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i=1}^\infty 5B_i\,.$$

Remark 2.8. Here $\mathcal{B}$ can be either a family of open balls or closed balls. In both cases the proof is the same.

Proof. Let $\sup\{\operatorname{diam} B : \ B \in \mathcal{B}\} = R < \infty$. Divide the balls in the family $\mathcal{B}$ according to their diameters:

$$\mathcal{F}_j = \left\{ B \in \mathcal{B} : \frac{R}{2^j} < \operatorname{diam} B \leq \frac{R}{2^{j-1}} \right\}.$$

Clearly $\mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{F}_j$.

First we select balls as large as possible so we choose balls from the family $\mathcal{F}_1$ and we take as many as we can: we define $\mathcal{B}_1 \subset \mathcal{F}_1$ to be a maximal family of pairwise disjoint balls.

Suppose the families $\mathcal{B}_1, \ldots, \mathcal{B}_{j-1}$ are already defined.

In the construction of $\mathcal{B}_j$ we want to select balls form $\mathcal{F}_j$, but we have to make sure that balls that we choose do not intersects with any of the previously selected balls. Thus we have to select balls from

$$\hat{\mathcal{F}}_j = \left\{ B \in \mathcal{F}_j : \ B \cap B' = \emptyset \text{ for all } B' \in \bigcup_{i=1}^{j-1} \mathcal{B}_i \right\}.$$

Then we define $\mathcal{B}_j$ to be the maximal family of pairwise disjoint balls in $\hat{\mathcal{F}}_j$

Finally we define $\mathcal{B}' = \bigcup_{j=1}^{\infty} \mathcal{B}_j$. It follows immediately from the definition of $\mathcal{B}_j$ that every ball $B \in \mathcal{F}_j$ intersects with a ball in $\bigcup_{i=1}^{j} \mathcal{B}_j$. If $B \cap B' \neq \emptyset$, $B' \in \bigcup_{i=1}^{j} \mathcal{B}_i$, then

$$\operatorname{diam} B \leq \frac{R}{2^{j-1}} = 2 \cdot \frac{R}{2^j} < 2 \operatorname{diam} B'$$

and hence $B \subset 5B'$. □

Proof of Theorem 2.2. (a) Let $f \in L^1(\mathbb{R}^n)$ and let $E_t = \{x : \mathcal{M}f(x) > t\}$. For $x \in E_t$, there is $r_x > 0$ such that

$$\fint_{B(x,r_x)} |f| > t \quad \text{so} \quad |B(x, r_x)| < t^{-1} \int_{B(x,r_x)} |f| \, .$$

Observe that $\sup_{x \in E_t} r_x < \infty$, because $f \in L^1(\mathbb{R}^n)$. The family of balls $\{B(x, r_x)\}_{x \in E_t}$ forms a covering of the set E_t so applying the $5r$-covering lemma there is a sequence of pairwise disjoint balls $B(x_i, r_{x_i})$, $i = 1, 2, \ldots$ such that $E_t \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_{x_i})$ and hence

$$|E_t| \leq 5^n \sum_{i=1}^{\infty} |B(x_i, r_{x_i})| \leq \frac{5^n}{t} \sum_{i=1}^{\infty} \int_{B(x_i, r_{x_i})} |f| \leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f| \, .$$

The proof is complete.

(b) Let $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$. Since $\|\mathcal{M}f\|_\infty \leq \|f\|_\infty$ we can assume that $1 < p < \infty$. Let $f = f_1 + f_2$, where

$$f_1 = f\chi_{\{|f| > t/2\}}, \qquad f_2 = f\chi_{\{|f| \leq t/2\}}$$

be a decomposition of $|f|$ into its 'lower' and 'upper' parts. It is easy to check that $f_1 \in L^1(\mathbb{R}^n)$ and clearly $f_2 \in L^\infty(\mathbb{R}^n)$.

The idea now if to apply the weak type estimate from (a) to f_1 and the estimate $\|\mathcal{M}f\|_\infty \leq \|f\|_\infty$ to f_2. This will give us an estimate for the size of the level set $|\{\mathcal{M}f > t\}|$

and the result will follow from Corollary 2.6. The same idea is the core of the so called real interpolation method and we will see it again in the proof of the Marcinkiewcz Interpolation Theorem 8.9.

Since $|f| \leq |f_1| + t/2$ we have $\mathcal{M}f \leq \mathcal{M}f_1 + t/2$ and hence

$$\{\mathcal{M}f > t\} \subset \{\mathcal{M}f_1 > t/2\}\,.$$

Thus

$$\begin{aligned} |E_t| &= |\{\mathcal{M}f > t\}| \leq \frac{2\cdot 5^n}{t}\int_{\mathbb{R}^n}|f_1(x)|\,dx \\ &= \frac{2\cdot 5^n}{t}\int_{\{|f|>t/2\}}|f(x)|\,dx\,. \end{aligned} \tag{2.4}$$

Cavalieri's principle gives

$$\begin{aligned} \int_{\mathbb{R}^n}|\mathcal{M}f(x)|^p\,dx &= p\int_0^\infty t^{p-1}|\{\mathcal{M}f>t\}|\,dt \\ &\leq p\int_0^\infty t^{p-1}\left(\frac{2\cdot 5^n}{t}\int_{\{|f|>t/2\}}|f(x)|\,dx\right)\,dt \\ &= 2\cdot 5^n p\int_{\mathbb{R}^n}|f(x)|\int_0^{2|f(x)|}t^{p-2}\,dt\,dx \\ &= 2^p\cdot 5^n\,\frac{p}{p-1}\int_{\mathbb{R}^n}|f(x)|^p\,dx \end{aligned}$$

and the results follows. □

Remark 2.9. Note that we proved in (2.4) the following inequality

$$|\{x:\,\mathcal{M}f(x)>t\}| \leq \frac{2\cdot 5^n}{t}\int_{\{|f|>t/2\}}|f(x)|\,dx \tag{2.5}$$

which is slightly stronger than (2.1). Later we will see that the measure of the set $|\{\mathcal{M}f > t\}|$ has a similar bound from below, see Proposition 2.33.

Remark 2.10. For a positive measure μ on $\mathbb{R}^n$ we define the maximal function by

$$\mathcal{M}\mu(x) = \sup_{r>0}\frac{\mu(B(x,r))}{|B(x,r)|}\,.$$

A minor modification of the proof of Theorem 2.2(a) leads to the following result.

Proposition 2.11. *If μ is a finite positive Borel measure on $\mathbb{R}^n$, then*

$$|\{x:\,\mathcal{M}\mu(x)>t\}| \leq \frac{5^n}{t}\,\mu(\mathbb{R}^n) \quad \textit{for all } t>0.$$

Remark 2.12. It is natural to inquire whether in the Euclidean case, there is a version of Theorem 2.7 for families of cubes. The answer is in the positive and it follows immediately from Theorem 2.7 and the fact that cubes with sides parallel to coordinate axes are balls in $\mathbb{R}^n$ with respect to the metric $d_\infty(x,y) = \max_i |x_i - y_i|$. In the following result $5Q$ will denote a cube concentric with Q and with five times the diameter.

Corollary 2.13. *Let $\mathcal{F}$ be a family of open or closed cubes in $\mathbb{R}^n$ with sides parallel to coordinate directions such that* $\sup\{\operatorname{diam} Q : Q \in \mathcal{F}\} < \infty$. *Then there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of pairwise disjoint cubes such that*

$$\bigcup_{Q\in\mathcal{F}} Q \subset \bigcup_{Q\in\mathcal{F}'} 5Q.$$

More precisely for every $Q \in \mathcal{F}$ there is $Q' \in \mathcal{F}'$ such that $Q \cap Q' \neq \emptyset$ and $Q \subset 5Q'$.

Although we are interested mostly in the Euclidean setting, it was important to formulate Theorem 2.7 in a the setting of metric spaces as we could conclude Corollary 2.13 directly from Theorem 2.7.

In the next two sections we will show applications of the maximal theorem.

2.2. **Fractional integration theorem.** As a first application of the maximal theorem we will prove a result about integrability of the Riesz potentials.

Definition 2.14. For $0 < \alpha < n$ and $n \geq 2$ we define the *Riesz potential* by

$$(I_\alpha f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy\,, \quad \text{where} \quad \gamma(\alpha) = \frac{\pi^{\frac{n}{2}}\, 2^\alpha\, \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}\,.$$

At this moment the particular value of the constant $\gamma(\alpha)$ is not important to us. We could even replace this constant by 1.[1]

Theorem 2.15 (Hardy-Littlewood-Sobolev)**.** *Let $\alpha > 0$, $1 < p < \infty$ and $\alpha p < n$. Then there is a constant $C = C(n, p, \alpha)$ such that*

$$\|I_\alpha f\|_{p^*} \leq C\|f\|_p \quad \textit{for all } f \in L^p(\mathbb{R}^n)\,,$$

where $p^* = np/(n - \alpha p)$.

Remark 2.16. It is not difficult to show directly[2] that if $I_\alpha : L^p \to L^q$ is bounded, then $q = p^*$. The idea is to use the change of variables $x \mapsto tx$, $t > 0$, and see how both sides of the inequality $\|I_\alpha f\|_q \leq C\|f\|_p$ change. This is so called a *scaling* argument.

We precede the proof with a technical lemma.

Lemma 2.17. *If $0 < \alpha < n$, and $\delta > 0$, then there is a constant $C = C(n, \alpha)$ such that*

$$\int_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \leq C\delta^\alpha \mathcal{M}f(x) \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof. For $x \in \mathbb{R}^n$ and $\delta > 0$ consider the annuli

$$A(k) = B\left(x, \frac{\delta}{2^k}\right) - B\left(x, \frac{\delta}{2^{k+1}}\right)\,.$$

[1] It will play an important role later: $I_2 f$ is the convolution with the fundamentals solution of $-\Delta$, see Theorem 5.60. Also Riesz potentials will be studied in Section 7 in connection with fractional powers of the Laplace operator.

[2] A nice exercise.

We have

$$\begin{aligned}\int_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy &= \sum_{k=0}^{\infty} \int_{A(k)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \\ &\leq \sum_{k=0}^{\infty} \left(\frac{\delta}{2^{k+1}}\right)^{\alpha-n} \int_{A(k)} |f(y)|\, dy \\ &\leq \omega_n \sum_{k=0}^{\infty} \left(\frac{\delta}{2^{k+1}}\right)^{\alpha-n} \left(\frac{\delta}{2^k}\right)^{n} \fint_{B(x,\delta/2^k)} |f(y)|\, dy \\ &\leq \omega_n \delta^{\alpha} \left(\frac{1}{2}\right)^{\alpha-n} \left(\sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}}\right) \mathcal{M}f(x)\,.\end{aligned}$$

The proof is complete. □

Proof of Theorem 2.15. Fix $\delta > 0$. Hölder's inequality and integration in polar coordinates yield

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy &\leq \|f\|_p \left(\int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{dy}{|x-y|^{(n-\alpha)p'}}\right)^{1/p'} \\ &= \|f\|_p \left(n\omega_n \int_{\delta}^{\infty} s^{n-1-(n-\alpha)p'}\, ds\right)^{1/p'} \\ &= C(n,p,\alpha)\delta^{\alpha-(n/p)}\|f\|_p\,,\end{aligned}$$

because $n\omega_n$ equals the $(n-1)$-dimensional measure of the unit sphere S^{n-1} and $n-(n-\alpha)p' < 0$. This and the lemma give

$$|I_\alpha f(x)| \leq C\left(\delta^\alpha Mf(x) + \delta^{\alpha-(n/p)}\|f\|_p\right)\,.$$

Taking[3]

$$\delta = \left(\frac{\mathcal{M}f(x)}{\|f\|_p}\right)^{-p/n}$$

yields

$$|I_\alpha f(x)| \leq C(\mathcal{M}f(x))^{1-\frac{\alpha p}{n}}\|f\|_p^{\frac{\alpha p}{n}}$$

which is equivalent to

$$|I_\alpha f(x)|^{p^*} \leq C(\mathcal{M}f(x))^{p}\|f\|_p^{\frac{\alpha p}{n}\,p^*}\,.$$

Integrating both sides over $\mathbb{R}^n$ and applying boundedness of the maximal function in L^p gives the result. □

2.3. **The Lebesgue differentiation theorem.** We will show now how to prove the Lebesgue differentiation theorem from the maximal theorem.

For $h \in \mathbb{R}^n$ we define $\tau_h f(x) = f(x+h)$.

Lemma 2.18. *If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $\|\tau_h f - f\|_p \to 0$ as $h \to 0$.*

[3]We apply here a standard trick: we take $\delta > 0$ such that $\delta^\alpha Mf(x) = \delta^{\alpha-(n/p)}\|f\|_p$.

This result is obvious if f is a compactly supported smooth function, because in that case $\tau_h f$ converges uniformly to f. The general case follows from the density of C_0^∞ in L^p.

Lemma 2.19. *Let $f \in L^1(\mathbb{R}^n)$ and let $f_r(x) = \fint_{B(x,r)} f(y)\, dy$. Then $f_r \to f$ in L^1 as $r \to 0$. In particular, there is a sequence $r_i \to 0$ such that*

$$\lim_{i\to\infty} \fint_{B(x,r_i)} f(y)\, dy = f(x) \quad a.e.$$

Proof. We have

$$\begin{aligned} \int_{\mathbb{R}^n} |f_r(x) - f(x)|\, dx &\leq \int_{\mathbb{R}^n} \fint_{B(x,r)} |f(y) - f(x)|\, dy\, dx \\ &= \int_{\mathbb{R}^n} \fint_{B(0,r)} |f(x+y) - f(x)|\, dy\, dx \\ &= \fint_{B(0,r)} \|\tau_y f - f\|_1\, dy\,. \end{aligned} \tag{2.6}$$

Since $\tau_y f \to f$ in L^1 as $y \to 0$, the right hand side of (2.6) converges to 0 as $r \to 0$. The existence of a sequence r_i follows from the fact that from a sequence convergent in L^1 we can extract a subsequence convergent a.e. □

The Lebesgue differentiation theorem gives much more.

Theorem 2.20 (Lebesgue differentiation theorem)**.** *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$\lim_{r\to 0} \fint_{B(x,r)} f(y)\, dy = f(x) \quad a.e.$$

Proof. Since the theorem is local in nature we can assume that $f \in L^1(\mathbb{R}^n)$. Let $f_r(x) = \fint_{B(x,r)} f(y)\, dy$ and define

$$\Omega f(x) = \limsup_{r\to 0} f_r(x) - \liminf_{r\to 0} f_r(x)\,.$$

It suffices to prove that $\Omega f = 0$ a.e. Indeed, this property means that $f_r \to g$ converges a.e. to a measurable function g. Since by Lemma 2.19, $f_{r_i} \to f$ a.e. we get $g = f$ a.e.

Observe that $\Omega f \leq 2\mathcal{M} f$, hence for any $\varepsilon > 0$, Theorem 2.2(a) yields

$$|\{x : \Omega f(x) > \varepsilon\}| \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^n} |f|\,.$$

Let h be a continuous function such that $\|f - h\|_1 < \varepsilon^2$. Continuity of h implies $\Omega h = 0$ everywhere and hence

$$\Omega f \leq \Omega(f - h) + \Omega h = \Omega(f - h)\,,$$

so

$$|\{\Omega f > \varepsilon\}| \leq |\{\Omega(f - h) > \varepsilon\}| \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^n} |f - h| \leq C\varepsilon\,.$$

Since $\varepsilon > 0$ can be arbitrarily small we conclude $\Omega f = 0$ a.e. □

If $f\in L^1_{\rm loc}(\mathbb{R}^n)$, then we can define f at *every* point by the formula

$$f(x) := \limsup_{r\to 0} \fint_{B(x,r)} f(y)\, dy\, . \tag{2.7}$$

According to the Lebesgue differentiation theorem this is a representative of f in the class of functions that coincide with f a.e.

Definition 2.21. Let $f\in L^1_{\rm loc}(\mathbb{R}^n)$. We say that $x\in\mathbb{R}^n$ is a *Lebesgue point of f* if

$$\lim_{r\to 0}\fint_{B(x,r)}|f(y)-f(x)|\, dy = 0\, ,$$

where $f(x)$ is defined by (2.7).

Theorem 2.22. *If $f\in L^1_{\rm loc}(\mathbb{R}^n)$, then the set of points that are* not *Lebesgue points of f has measure zero.*

Proof. For $c\in\mathbb{Q}$ let E_c be the set of points for which

$$\lim_{r\to 0}\fint_{B(x,r)}|f(y)-c|\, dy = |f(x)-c| \tag{2.8}$$

does *not* hold. Clearly $|E_c|=0$ and hence the set $E=\bigcup_{c\in\mathbb{Q}}E_c$ has measure zero too. Thus for $x\in\mathbb{R}^n\setminus E$ and all $c\in\mathbb{Q}$, (2.8) is satisfied.

If $x\in\mathbb{R}^n\setminus E$ and $f(x)\in\mathbb{R}$, approximating $f(x)$ by rational numbers one can easily check that

$$\lim_{r\to 0}\fint_{B(x,r)}|f(y)-f(x)|\, dy = |f(x)-f(x)| = 0\, .$$

Indeed, if $\varepsilon>0$ is arbitrary and $c\in\mathbb{Q}$ is such that $|f(x)-c|<\varepsilon/3$, then there is $\delta>0$ such that for $0<r<\delta$

$$\begin{aligned}\fint_{B(x,r)}|f(y)-f(x)|\, dy &\leq \fint_{B(x,r)}|f(y)-c|\, dy + |c-f(x)|\\ &< \left(|f(x)-c|+\frac{\varepsilon}{3}\right)+|c-f(x)|<\varepsilon.\end{aligned}$$

The proof is complete. ☐

Definition 2.23. We say that $x\in\mathbb{R}^n$ is a *p-Lebesgue point* of $f\in L^p_{\rm loc}$, $1\leq p<\infty$ if

$$\lim_{r\to 0}\fint_{B(x,r)}|f(y)-f(x)|^p\, dy = 0\, .$$

The same method as above leads to the following result that we leave as an exercise.

Theorem 2.24. *If $f\in L^p_{\rm loc}(\mathbb{R}^n)$, $1\leq p<\infty$, then the set of points that are* not *p-Lebesgue points of f has measure zero.*

Definition 2.25. Let $E\subset\mathbb{R}^n$ be a measurable set. We say that $x\in\mathbb{R}^n$ is a density point of E if

$$\lim_{r\to 0}\frac{|B(x,r)\cap E|}{|B(x,r)|} = 1\, .$$

Applying the Lebesgue theorem to the characteristic function of the set E, i.e., $f = \chi_E$ we obtain

Theorem 2.26. *Almost every point of a measurable set $E \subset \mathbb{R}^n$ is its density point and a.e. point of $\mathbb{R}^n \setminus E$ is not a density point of E.*

The Lebesgue theorem shows that for almost all x, the averages of f over balls centered as x converge to $f(x)$. It is natural to inquire whether we can replace balls by other sets like cubes or even balls, but not centered at x.

Definition 2.27. We say that a family $\mathcal{F}$ of measurable subsets of $\mathbb{R}^n$ is *regular at $x \in \mathbb{R}^n$* if

(a) The sets are bounded and have positive measure.
(b) There is a sequence $\{S_i\} \subset \mathcal{F}$ with $|S_i| \to 0$ as $i \to \infty$.
(c) There is a constant $C > 0$ such that for all $S \in \mathcal{F}$ we have $|S| \geq C|B_S|$, where B_S is the smallest closed ball centered at x that contains S.

Note that if $|S_i| \to 0$, then the radius of B_{S_i} approaches to zero so the sets S_i approach to x.

Examples of families regular at x include:

- Family of all balls that contain x.
- Family of all cubes centered at x.
- Family of all cubes that contain x.

Theorem 2.28. *If $f \in L^1_{\rm loc}(\mathbb{R}^n)$, x is a Lebesgue point of f, and if $\mathcal{F}$ is a family regular at x, then*

$$\lim_{\substack{S\in\mathcal{F}\\ |S|\to 0}} ⨍_S f(y)\, dy = f(x)\, .$$

Proof. We have

$$\begin{aligned} \left| ⨍_S f(y)\, dy - f(x)\right| &\leq ⨍_S |f(y)-f(x)|\, dy \leq \frac{1}{|S|}\int_{B_S} |f(y)-f(x)|\, dy \\ &\leq \frac{|B_S|}{|S|} ⨍_{B_S} |f(y)-f(x)|\, dy \leq \frac{1}{C} ⨍_{B_S} |f(y)-f(x)|\, dy \to 0 \end{aligned}$$

as $|S| \to 0$. □

Note that if $\mathcal{F}$ is a family regular at 0 and if we define the maximal function associated with $\mathcal{F}$ by

$$\mathcal{M}_{\mathcal{F}} f(x) = \sup_{S\in\mathcal{F}} ⨍_S |f(x-y)|\, dy\, ,$$

then a routine calculation shows that

$$\mathcal{M}_{\mathcal{F}} f(x) \leq C\mathcal{M} f(x)\, , \tag{2.9}$$

so $\mathcal{M}_{\mathcal{F}}$ satisfies the claim of Theorem 2.2 (with different constants). In particular $\mathcal{M}_{\mathcal{F}}$ is a bounded operator in L^p, $1 < p \leq \infty$.

Let $\mathcal{F}$ be the family of all rectangular boxes in $\mathbb{R}^n$ that contain the origin and have sides parallel to the coordinate axes. With the family we can associate the maximal function

$$\widetilde{\mathcal{M}}f(x) = \sup_{S\in\mathcal{F}} \fint_S |f(x-y)|\, dy\, .$$

Note that the family $\mathcal{F}$ is *not* regular at 0 and hence the boundedness of $\widetilde{\mathcal{M}}$ in L^p, $1 < p \leq \infty$ cannot be concluded from (2.9). However, we have

Theorem 2.29 (Zygmund)**.** *For $1 < p < \infty$ there is a constant $C = C(n,p) > 0$ such that*

$$\|\widetilde{\mathcal{M}}f\|_p \leq C\|f\|_p\, . \tag{2.10}$$

Moreover, if $f \in L^p_{\rm loc}$, $1 < p < \infty$, then

$$\lim_{\substack{\operatorname{diam} S\to 0\\ S\in\mathcal{F}}} \fint_S f(x-y)\, dy = f(x) \quad a.e. \tag{2.11}$$

Proof. First we will prove how to conclude (2.11) from (2.10). Note that since the family is not regular at 0, (2.11) is not a consequence of Theorem 2.28 (see also Theorem 2.30). However,

$$0 \leq \limsup_{\substack{\operatorname{diam} S\to 0\\ S\in\mathcal{F}}} \fint_S f(x-y)\, dy - \liminf_{\substack{\operatorname{diam} S\to 0\\ S\in\mathcal{F}}} \fint_S f(x-y)\, dy \leq 2\widetilde{\mathcal{M}}f(x)$$

and hence (2.11) follows from (2.10) by almost the same argument as the one used to deduce the Lebesgue Differentiation Theorem from Theorem 2.2. We leave details to the reader.

Thus it remains to prove (2.10). In dimension one

$$\widetilde{\mathcal{M}}f(x) = \sup_{a,b>0} \fint_{x-a}^{x+b} |f(y)|\, dy$$

is the so called *non-centered* Hardy-Littlewood maximal function. Clearly $\widetilde{\mathcal{M}}f \leq C\mathcal{M}f$ which gives (2.10). For simplicity of notation we will prove the higher dimensional result for $n = 2$ only. The proof in the case of general $n > 1$ is similar. We have

$$\begin{aligned}
\widetilde{\mathcal{M}}f(x_1,x_2) &= \sup_{\substack{a_1,b_1>0\\ a_2,b_2>0}} \fint_{x_2-a_2}^{x_2+b_2} \fint_{x_1-a_1}^{x_1+b_1} |f(y_1,y_2)|\, dy_1\, dy_2 \\
&\leq \sup_{a_2,b_2>0} \fint_{x_2-a_2}^{x_2+b_2} \left(\sup_{a_1,b_1>0} \fint_{x_1-a_1}^{x_1+b_1} |f(y_1,y_2)|\, dy_1 \right) dy_2\, .
\end{aligned}$$

On the right hand side we have iteration of one dimensional non-centered maximal functions. First we apply the maximal function to variable y_1 and evaluate it at x_1 and then we apply the maximal function to the variable y_2 and evaluate it at x_2. It is easy to see now that inequality (2.10) follows from the one dimensional case applied twice and the Fubini theorem. $\square$

Surprisingly (2.11) does not hold for $p = 1$ and hence the maximal function $\widetilde{M}f$ does not satisfy the weak type estimate (2.1).[4] Namely one can prove the following result that we leave without a proof.

Theorem 2.30 (Saks)**.** *Let $\mathcal{F}$ be the family of all rectangular boxes in $\mathbb{R}^n$ that contain the origin and have sides parallel to the coordinate axes. Then the set of functions $f \in L^1(\mathbb{R}^n)$ such that*

$$\limsup_{\substack{\operatorname{diam} S \to 0 \\ S \in \mathcal{F}}} \fint_S f(x-y)\, dy = \infty \quad \textit{for all } x \in \mathbb{R}^n$$

is a dense G_δ subset of $L^1(\mathbb{R}^n)$. In particular it is not empty.

2.4. **The Calderón-Zygmund decomposition.** The following result will play an important role in what follows.

Theorem 2.31 (Calderón-Zygmund decomposition)**.** *Suppose $f \in L^1(\mathbb{R}^n)$, $f \geq 0$ and $\alpha > 0$. Then there is an open set Ω and a closed set F such that*

(a) $\mathbb{R}^n = \Omega \cup F$, $\Omega \cap F = \emptyset$;
(b) $f \leq \alpha$ *a.e. in* F;
(c) Ω *can be decomposed into cubes* $\Omega = \bigcup_{k=1}^\infty Q_k$ *with pairwise disjoint interiors such that*

$$\alpha < \fint_{Q_k} f(x)\, dx \leq 2^n \alpha\, , \quad k = 1, 2, 3, \ldots$$

Proof. Decompose $\mathbb{R}^n$ into a grid of identical cubes, large enough to have

$$\fint_Q f(x)\, dx \leq \alpha$$

for each cube in the grid. Take a cube Q from the grid and divide it into 2^n identical cubes. Let Q' be one of the cubes from this partition. We have two cases:

$$\fint_{Q'} f(x)\, dx > \alpha \qquad \text{or} \qquad \fint_{Q'} f(x)\, dx \leq \alpha\, .$$

If the fist case holds we include the open cube Q' to the family $\{Q_k\}$. Note that

$$\alpha < \fint_{Q'} f = 2^n |Q|^{-1} \int_{Q'} f \leq 2^n \fint_Q f \leq 2^n \alpha$$

so the condition (c) is satisfied. If the second case holds we divide Q' into 2^n identical cubes and proceed as above. We continue this process infinitely many times or until it is terminated. We apply it to all the cubes of the original grid. Let $\Omega = \bigcup_{k=1}^\infty Q_k$, where the cubes are defined by the first case of the process. It remains to prove that $f \leq \alpha$ a.e. in the set $F = \mathbb{R}^n \setminus \Omega$. The set F consists of faces of the cubes (this set has measure zero) and points x such that there is a sequence of cubes $\tilde{Q}_i$ with the property that $x \in \tilde{Q}_i$, $\operatorname{diam} \tilde{Q}_i \to 0$, $\fint_{\tilde{Q}_i} f \leq \alpha$. According to Theorem 2.28 for a.e. such x we have $\fint_{\tilde{Q}_i} f \to f(x)$ and hence $f \leq \alpha$ a.e. in F. □

Corollary 2.32. *Let f, α and Ω be as in Theorem 2.31. Then $|\Omega| < \alpha^{-1} \|f\|_1$.*

[4] Such estimate would imply (2.11).

Proof. We have

$$|\Omega| = \sum_{k=1}^{\infty} |Q_k| \le \sum_{k=1}^{\infty} \alpha^{-1} \int_{Q_k} |f| \le \alpha^{-1} \|f\|_1 \, .$$

The proof is complete. □

In Remark 2.9 we obtained an upper bound for the measure of the level set of the maximal function which improves the weak type estimate from Theorem 2.2. We will prove now a similar lower bound.

Proposition 2.33. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\frac{2^{-n}C(n)^{-1}}{t} \int_{\{|f|>C(n)t\}} |f(x)|\, dx \le |\{x : \mathcal{M}f(x) > t\}| \le \frac{2\cdot 5^n}{t} \int_{\{|f|>t/2\}} |f(x)|\, dx.$$

where we can take $C(n) = \omega_n n^{n/2}$.

Proof. For the proof of the right inequality see (2.4). In order to prove the left inequality we apply the Calderón-Zygmnud decomposition to the function $|f|$ and $\alpha = s > 0$. If $\Omega = \bigcup_k Q_k$ is an open set as in Theorem 2.31, then

$$s < \fint_{Q_k} |f(x)|\, dx \le 2^n s \quad \text{for all } k.$$

Let ℓ_k be the edge-length of the cube Q_k. Then for any $x \in Q_k$, $Q_k \subset B_x = B(x, \sqrt{n}\ell_k)$ so

$$s < \fint_{Q_k} |f(y)|\, dy \le \frac{|B_x|}{|Q_k|} \fint_{B_x} |f(y)|\, dy \le \omega_n n^{\frac{n}{2}} \mathcal{M}f(x).$$

Hence

$$\begin{aligned} |\{x : \mathcal{M}f(x) > s\omega_n^{-1} n^{-\frac{n}{2}}\}| &\ge \sum_k |Q_k| \ge \frac{2^{-n}}{s} \sum \int_{Q_k} |f(x)|\, dx \\ &\ge \frac{2^{-n}}{s} \int_{\{|f|>s\}} |f(y)|\, dy \end{aligned}$$

where the last inequality follows from the fact that $|f| \le s$ for almost all $x \notin \Omega$ which means that the set $\{|f| > s\}$ is contained in $\Omega = \bigcup_k Q_k$. Now it suffices to take $s = t\omega_n n^{n/2}$. □

2.5. **Integrability of the maximal function.** In Remark 2.3 we observed that if $f \in L^1(\mathbb{R}^n)$, then $\mathcal{M}f \notin L^1(\mathbb{R}^n)$. However, we showed that the function cannot be *globally* integrable. It turns out that, in general, the maximal function need not be even locally integrable. We will actually characterize all functions such that the maximal function is locally integrable.

Definition 2.34. We say that a measurable function f belongs to the *Zygmund space* $L\log L$ if $|f|\log(e + |f|) \in L^1$.

It is easy to see that for a space with finite measure we have

$$\bigcap_{p>1} L^p \subset L\log L \subset L^1 \, ,$$

so the Zygmund space is an intermediate space between all L^p for $p>1$ and L^1.

Theorem 2.35 (Stein). *Suppose that a measurable function f defined in $\mathbb{R}^n$ equals zero outside a ball B. Then $\mathcal{M}f \in L^1(B)$ if and only if $f \in L\log L(B)$.*

Proof. Suppose that $f \in L\log L(B)$.[5] Cavalieri's principle (Theorem 2.5) and weak type estimates from Proposition 2.33 give

$$\begin{aligned}
\int_B \mathcal{M}f(x)\,dx &= \int_0^\infty |\{x\in B:\, \mathcal{M}f(x)>t\}|\,dt \\
&\leq |B| + \int_1^\infty |\{x\in B:\, \mathcal{M}f(x)>t\}|\,dt \\
&\leq |B| + \int_1^\infty \Big(\frac{2\cdot 5^n}{t}\int_{\{|f|>t/2\}} |f(x)|\,dx\Big)\,dt \\
&= |B| + C(n)\int_B \Big(\int_1^{\max\{2|f(x)|,1\}} \frac{dt}{t}\Big)\,|f(x)|\,dx \\
&= |B| + C(n)\int_B |f(x)|\max\{0,\log(2|f(x)|)\}\,dx \\
&\leq |B| + 2C(n)\int_B |f(x)|\log(e+|f(x)|)\,dx < \infty.
\end{aligned}$$

To prove the other implication we assume $\mathcal{M}f \in L^1(B)$ and we want to prove that $f \in L\log L(B)$. In the proof of the first implication we used the right inequality from Proposition 2.33, but now we want to use the left inequality. However, the following calculations seem to disprove integrability of $\mathcal{M}f$ on B.

$$\begin{aligned}
\int_B \mathcal{M}f(x)\,dx &= \int_0^\infty |\{x\in B:\, \mathcal{M}f(x)>t\}|\,dx \geq \int_0^\infty \Big(\frac{C_1}{t}\int_{\{|f(x)|>C_2 t\}} |f(x)|\,dx\Big)\,dt \\
&= C_1\int_B \Big(\underbrace{\int_0^{|f(x)|/C_2}\frac{dt}{t}}_{=+\infty \text{ if } f(x)\neq 0.}\Big)|f(x)|\,dx = +\infty.
\end{aligned}$$

These calculations are erroneous. Although the function f is supported in the ball B only, the maximal function $\mathcal{M}f$ is positive on all of $\mathbb{R}^n$ and in the left estimate in Proposition 2.33 we deal with the measure of the set

$$\{x\in\mathbb{R}^n:\, \mathcal{M}f(x)>t\} \quad \text{and not of the set} \quad \{x\in B:\, \mathcal{M}f(x)>t\}.$$

We will show not how to overcome this difficulty.

Note that $\mathcal{M}f \in L^1_{\rm loc}(\mathbb{R}^n)$. Indeed, $\mathcal{M}f$ is integrable in B and locally bounded outside the closed ball B, so we need to verify integrability of $\mathcal{M}f$ in a neighborhood of the boundary of B.[6] But it is easy to see that if x is near the boundary and outside the ball,

[5] The proof of integrability of $\mathcal{M}f$ will be similar to the proof of L^p integrability of the maximal function when $f\in L^p$, $p>1$.

[6] Integrability on B and local boundedness outside B does not imply local integrability in $\mathbb{R}^n$. The function $f(x)=0$ for $x\leq 0$ and $f(x)=1/x$ for $x>0$ is integrable on $(-\infty,0]$ and is locally bounded on $(0,\infty)$.

we can estimate the value of $\mathcal{M}f(x)$ by (constant times) the value of the maximal function in a point being the reflection of x across the boundary. Thus the integrability of $\mathcal{M}f$ near the boundary follows from the integrability of $\mathcal{M}f$ in B. Note also that the set $\{\mathcal{M}f > 1\}$ is bounded, because $\mathcal{M}f(x)$ decays to zero as $x \to \infty$. Thus local integrability of $\mathcal{M}f$ and boundedness of the set $\{\mathcal{M}f > 1\}$ implies that the function $\mathcal{M}f$ in integrable in $\{\mathcal{M}f > 1\}$. With these remarks we can complete the proof as follows.

$$
\begin{aligned}
\infty &> \int_{\{\mathcal{M}f>1\}} \mathcal{M}f(x)\,dx \\
&\geq \int_1^\infty |\{\mathcal{M}f > t\}|\,dt \\
&\geq \int_1^\infty \frac{2^{-n}C^{-1}}{t} \int_{\{|f|>Ct\}} |f(x)|\,dx\,dt \\
&= 2^{-n}C^{-1} \int_B \left(\int_1^{\max\{|f(x)|/C,1\}} \frac{dt}{t} \right) |f(x)|\,dx \\
&= 2^{-n}C^{-1} \int_B |f(x)| \max\{0, \log(|f(x)|/C)\}\,dx\,.
\end{aligned}
$$

This easily implies that $f \in L\log L$. □

Remark 2.36. A more careful analysis leads to the following version of Stein's theorem. In an open set $\Omega \subset \mathbb{R}^n$ we define the local maximal function by

$$\mathcal{M}_\Omega f(x) = \sup \left\{ ⨍_Q |f| \,:\, x \in Q \subset \Omega \right\},$$

where the supremum is taken over all cubes Q in Ω that contain x.

Theorem 2.37 (Stein)**.** *Let $Q \subset \mathbb{R}^n$ be a cube. Then $\mathcal{M}_Q f \in L^1(Q)$ if and only if $f \in L\log L(Q)$. Moreover*

$$5^{-(n+1)} \int_Q \mathcal{M}_Q f \leq \int_Q |f| \log\left(e + \frac{|f|}{|f|_Q} \right) \leq 2^{n+2} \int_Q \mathcal{M}_Q f\,,$$

where

$$|f|_Q = ⨍_Q |f|\,.$$

2.6. **The Minkowski integral inequality.** Let $F_k \in L^p(Y,\nu)$ and denote the variable by y. If we write $F(k,y)$ instead of $F_k(y)$, then the classical Minkowski inequality $\||F_1| + \ldots + |F_m|\|_p \leq \|F_1\|_p + \ldots + \|F_m\|_p$ can be rewritten as

$$\left(\int_Y \Big(\sum_k |F(k,y)| \Big)^p d\nu(y) \right)^{1/p} \leq \sum_k \left(\int_Y |F(k,y)|^p\,d\nu(y) \right)^{1/p}. \tag{2.12}$$

Since the sum over k can be regarded as an integral with respect to the counting measure, the following result is a natural generalization of the Minkowski inequality.

Theorem 2.38 (Minkowski's integral inequality)**.** *If μ and ν are σ-finite measures on X and Y respectively and if $F : X \times Y \to \mathbb{R}$ is measurable, then for $1 \leq p < \infty$ we have*

$$\left(\int_Y \left(\int_X |F(x,y)|\, d\mu(x)\right)^p d\nu(y)\right)^{1/p} \leq \int_X \left(\int_Y |F(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x)\,.$$

Remark 2.39. It would be natural to try approximate F by simple functions in x and apply the discrete version of the inequality (2.12). The proof given below uses however, a very different and much more elegant argument.

Proof. If $p = 1$ then we actually have equality by Fubini's theorem. Thus assume that $1 < p < \infty$. We will use the fact that the $L^p(Y,\nu)$ norm of the function $y \mapsto \int_X |F(x,y)|\, d\mu(x)$ equals to the norm of the corresponding functional on $L^q(Y,\nu)$, $p^{-1} + q^{-1} = 1$.

$$\begin{aligned}
&\left(\int_Y \left(\int_X |F(x,y)|\, d\mu(x)\right)^p d\nu(y)\right)^{1/p} \\
&= \sup_{\substack{h \in L^q(\nu) \\ \|h\|_q = 1}} \int_Y h(y) \left(\int_X |F(x,y)|\, d\mu(x)\right) d\nu(y) \\
&= \sup_{\substack{h \in L^q(\nu) \\ \|h\|_q = 1}} \int_X \left(\int_Y h(y)|F(x,y)|\, d\nu(y)\right) d\mu(x) \\
&\leq \sup_{\substack{h \in L^q(\nu) \\ \|h\|_q = 1}} \int_X \underbrace{\left(\int_Y |h(y)|^q\, d\nu(y)\right)^{1/q}}_{1} \left(\int_Y |F(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x) \\
&= \int_X \left(\int_Y |F(x,y)|^p\, d\nu(y)\right)^{1/p} d\mu(x)\,.
\end{aligned}$$

The proof is complete. □

2.7. **Convergence of averages.** In this section we will show that a large class of averages of f converge to f. Our approach will be based on the maximal function estimates. This will lead to far reaching generalizations of the Lebesgue Differentiation Theorem 2.20. Note that in the proofs of Theorem 2.20 and its generalizations discussed above (Theorems 2.28 and 2.29), maximal function also plays an important role. The results of this section will be used later in Section 3.3

Since many averages are constructed with the help of convolution, let us recall that the *convolution* of measurable functions f and g is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\, dy = \int_{\mathbb{R}^n} f(y)g(x-y)\, dy.$$

The equality of the integrals follows from a linear change of variables. This equality actually means that $f * g = g * f$. One can also easily check that $(f * g) * h = f * (g * h)$.

Theorem 2.40. *Let $g \in L^1(\mathbb{R}^n)$. If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $f * g \in L^p(\mathbb{R}^n)$ and*

$$\|f * g\|_p \leq \|f\|_p \|g\|_1. \tag{2.13}$$

If $f \in C_0(\mathbb{R}^n)$, then $f * g \in C_0(\mathbb{R}^n)$ and

$$\|f * g\|_\infty \leq \|f\|_\infty \|g\|_1. \tag{2.14}$$

Proof. Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. The inequality can be obtained from the Minkowski Integral Inequality (Theorem 2.38) as follows

$$\begin{aligned} \|f * g\|_p &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)\, dy \right|^p dx \right)^{1/p} \\ &\leq \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)| \underbrace{|g(y)|\, dy}_{d\mu} \right)^p \underbrace{dx}_{d\nu} \right)^{1/p} \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^p\, dx \right)^{1/p} |g(y)|\, dy \\ &= \|f\|_p \|g\|_1 . \end{aligned}$$

The proof in the case of $f \in C_0(\mathbb{R}^n)$ is left to the reader as an exercise. □

Remark 2.41. In the above proof one could use the classical Minkowski inequality instead of the integral one via the following estimate

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dy &= \int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|^{1/p} |g(y)|^{1/q}\, dy \\ &\leq \left(\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|\, dy \right)^{1/p} \left(\int_{\mathbb{R}^n} |g(y)|\, dy \right)^{1/q} . \end{aligned}$$

We leave details to the reader.

Theorem 2.40 is a special case of a more general Young's inequality. We leave the proof as an exercise.

Theorem 2.42 (Young's inequality)**.** *If $1 \leq p, q, r \leq \infty$ and $q^{-1} = p^{-1} + r^{-1} - 1$, then*

$$\|f * g\|_q \leq \|f\|_p \|g\|_r .$$

For $\varphi \in L^1(\mathbb{R}^n)$ and $t > 0$ we define

$$\varphi_t(x) = t^{-n} \varphi\left(\frac{x}{t}\right).$$

Note that $\int_{\mathbb{R}^n} \varphi_t = \int_{\mathbb{R}^n} \varphi$.

Example 2.43. If $\varphi = \omega_n^{-1} \chi_{B(0,1)}$, then $\varphi_t = (\omega_n t^n)^{-1} \chi_{B(0,t)}$ so

$$f * \varphi_t(x) = \frac{1}{\omega_n t^n} \int_{B(0,t)} f(x-y)\, dy = \fint_{B(x,t)} f(y)\, dy.$$

Thus[7] in that case $f * \varphi_t \to f$ a.e. and in L^1, provided $f \in L^1$. Note also that

$$\sup_{t>0} (|f| * \varphi_t)(x) = \mathcal{M} f(x). \tag{2.15}$$

[7]See Lemma 2.19 and Theorem 2.20.

Actually this identity[8] plays a crucial role in the proof of the pointwise convergence $f*\varphi_t \to f$ a.e. In what follows we will study convergence of $f*\varphi_t$ for more general functions φ. Not surprisingly we will be looking for conditions that will guarantee an estimate similar to (2.15).[9]

Theorem 2.44. *Let $\varphi \in L^1(\mathbb{R}^n)$ and let $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$. If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ or $f \in C_0(\mathbb{R}^n)$ and $p = \infty$, then*

$$\lim_{t\to 0^+} \|f*\varphi_t - af\|_p = 0, \quad \textit{where} \quad a = \int_{\mathbb{R}^n} \varphi(x)\, dx. \tag{2.16}$$

Remark 2.45. The two most interesting cases are when $a = 1$ and when $a = 0$.

Remark 2.46. If $f \in C_0(\mathbb{R}^n)$, then $f*\varphi_t \in C_0(\mathbb{R}^n)$[10] and (2.16) with $p = \infty$ means that $f*\varphi_t$ converges to af uniformly.

Remark 2.47. If $a = 1$, then (2.16) means that $f*\varphi_t \to f$ in L^p so the operators $f \mapsto f*\varphi_t$ converge pointwise, i.e. for every f, to the identity operator $f \mapsto f$. For that reason the family of functions $\{\varphi_t : t > 0\}$ is often called an *approximation of identity* (provided $a = 1$).

Proof. We will prove the result when $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, leaving the case of $f \in C_0(\mathbb{R}^n)$ to the reader. We have

$$|f*\varphi_t(x) - af(x)| = \left| \int_{\mathbb{R}^n} (f(x-y) - f(x))\, \varphi_t(y)\, dy \right|.$$

Given $\delta > 0$, the Minkowski Integral Inequality yields

$$\begin{aligned} \|f*\varphi_t - af\|_p &\leq \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p\, dx \Big)^{1/p} |\varphi_t(y)|\, dy \\ &= \int_{\mathbb{R}^n} \|\tau_{-y}f - f\|_p |\varphi_t(y)|\, dy \\ &= \int_{\mathbb{R}^n} \|\tau_{-ty}f - f\|_p |\varphi(y)|\, dy, \end{aligned}$$

where the last equality follows from a simple linear change of variables.

Note that $\|\tau_{-ty}f - f\|_p|\varphi(y)| \leq 2\|f\|_p|\varphi(y) \in L^1$ and for every $y \in \mathbb{R}^n$, $\|\tau_{-ty}f - f\|_p \to 0$ as $t \to 0$ (Lemma 2.18) so

$$\|f*\varphi_t - af\|_p \leq \int_{\mathbb{R}^n} \|\tau_{-ty}f - f\|_p|\varphi(y)|\, dy \to 0 \quad \text{as } t \to 0$$

by the Dominated Convergence Theorem. □

Under assumptions of Theorem 2.44, if $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, there is a sequence $t_i \to 0$ such that $f*\varphi_{t_i} \to af$ a.e. as $i \to \infty$, but a more challenging question is to find conditions that would guarantee $f*\varphi_t \to af$ a.e. as $t \to 0$. That would be a true generalization of the Lebesgue Differentiation Theorem.

[8] Along with weak type estimates for the maximal function, see Theorem 2.20.

[9] See Theorem 2.49.

[10] See Theorem 2.40.

Definition 2.48. Let $\varphi \in L^1(\mathbb{R}^n)$. We say that Ψ is a *radially decreasing majorant of* φ if

(a) $\Psi(x) = \eta(|x|)$ for some[11] $\eta : [0, \infty) \to [0, \infty]$.[12]
(b) η is decreasing.[13].
(c) $|\varphi(x)| \leq \Psi(x)$ a.e.

Every function $\varphi \in L^1$ has the least radially decreasing majorant[14]

$$\Psi_0(x) = \operatorname*{ess\,sup}_{|y| \geq |x|} |\varphi(y)|.$$

Thus the existence of an integrable radially decreasing majorant Ψ of φ is equivalent to $\Psi_0 \in L^1$.

Theorem 2.49. *Suppose that* $\varphi \in L^1(\mathbb{R}^n)$ *has an integrable radially decreasing majorant*

$$\Psi(x) = \eta(|x|) \in L^1(\mathbb{R}^n)\,.$$

Then for $f \in L^1_{\rm loc}(\mathbb{R}^n)$ *and all* $x \in \mathbb{R}^n$

$$\left|\sup_{t>0} (f * \varphi_t)(x)\right| \leq \|\Psi\|_1 \mathcal{M} f(x)\,. \tag{2.17}$$

If in addition $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, *and* $\int_{\mathbb{R}^n} \varphi(x)\, dx = a$, *then*

$$f * \varphi_t \to af \quad \textit{a.e. as } t \to 0. \tag{2.18}$$

Proof. In order to prove (2.17) it suffices to prove that for a radially decreasing and integrable function Ψ we have

$$|f| * \Psi(x) \leq \|\Psi\|_1 \mathcal{M} f(x). \tag{2.19}$$

Indeed, since Ψ_t is also radially decreasing and integrable, applying (2.19) to Ψ_t will give

$$|f * \varphi_t(x)| \leq |f| * \Psi_t(x) \leq \|\Psi_t\|_1 \mathcal{M} f(x) = \|\Psi\|_1 \mathcal{M} f(x).$$

Thus it remains to prove (2.19).

For a positive integer k let

$$s_k(x) = \sum_{i=1}^{k\cdot 2^k} \frac{1}{2^k} \chi_{\{\Psi \geq i/2^k\}}(x) = \sum_{i=1}^{k \cdot 2^k} \frac{1}{2^k} \chi_{B_i}(x)\,.$$

Since Ψ is radially symmetric the sets

$$B_i = \left\{ x : \ \Psi(x) \geq \frac{i}{2^k} \right\}$$

are balls[15] centered at zero.

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[11] Functions of this form, i.e. functions constant on spheres $S^{n-1}(0, r)$ are called *radially symmetric*.
[12] It may happen that $\eta(t) = +\infty$ for all t.
[13] i.e. non-increasing
[14] However, it may happen that $\Psi_0 = +\infty$ everywhere.
[15] Open or closed – find an example such that for some t the balls are open while for other values of t they are closed.

The sequence $\{s_k\}$ is increasing and convergent to Ψ almost everywhere. Indeed, for $x \neq 0$, $\Psi(x) < \infty$[16] so $\Psi(x) < k_0$, for some $k_0 \in \mathbb{N}$. Then for $k \geq k_0$

$$\frac{\ell}{2^k} \leq \Psi(x) < \frac{\ell+1}{2^k}, \quad \text{for some } \ell = 0, 1, 2, \ldots k \cdot 2^k - 1.$$

Since the inequality $\Psi \geq i/2^k$ is satisfied for $i = 1, 2, \ldots, \ell$, the definition of the function s_k gives that $s_k(x) = \ell/2^k$ so

$$\Psi(x) - \frac{1}{2^k} < s_k(x) \leq \Psi(x).$$

That means $s_k(x) \to \Psi(x)$ for all all $x \neq 0$. Observe also that the sequence $\{s_k\}$ is increasing, $s_k \leq s_{k+1}$. Indeed, if $s_k(x) = \ell/2^k$, then $\Psi(x) \geq \ell/2^k = (2\ell)/(2^{k+1})$ so the definition of s_{k+1} gives

$$s_{k+1}(x) \geq \frac{1}{2^{k+1}} \cdot 2\ell = \frac{\ell}{2^k} = s_k(x).$$

Thus if we can prove

$$|f| * s_k(x) \leq \|\Psi\|_1 \mathcal{M}f(x), \tag{2.20}$$

inequality (2.19) will follow from the Monotone Convergence Theorem.

We have

$$|f| * s_k(x) = \sum_{i=1}^{k \cdot 2^k} \frac{|B_i|}{2^k} \, |f| * \frac{\chi_{B_i}}{|B_i|}(x).$$

Since[17]

$$|f| * \frac{\chi_{B_i}}{|B_i|}(x) = \fint_{B_i} |f(x-y)| \, dy \leq \mathcal{M}f(x) \quad \text{and} \quad \sum_{i=1}^{k \cdot 2^k} \frac{|B_i|}{2^k} = \|s_k\|_1 \leq \|\Psi\|_1,$$

inequality (2.20) follows. The proof of (2.17) is complete.

It remains now to prove convergence (2.18).

If $p = 1$, then

$$\Omega f(x) := \limsup_{t \to 0^+} f * \varphi_t - \liminf_{t \to 0^+} f * \varphi_t \leq 2\|\Psi\|_1 \mathcal{M}f(x) \tag{2.21}$$

and almost the same argument as in the proof of Theorem 2.20 yields that[18] $\Omega f = 0$ a.e. so $f * \varphi_t$ converges a.e. to a measurable function. Since $f * \varphi_t \to af$ in L^1 (Theorem 2.44), (2.18) follows.

The case $1 < p < \infty$ uses a similar argument. If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then Chebyschev's inequality (2.2) and Theorem 2.2(b) yield

$$\begin{aligned} |\{x : \mathcal{M}f(x) > t\}| &= |\{x : |\mathcal{M}f(x)|^p > t^p\}| \leq \frac{1}{t^p} \int_{\mathbb{R}^n} |\mathcal{M}f(x)|^p \, dx \\ &\leq \frac{C(n,p)}{t^p} \int_{\mathbb{R}^n} |f(x)|^p \, dx. \end{aligned}$$

[16]Because $\Psi \in L^1$.

[17]Because the balls B_i are centered at the origin.

[18]We will actually repeat this argument in the case of $1 < p < \infty$.

Hence for $\varepsilon > 0$, inequality (2.21) yields

$$|\{x : \Omega f(x) > \varepsilon\}| \leq \frac{C}{\varepsilon^p} \int_{\mathbb{R}^n} |f(x)|^p \, dx,$$

where the constant C depends on n, p and $\|\Psi\|_1$. If $h \in C_0(\mathbb{R}^n)$, then $\Omega h=0$.[19] Taking $h \in C_0(\mathbb{R}^n)$ such that $\|f - h\|_p^p \leq \varepsilon^{p+1}$ we conclude that

$$|\{x : \Omega f(x) > \varepsilon\}| = |\{x : \Omega(f - h)(x) > \varepsilon\}| < C\varepsilon$$

so $\Omega f = 0$ a.e. and exactly as in the case of $p = 1$, (2.18) follows.

□

[19]Because $h * \varphi_t$ converges uniformly (and hence pointwise) to h.

3. The Fourier transform

3.1. **Fourier transform.**

Definition 3.1. For $f \in L^1(\mathbb{R}^n)$ we define the *Fourier transform* as

$$\mathcal{F}(f)(x) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx \quad \text{where} \quad x\cdot\xi = \sum_{j=1}^{n} x_j\xi_j\,.$$

Theorem 3.2. *The Fourier transform has the following properties*

(a) $\hat{}: L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ *is a bounded linear operator with* $\|\hat{f}\|_\infty \le \|f\|_1$.

(b) $(f*g)\hat{} = \hat{f}\hat{g}$ *for* $f, g \in L^1(\mathbb{R}^n)$.

(c) *If* $f, g \in L^1(\mathbb{R}^n)$, *then* $\hat{f}g, f\hat{g} \in L^1(\mathbb{R}^n)$ *and*

$$\int_{\mathbb{R}^n} \hat{f}(x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx.$$

(d) *If* $\tau_h f(x) = f(x+h)$, *then*

$$(\tau_h f)\hat{}(\xi) = \hat{f}(\xi)e^{2\pi i h\cdot\xi} \quad \textit{and} \quad (f(x)e^{2\pi i h\cdot x})\hat{}(\xi) = \hat{f}(\xi - h).$$

(e) *If* $f_t(x) = t^{-n}f(x/t)$, *then*

$$(f_t)\hat{}(\xi) = \hat{f}(t\xi) \quad \textit{and} \quad (f(tx))\hat{}(\xi) = (\hat{f})_t(\xi)\,.$$

(f) *If* $\rho \in O(n)$ *is an orthogonal transformation, then*

$$(f(\rho\,\cdot))\hat{}(\xi) = \hat{f}(\rho\xi).$$

Proof. (a) Clearly, $\hat{}: L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is a bounded linear mapping with $\|\hat{f}\|_\infty \le \|f\|_1$. Indeed,

$$|\hat{f}(\xi)| \le \int_{\mathbb{R}^n} \left|f(x)e^{-2\pi i x\cdot\xi}\right| dx = \|f\|_1.$$

The Dominated Convergence Theorem implies that the function $\hat{f}$ is continuous. It remains to prove that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. Let $\xi \ne 0$. Since $e^{\pi i} = -1$ we have

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx = -\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}e^{\pi i}\,dx \\ &= -\int_{\mathbb{R}^n} f(x)\exp\Big(-2\pi i\Big(x - \frac{\xi}{2|\xi|^2}\Big)\cdot\xi\Big)\,dx\end{aligned}$$

$$= \; -\int_{\mathbb{R}^n} f\Big(x+\frac{\xi}{2|\xi|^2}\Big)\, e^{-2\pi i x\cdot\xi}\, dx\, .$$

Hence

$$\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^n} \Big(f(x) - f\Big(x+\frac{\xi}{2|\xi|^2}\Big)\Big)\, e^{-2\pi i x\cdot\xi}\, dx$$

and thus Lemma 2.18 yields

$$|\hat{f}(\xi)| \leq \frac{1}{2}\Big\| f - \tau_{\frac{\xi}{2|\xi|^2}} f\Big\|_1 \to 0 \quad \text{as } |\xi|\to\infty.$$

(b)

$$\begin{aligned}(f*g)\hat{\ }(\xi) &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x-y)g(y)\, dy\Big)\, e^{-2\pi i x\cdot\xi}\, dx\\ &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x-y)e^{-2\pi i(x-y)\cdot\xi}\, dx\Big)\, g(y)e^{-2\pi i y\cdot\xi}\, dy\\ &= \hat{f}(\xi)\hat{g}(\xi)\, .\end{aligned}$$

(c) $\hat{f}g, f\hat{g}\in L^1$, because the functions $\hat{f},\hat{g}$ are bounded and the equality of the integrals easily follows from the Fubini theorem.

(d)

$$\begin{aligned}(\tau_h f)\hat{\ }(\xi) &= \int_{\mathbb{R}^n} f(x+h)e^{-2\pi i x\cdot\xi}\, dx = \int_{\mathbb{R}^n} f(x)e^{-2\pi i(x-h)\cdot\xi}\, dx\\ &= e^{2\pi i h\cdot\xi}\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\, dx = e^{2\pi i h\cdot\xi}\hat{f}(\xi)\, .\end{aligned}$$

The second equality follows from a similar argument.

(e)

$$\begin{aligned}(f_t)\hat{\ }(\xi) &= \int_{\mathbb{R}^n} t^{-n} f\Big(\frac{x}{t}\Big)\, e^{-2\pi i x\cdot\xi}\, dx = \int_{\mathbb{R}^n} f(y)e^{-2\pi i(ty)\cdot\xi}\, dy\\ &= \int_{\mathbb{R}^n} f(y)e^{-2\pi i y\cdot(t\xi)}\, dy = \hat{f}(t\xi)\, .\end{aligned}$$

The second formula follows from the first one if we replace t by t^{-1}.

(f)[20]

$$\begin{aligned}(f(\rho\,\cdot))\hat{\ }(\xi) &= \int_{\mathbb{R}^n} f(\rho x)e^{-2\pi i x\cdot\xi}\, dx = \int_{\mathbb{R}^n} f(y)e^{-2\pi i(\rho^{-1}y)\cdot\xi}\, dy\\ &= \int_{\mathbb{R}^n} f(y)e^{-2\pi i y\cdot(\rho\xi)}\, dy = \hat{f}(\rho\xi)\, .\end{aligned}$$

Equality $(\rho^{-1}y)\cdot\xi = y\cdot(\rho\xi)$ follows from the fact that the mapping $x\mapsto\rho x$ is an isometry. □

[20]This property is in some sense obvious: the Fourier transform does not depend on the choice of the coordinate system since it depends on the scalar product $x\cdot\xi$. Thus the Fourier transform should stay the same if we rotate the coordinate system by ρ.

Theorem 3.3. *Suppose $f \in L^1(\mathbb{R}^n)$ and $x_k f(x) \in L^1(\mathbb{R}^n)$, where x_k is the k-th coordinate function. Then $\hat{f}$ is differentiable with respect to ξ_k and*

$$(-2\pi i x_k f(x))\hat{} = \frac{\partial \hat{f}}{\partial \xi_k}(\xi)\,.$$

Proof. Let e_k be the unit vector along the k-th coordinate. Then the second part of Theorem 3.2(d) gives

$$\frac{\hat{f}(\xi + he_k) - \hat{f}(\xi)}{h} = \left(\frac{e^{-2\pi i (he_k)\cdot x} - 1}{h}\, f(x) \right)^{\wedge}(\xi) \to (-2\pi i x_k f(x))\hat{}(\xi)\,.$$

The convergence follows from the continuity of the Fourier transform in L^1. □

Definition 3.4. We say that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is *L^p-differentiable with respect to x_k* if there is $g \in L^p(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \left| \frac{f(x+he_k) - f(x)}{h} - g(x) \right|^p dx \to 0 \quad \text{as } h \to 0.$$

The function g is called the *L^p-partial derivative of f with respect to x_k* and is denoted by $g = \partial f / \partial x_k$.

Theorem 3.5. *If $f \in L^1(\mathbb{R}^n)$ is L^1-differentiable with respect to x_k, then*

$$\left(\frac{\partial f}{\partial x_k} \right)^{\wedge} = 2\pi i \xi_k \hat{f}(\xi)\,.$$

Proof. The first part of Theorem 3.2(d) and the L^1-differentiability of f give

$$\left(\frac{\partial f}{\partial x_k} \right)^{\wedge} - \hat{f}(\xi)\, \frac{e^{2\pi i (he_k)\cdot \xi} - 1}{h} = \left(\frac{\partial f}{\partial x_k} - \frac{f(x+he_k) - f(x)}{h} \right)^{\wedge} \to 0$$

as $h \to 0$, so

$$\left(\frac{\partial f}{\partial x_k} \right)^{\wedge}(\xi) = \lim_{h\to 0} \hat{f}(\xi)\, \frac{e^{2\pi i (he_k)\cdot \xi} - 1}{h} = 2\pi i \xi_k \hat{f}(\xi)\,.$$

The proof is complete. □

Remark 3.6. With each polynomial

$$P(x) = \sum_{|\alpha| \le m} a_\alpha x^\alpha$$

of variables $x_1, \dots, x_n$ we associate a differential operator

$$P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha = \sum_{|\alpha| \le m} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}\,.$$

Then under suitable assumptions Theorems 3.3 and 3.5 have the following higher order generalizations

$$P(D)\hat{f}(\xi) = (P(-2\pi i x) f(x))\hat{}(\xi), \quad (P(D)f)\hat{}(\xi) = P(2\pi i \xi)\hat{f}(\xi). \tag{3.1}$$

The next result is our first (and actually one of the most important) example of the Fourier transform.

Theorem 3.7. *If $f(x) = e^{-\pi|x|^2}$, then $\hat{f}(\xi) = e^{-\pi|\xi|^2}$.*

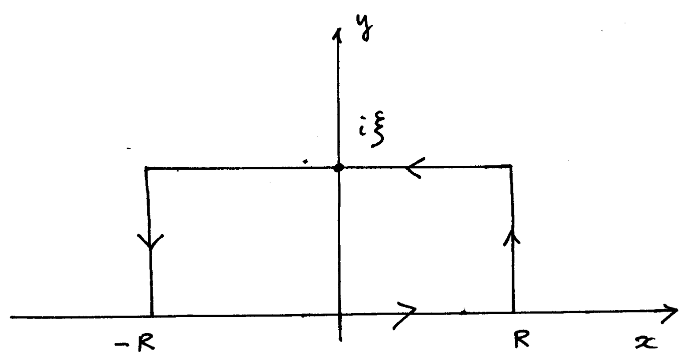
Proof. First observe that it suffices to prove the result in dimension 1. Indeed, assuming it for $n = 1$, the case of general n follows from Fubini's theorem

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\pi|x|^2} e^{-2\pi i x\cdot\xi}\, dx = \prod_{k=1}^{n} \int_{-\infty}^{\infty} e^{-\pi x_k^2} e^{-2\pi i x_k \xi_k}\, dx_k = \prod_{k=1}^{n} e^{-\pi\xi_k^2} = e^{-\pi|\xi|^2}\,.$$

In the case $n = 1$ we will show two different proofs. The first one will use the contour integration while the second one will use differential equations.

Thus in the remaining part of the proof we will assume that $f(x) = e^{-\pi x^2}$ is a function of one variable.

Proof by contour integration. The function $e^{-\pi z^2}$ is holomorphic and hence its integral along the following curve equals zero.



Letting $R \to \infty$ we obtain

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\, dx\,.$$

The left hand side equals 1, so

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} e^{\pi\xi^2}\, dx\,.$$

Hence

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx = \hat{f}(\xi)\,.$$

The proof is complete.

Proof by differential equations. We have

$$f'(x) = -2\pi x f(x), \qquad -i(-2\pi i x f(x)) = f'(x)$$

so Theorems 3.3 and 3.5 yield

$$-i\underbrace{\big(-2\pi i x f(x)\big)^{\wedge}(\xi)}_{(\hat{f})'(\xi)} = \underbrace{\widehat{f'}(\xi)}_{2\pi i\xi\hat{f}(\xi)}\,, \qquad (\hat{f})'(\xi) = -2\pi\xi\hat{f}(\xi).$$

Solving this differential equation for the unknown function $\xi \mapsto \hat{f}(\xi)$ yields

$$\hat{f}(\xi) = \hat{f}(0)e^{-\pi\xi^2} = e^{-\pi\xi^2}, \quad \text{because} \quad \hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1.$$

□

3.2. **Measures of finite total variation.** Let us first recall basic facts regarding measures of finite total variation.

If μ is a complex (Borel) measure on $\mathbb{R}^n$, then there is a unique positive measure $|\mu|$ called the *total variation of measure* μ and a Borel function $h : \mathbb{R}^n \to \mathbb{C}$, $|h(x)| = 1$ for all $x \in \mathbb{R}^n$ such that

$$\mu(E) = \int_E h(x)\, d|\mu|(x) \quad \text{for all Borel sets } E \subset \mathbb{R}^n.$$

The space $\mathcal{B}(\mathbb{R}^n)$ of complex measures of finite total variation $\|\mu\| := |\mu|(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|$.

Note that $f \in L^1(\mathbb{R}^n)$ defines a measure of finite total variation by

$$\mu(E) = \int_E f(x)\, dx.$$

In that case

$$|\mu|(E) = \int_E |f(x)|\, dx \quad \text{so} \quad \|\mu\| = \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_1 < \infty.$$

This proves that $L^1(\mathbb{R}^n)$ is a closed subspace of $\mathcal{B}(\mathbb{R}^n)$.

In a special case when μ is a real valued measure, called *signed measure*, we have $h : \mathbb{R}^n \to \{\pm 1\}$ so if we define $h^+ = h\chi_{\{h=1\}}$ and $h^- = h\chi_{\{h=-1\}}$, then

$$\mu^{\pm}(E) = \int_E h^{\pm}(x)\, d|\mu|(x)$$

are positive measures such that

$$\mu = \mu^+ - \mu^- \tag{3.2}$$

and

$$|\mu| = \mu^+ + \mu^-.$$

The representation (3.2) is called the *Hahn decomposition of* μ. Note that the measures μ^+ and μ^- are supported on disjoint sets.

Theorem 3.8 (Riesz representation theorem)**.** *The dual space to $C_0(\mathbb{R}^n)$ is isometrically isomorphic to the space of measures of finite total variation. More precisely, if $\Phi \in (C_0(\mathbb{R}^n))^*$, then there is a unique measure μ of finite total variation such that*

$$\Phi(f) = \int_{\mathbb{R}^n} f\, d\mu \quad \textit{for } f \in C_0(\mathbb{R}^n).$$

Moreover $\|\Phi\| = \|\mu\| = |\mu|(\mathbb{R}^n)$.

This result allows us to define convolution of measures.

If $f,g\in L^1(\mathbb{R}^n)$, then $f*g\in L^1(\mathbb{R}^n)\subset\mathcal{B}(\mathbb{R}^n)$ and hence it acts as a functional on $C_0(\mathbb{R}^n)$ by the formula

$$\begin{aligned}\Phi(h) &= \int_{\mathbb{R}^n} h(x)(f*g)(x)\,dx = \int_{\mathbb{R}^n} h(x)\Big(\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\Big)\,dx \\ &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} h(x)f(x-y)\,dx\Big)\,g(y)\,dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x+y)f(x)g(y)\,dx\,dy\,.\end{aligned}$$

This suggests how to define convolution of measures.

If $\mu_1,\mu_2\in\mathcal{B}(\mathbb{R}^n)$, then

$$\Phi(h)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x+y)\,d\mu_1(x)\,d\mu_2(y)$$

defines a functional on $C_0(\mathbb{R}^n)$ and hence there is a unique measure $\mu\in\mathcal{B}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x+y)\,d\mu_1(x)\,d\mu_2(y)=\int_{\mathbb{R}^n} h(x)\,d\mu(x)\quad\text{for all } h\in C_0(\mathbb{R}^n).$$

Definition 3.9. We denote the measure μ by

$$\mu=\mu_1*\mu_2$$

and call it *convolution of measures.*

Clearly

$$\mu_1*\mu_2=\mu_2*\mu_1\quad\text{and}\quad\|\mu_1*\mu_2\|\leq\|\mu_1\|\,\|\mu_2\|.$$

If $d\mu_1=f\,dx$, $d\mu_2=g\,dx$, then

$$d(\mu_1*\mu_2)=(f*g)\,dx,$$

so the convolution of measures extends the notion of convolution of functions. If $d\mu_1=f\,dx$ and $\mu\in\mathcal{B}(\mathbb{R}^n)$, then

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x+y)\,d\mu_1(x)\,d\mu(y) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x+y)f(x)\,dx\,d\mu(y) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x)f(x-y)\,dx\,d\mu(y) \\ &= \int_{\mathbb{R}^n} h(x)\Big(\int_{\mathbb{R}^n} f(x-y)\,d\mu(y)\Big)\,dx.\end{aligned}$$

Thus $\mu_1*\mu$ can be identified with a function

$$x\mapsto\int_{\mathbb{R}^n} f(x-y)\,d\mu(y)\in L^1(\mathbb{R}^n),$$

so we can write

$$f*\mu=\int_{\mathbb{R}^n} f(x-y)\,d\mu(y),\qquad\|f*\mu\|_1\leq\|f\|_1\,\|\mu\|.$$

Theorem 3.10. *If $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $\mu \in \mathcal{B}(\mathbb{R}^n)$, then*

$$\|f * \mu\|_p \leq \|f\|_p \, \|\mu\|.$$

Proof is almost the same as that for Theorem 2.40 as we leave it to the reader.

Definition 3.11. The Fourier transform of a measure of finite total variation $\mu \in \mathcal{B}(\mathbb{R}^n)$ is defined by

$$\hat{\mu}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \, d\mu.$$

Proposition 3.12. *If $\mu \in \mathcal{B}(\mathbb{R}^n)$, then $\hat{\mu}$ is a bounded and continuous function such that $\|\hat{\mu}\|_\infty \leq \|\mu\|$.*

Proof. Boundedness of $\hat{\mu}$ is obvious and continuity follows from the Dominated Convergence Theorem. □

Proposition 3.13. *If $\mu_1, \mu_2 \in \mathcal{B}(\mathbb{R}^n)$, then $\widehat{\mu_1 * \mu_2} = \hat{\mu}_1 \cdot \hat{\mu}_2$.*

We leave the proof as an exercise.

Remark 3.14. Convolution of measures and the Fourier transform of measures play a fundamental role in probability.

3.3. **Summability methods.** The answer to an important question of how to reconstruct a function f from its Fourier transform $\hat{f}$ is provided by the *inversion formula*[21]

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) \, e^{2\pi i x \cdot \xi} \, d\xi \, . \tag{3.3}$$

It seems that this formula requires both f and $\hat{f}$ to be integrable. Integrability of f is needed for the existence of $\hat{f}$ and integrability of $\hat{f}$ is needed for the integral in (3.3) to make sense. We will actually prove in Corollary 3.26 that if $f, \hat{f} \in L^1$, then (3.3) is true a.e. There is, however, a problem here. Equality (3.3) means that $f(x)$ is the Fourier transform of $\hat{f} \in L^1$ evaluated at $-x$. Since the Fourier transform of an integrable function $\hat{f} \in L^1$ belongs to $C_0(\mathbb{R}^n)$, it follows that $f \in C_0(\mathbb{R}^n)$. That means, if $f \in L^1 \setminus C_0$, then $\hat{f}$ cannot be integrable so the right hand side of (3.3) does not make much sense. Is there any chance to give meaning to the integral at (3.3), beyond the class of integrable Fourier transforms so that the equality at (3.3) would be true for all $f \in L^1$? The answer is in the positive but we need to interpret the integral at (3.3) using suitable *summability methods*.[22]

[21]Compare it with the formula for Fourier series:

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i x n} \, dx, \qquad f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i x n}.$$

[22]An example of a summability method for sequences is given by the so called *Cesàro method*: with each sequence $\{a_n\}$ we associate the limit of $(a_1 + \ldots + a_n)/n$. This extends the notion of limit far beyond the class of convergent sequences.

If $\Phi\in C_0(\mathbb{R}^n)$ is such that $\Phi(0)=1$, then for $t>0$ we define the Φ-*mean* of a function f as

$$M_{\Phi,t}f=M_tf=\int_{\mathbb{R}^n} f(x)\Phi(tx)\,dx.$$

If $f\in L^1(\mathbb{R}^n)$, then $M_{\Phi,t}f\to\int_{\mathbb{R}^n}f(x)\,dx$ as $t\to 0^+$, but the limit of M_tf might also exist for some non-integrable functions f.

Definition 3.15. We say that $\int_{\mathbb{R}^n} f$ is Φ-*summable to* $\ell\in\mathbb{R}$ if $\lim_{t\to 0^+}M_{\Phi,t}f=\ell$.

In the case of $\Phi(x)=e^{-|x|}$ and $\Phi(x)=e^{-|x|^2}$ we obtain the so called the *Abel* and the *Gauss summability* methods. More precisely we have

Definition 3.16. For each $t>0$ the *Abel mean* and the *Gauss mean* of a function f are

$$A_t(f)=\int_{\mathbb{R}^n} f(x)e^{-t|x|}\,dx \quad\text{and}\quad G_t(f)=\int_{\mathbb{R}^n} f(x)\,e^{-t|x|^2}\,dx\,.$$

We say that $\int_{\mathbb{R}^n}f(x)\,dx$ is *Abel summable*[23] (*Gauss summable*) to $\ell\in\mathbb{R}$ if $\lim_{t\to 0}A_t(f)=\ell$ ($\lim_{t\to 0}G_t(f)=\ell$).

Remark 3.17. If $\Phi(x)=e^{-|x|^2}$, then $\Phi(tx)=e^{-t^2|x|^2}$ so $M_{\Phi,t}f=G_{t^2}(f)$. This however, does not cause any problems when you compare notions of the Φ-summability and the Gauss summability because $\lim_{t\to 0^+}M_{\Phi,t}f=\lim_{t\to o}G_{t^2}(f)$.

Remark 3.18. Clearly, in the case of integrable functions, the Abel and the Gauss methods give the usual integral. However, the means $A_t(f)$ and $G_t(f)$ are finite whenever f is bounded so for some bounded, but non-integrable f, the integral $\int_{\mathbb{R}^n}f(x)\,dx$ might still be Abel or Gauss summable to a finite value. For example if $f\in L^1(\mathbb{R}^n)\setminus C_0(\mathbb{R}^n)$, the the function $\xi\mapsto\hat{f}(\xi)e^{2\pi ix\cdot\xi}$ is in $C_0(\mathbb{R}^n)$ but it is not integrable. However, as we will see (Theorems 3.23 and 3.30) for almost all $x\in\mathbb{R}^n$, the integral $\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi ix\cdot\xi}\,d\xi$ is both Abel and Gauss summable to $f(x)$. This gives some meaning of the inversion formula (3.3) for generic $f\in L^1(\mathbb{R}^n)$.

We want to find a reasonably large class of Φ-means (including the Abel and the Gauss means) such that

$$\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi ix\cdot\xi}\,d\xi$$

is Φ-summable to $f(x)$ i.e.

$$M_t(x)=\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi ix\cdot\xi}\Phi(t\xi)\,d\xi\to f(x)\quad\text{as } t\to 0^+$$

and we can consider both, convergence $M_t\to f$ almost everywhere and in L^1 (with respect to variable x). The main result reads as follows

Theorem 3.19. *Let $\Phi\in L^1(\mathbb{R}^n)$ be such that $\varphi=\hat{\Phi}\in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$. For $f\in L^1(\mathbb{R}^n)$ let*

$$M_t(x)=\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi ix\cdot\xi}\,\Phi(t\xi)\,d\xi$$

[23]**Exercise.** *Prove that if $\lim_{a\to\infty}\int_0^a f(x)\,dx=\ell$, then $A_t=\int_0^\infty f(x)e^{-tx}\,dx$ converges to ℓ as $t\to 0$.*

be the Φ*-mean of the integral* (3.3). *Then*

$$M_t \to f \text{ in } L^1 \text{ as } t \to 0. \tag{3.4}$$

If in addition $\varphi = \hat{\Phi}$ *has an integrable radially decreasing majorant, then*

$$M_t \to f \text{ a.e. as } t \to 0. \tag{3.5}$$

Proof. The lemma below is a link between the Φ-mean and the approximation by convolution discussed in Section 2.7.

Lemma 3.20. *If* $f, \Phi \in L^1(\mathbb{R}^n)$ *and* $\varphi = \hat{\Phi}$, *then*

$$\int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} \Phi(t\xi)\, d\xi = \int_{\mathbb{R}^n} f(x-y)\varphi_t(-y)\, dy = f * \tilde{\varphi}_t(x), \quad \text{where } \tilde{\varphi}(y) = \varphi(-y).$$

Proof. For $h \in \mathbb{R}^n$ define

$$g^{(h)}(x) = e^{2\pi i x\cdot h}\Phi(tx) \in L^1(\mathbb{R}^n)\,.$$

Recall that[24]

$$(w(x)e^{2\pi i h\cdot x})^\wedge(\xi) = \hat{w}(\xi - h) \quad \text{and} \quad (w(tx))^\wedge(\xi) = (\hat{w})_t(\xi).$$

Hence

$$(g^{(h)})^\wedge(\xi) = (\hat{\Phi})_t(\xi - h) = \varphi_t(\xi - h)$$

so replacing h by x we get

$$(g^{(x)})^\wedge(\xi) = \varphi_t(\xi - x).$$

We have

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x\cdot\xi}\Phi(t\xi)\, d\xi &= \int_{\mathbb{R}^n} \hat{f}(\xi) g^{(x)}(\xi)\, d\xi = \int_{\mathbb{R}^n} f(\xi)(g^{(x)})^\wedge(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} f(\xi)\varphi_t(\xi - x)\, d\xi = \int_{\mathbb{R}^n} f(x-y)\varphi_t(-y)\, dx.\end{aligned}$$

We used here Theorem 3.2(c) and in the last equality the change of variables $-y = \xi - x$. □

Now (3.4) follows directly from Theorem 2.44 and (3.5) follows from Theorem 2.49. □

Remark 3.21. Theorem 3.19 is not easy to apply since given $\Phi \in L^1$, we have to compute $\hat{\Phi}$ and show its integrability. In the case of the Gauss mean $\Phi(x) = e^{-|x|^2}$ it can be easily done with the help of Theorem 3.7, but the case of the Abel mean $\Phi(x) = e^{-|x|}$ is *much* harder due to difficulty of computing the Fourier transform of $\Phi(x) = e^{-|x|}$.

In Theorems 3.23 and 3.30 we will investigate the case of the the Gauss and the Abel and methods.

Theorem 3.22. *Let* $f(x) = e^{-4\pi^2 t|x|^2}$, $t > 0$. *Then*

(a) $W(x,t) := \hat{f}(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$.
(b) *The function* W *has the following scaling property with respect to* t*: if* $\varphi(x) = W(x,1)$, *then* $W(x,t) = \varphi_{t^{1/2}}(x)$.

[24]see Theorem 3.2.

(c)
$$\int_{\mathbb{R}^n} W(x,t)\,dx = 1 \quad \textit{for all } t>0.$$

Proof. We can write
$$f(x) = e^{-4\pi^2 t|x|^2} = u((4\pi t)^{1/2}x), \quad \text{where} \quad u(x) = e^{-\pi|x|^2}.$$
Since $(\psi(tx))^\wedge(\xi) = (\hat{\psi})_t(\xi)$ and $\hat{u}(\xi) = u(\xi) = e^{-\pi|\xi|^2}$ we obtain[25]
$$W(\xi,t) = \hat{f}(\xi) = (\hat{u})_{(4\pi t)^{1/2}}(\xi) = u_{(4\pi t)^{1/2}}(\xi) = (4\pi t)^{-n/2}e^{-|\xi|^2/(4t)}$$
which is (a). In particular
$$\varphi(\xi) = W(\xi,1) = u_{(4\pi)^{1/2}}(\xi).$$
Since $(\psi_{t_1})_{t_2}(x) = \psi_{t_1t_2}(x)$ we get
$$W(\xi,t) = u_{(4\pi t)^{1/2}}(\xi) = \left(u_{(4\pi)^{1/2}}\right)_{t^{1/2}}(\xi) = \varphi_{t^{1/2}}(\xi)$$
which is (b). Finally equality $\int_{\mathbb{R}^n}\psi_{t_1} = \int_{\mathbb{R}^n}\psi_{t_2}$, $t_1,t_2>0$ along with $W(\xi,t) = \varphi_{t^{1/2}}(\xi)$ yields (c):
$$\int_{\mathbb{R}^n} W(\xi,t)\,d\xi = \int_{\mathbb{R}^n} W\Big(\xi,\frac{1}{4\pi}\Big)\,d\xi = \int_{\mathbb{R}^n} e^{-\pi|\xi|^2}\,d\xi = 1.$$
□

As an application we obtain

Theorem 3.23 (The Gauss-Weierstrass summability method)**.** *If $f\in L^1(\mathbb{R}^n)$, then*
$$\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi ix\cdot\xi}e^{-4\pi^2t|\xi|^2}\,d\xi = \int_{\mathbb{R}^n} f(y)W(x-y,t)\,dy \to f \quad \textit{as } t\to 0^+ \tag{3.6}$$
both in $L^1(\mathbb{R}^n)$ and almost everywhere.

Proof. First we will prove equality. Let $\Phi(x) = e^{-4\pi^2|x|^2}$. Using notation from Theorem 3.22 $\hat{\Phi}(x) = W(x,1) = \varphi(x)$. Since $\varphi(x) = \varphi(-x)$, $\tilde{\varphi}_{t^{1/2}}(x) = \varphi_{t^{1/2}}(x) = W(x,t)$. Thus Lemma 3.20 yields
$$\begin{aligned}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi ix\cdot\xi}e^{-4\pi^2t|\xi|^2}\,d\xi &= \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi ix\cdot\xi}\Phi(t^{1.2}\xi)\,d\xi \\ &= f*\tilde{\varphi}_{t^{1/2}}(x) = \int_{\mathbb{R}^n} W(x-y,t)f(y)\,dy.\end{aligned}$$
Theorem 3.22 shows that the function Φ satisfies the assumptions of Theorem 3.19 so both convergences in L^1 and almost everywhere follow. □

Remark 3.24. The function $W(x,t) = \varphi_{t^{1/2}}(x)$ satisfies the assumptions of Theorem 2.49 with $a=1$ so for $f\in L^p$, $1\le p<\infty$
$$\int_{\mathbb{R}^n} f(y)W(x-y,t)\,dy \to f \quad \text{a.e. and in } L^p \text{ as } t\to 0^+. \tag{3.7}$$

[25]The function W is defined as a Fourier transform so it will be more convenient for us to prove (a)-(c) with the variable x replaced by ξ.

Remark 3.25. $W(x,t)$ is called the *Gauss-Weierstrass kernel.* Under suitable assumptions about f, differentiation under the sign of the integral shows that the function

$$w(x,t)=\int_{\mathbb{R}^n} W(x-y,t)f(y)\,dy=\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\,f(y)\,dy$$

satisfies the *heat equation* $\frac{\partial w}{\partial t}=\Delta_x w$ in the interior of $\mathbb{R}^{n+1}_+$. As we showed, the function $w(x,t)$ converges to $f(x)$ as $t\to 0$, so w is a solution to

$$\begin{cases} \frac{\partial w}{\partial t}=\Delta_x w & \text{on } \mathbb{R}^{n+1}_+,\\ w(x,0)=f(x), & x\in\mathbb{R}^n. \end{cases}$$

Corollary 3.26. *If both f and $\hat{f}$ are integrable*[26]*, then*

$$f(x)=\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi \quad a.e. \tag{3.8}$$

Proof. Since $\hat{f}\in L^1$, the left hand side of (3.6) converges to $\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi$ so the result follows from Theorem 3.23. □

The above inversion formula motivates the following definitions.

Definition 3.27. The *inverse Fourier* transform is defined by

$$\check{f}(x)=\mathcal{F}^{-1}(f)(x)=\int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}\,d\xi.$$

Corollary 3.28. *If $f_1,f_2\in L^1(\mathbb{R}^n)$ and $\hat{f}_1=\hat{f}_2$ on $\mathbb{R}^n$, then $f_1=f_2$ a.e.*

Proof. Let $f=f_1-f_2$. Then $f\in L^1$ and $\hat{f}=0\in L^1$ so (3.8) yields that $f=0$ a.e. □

The following result provides another example of a function that satisfies the assumptions of Theorem 3.19.

Theorem 3.29. *Let $f(x)=e^{-2\pi|x|t}$, $t>0$. Then*

(a)
$$P(x,t):=\hat{f}(x)=c_n\,\frac{t}{(t^2+|x|^2)^{(n+1)/2}},$$
where
$$c_n=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.$$

(b) *The function P has the following scaling property with respect to t: if $\varphi(x)=P(x,1)$, then $P(x,t)=\varphi_t(x)$.*

(c)
$$\int_{\mathbb{R}^n} P(x,t)\,dx=1 \quad \textit{for all } t>0.$$

By the same arguments as before we obtain.

[26] The assumption about integrability of $\hat{f}$ is very strong. As already observed, the equality (3.8) implies that f equals a.e. to a function in $C_0(\mathbb{R}^n)$.

Theorem 3.30 (The Abel summability method)**.** *If $f \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi} e^{-2\pi|\xi|t}\, d\xi = \int_{\mathbb{R}^n} f(y)P(x-y,t)\, dy \to f \quad \text{as } t\to 0^+$$

both in $L^1(\mathbb{R}^n)$ and almost everywhere.

Remark 3.31. The function $P(x,t) = \varphi_t(x)$ satisfies the assumptions of Theorem 2.49 with $a = 1$ so for $f \in L^p$, $1 \le p < \infty$

$$\int_{\mathbb{R}^n} f(y)P(x-y,t)\, dy \to f \quad \text{a.e. and in } L^p \text{ as } t \to 0^+.$$

Remark 3.32. $P(x,t)$ is called the *Poisson kernel*. Under suitable assumptions about f, differentiation under the sign of the integral shows that the function

$$u(x,t) = \int_{\mathbb{R}^n} P(x-y,t)f(y)\, dy$$

satisfies the *Laplace equation* $\Delta_{(x,t)}u = 0$ in the interior of $\mathbb{R}^{n+1}_+$. As we showed $u(x,t)$ converges to f as $t \to 0^+$ so $u(x,t)$ is a solution to the *Dirichlet problem* in the half-space

$$\begin{cases} \Delta_{(x,t)}u = 0 & \text{on } \mathbb{R}^{n+1}_+, \\ u(x,0) = f(x), & x \in \mathbb{R}^n \end{cases}$$

The proof of Theorem 3.29 is substantially more difficult than that of Theorem 3.22. Since the formula for the Fourier transform involves the Γ function we need to recall its basic properties.

Definition 3.33. For $0 < x < \infty$ we define

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt\,.$$

Theorem 3.34.

(a) $\Gamma(x+1) = x\Gamma(x)$ *for all* $0 < x < \infty$.
(b) $\Gamma(n+1) = n!$ *for all* $n = 0, 1, 2, 3, \ldots$
(c) $\Gamma(1/2) = \sqrt{\pi}$.

Proof. (a) follows from the integration by parts. Since $\Gamma(1) = 1$, (b) follows from (a) by induction. The substitution $t = s^2$ gives

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt = \int_0^\infty s^{2(x-1)}e^{-s^2}2s\, ds = 2\int_0^\infty s^{2x-1}e^{-s^2}\, ds$$

and hence

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-s^2}\, ds = \sqrt{\pi}\,.$$

□

The formula $\Gamma(x) = \Gamma(x+1)/x$ allows us to define $\Gamma(x)$ for all negative non-integer values of x. For example

$$\Gamma\Big(-\frac{3}{2}\Big) = \frac{\Gamma(-1/2)}{-3/2} = \frac{\Gamma(1/2)}{(-3/2)(-1/2)} = \frac{4\sqrt{\pi}}{3}.$$

Definition 3.35. For $x \in (-\infty, 0) \setminus \{-1, -2, -3, \ldots\}$, the function $\Gamma(x)$ is defined by

$$\Gamma(x) = \frac{\Gamma(x+k)}{x(x+1)\cdot \ldots \cdot (x+k-1)},$$

where k is a positive integer such that $x + k \in (0, 1)$.

Lemma 3.36.

$$\int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} \quad \textit{for } n = 1, 2, 3, \ldots$$

Proof. Denote the left hand side by a_n and the right hand side by b_n. Easy one time integration by parts[27] shows that

$$a_{n+2} = (n+1)(a_n - a_{n+2}), \quad a_{n+2} = \frac{n+1}{n+2}\, a_n \, .$$

Also elementary properties of the Γ function show that

$$b_{n+2} = \frac{n+1}{n+2}\, b_n$$

and now it is enough to observe that $a_1 = 1 = b_1 = 1$, $a_2 = \pi/4 = b_2$. □

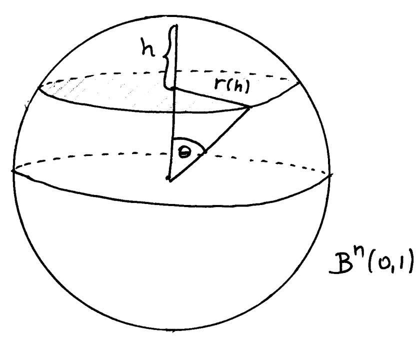
Lemma 3.37.

(a) *The volume of the unit ball in* $\mathbb{R}^n$ *equals*

(3.9)
$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \, .$$

(b) *The* $(n-1)$*-dimensional measure of the unit sphere in* $\mathbb{R}^n$ *equals* $n\omega_n$

Proof.



It follows from the picture and the Fubini theorem that the volume of the upper half of the ball equals

$$\frac{1}{2}\omega_n = \int_0^1 \omega_{n-1} r(h)^{n-1} \, dh \, .$$

The substitution

$$h = 1 - \cos\theta, \quad dh = \sin\theta \, d\theta, \quad r(h) = \sin\theta$$

[27] Use $\sin^{n+2}\Theta = (-\cos\theta)' \sin^{n+1}\Theta$.

and Lemma 3.36 give

$$\frac{1}{2}\omega_n = \omega_{n-1}\int_0^{\pi/2} \sin^n\theta\, d\theta = \omega_{n-1}\,\frac{\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)}\,,$$

so

$$\omega_n = \frac{2\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)}\,\omega_{n-1}\,.$$

If

$$a_n = \frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)}$$

then a direct computation shows that $a_1 = 2 = \omega_1$ and that a_n satisfies the same recurrence relationship as ω_n, so $\omega_n = a_n$ for all n. The second equality in (3.9) follows from Theorem 3.34(a).

(b) follows from the fact that the $(n-1)$-dimensional measure of a sphere of radius r equals to the derivative with respect to r of the volume of an n-dimensional ball of radius r. $\square$

We will also need the following result.

Lemma 3.38. *For $\beta > 0$ we have*

$$e^{-\beta} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, e^{-\beta^2/(4u)}\, du\,.$$

Proof. Applying the theory of residues to the function $e^{i\beta z}/(1+z^2)$ one can easily prove that

$$e^{-\beta} = \frac{2}{\pi}\int_0^\infty \frac{\cos\beta x}{1+x^2}\, dx\,.$$

This and an obvious identity

$$\frac{1}{1+x^2} = \int_0^\infty e^{-(1+x^2)u}\, du$$

yields

$$\begin{aligned}
e^{-\beta} &= \frac{2}{\pi}\int_0^\infty \frac{\cos\beta x}{1+x^2}\, dx \\
&= \frac{2}{\pi}\int_0^\infty \cos\beta x\Big(\int_0^\infty e^{-u}e^{-ux^2}\, du\Big)\, dx \\
&= \frac{2}{\pi}\int_0^\infty e^{-u}\Big(\int_0^\infty e^{-ux^2}\cos\beta x\, dx\Big)\, du \\
&= \frac{2}{\pi}\int_0^\infty e^{-u}\Big(\frac{1}{2}\int_{-\infty}^\infty e^{-ux^2}e^{i\beta x}\, dx\Big)\, du \\
&\overset{(x=-2\pi y)}{=} \frac{2}{\pi}\int_0^\infty e^{-u}\Big(\pi\underbrace{\int_{-\infty}^\infty e^{-4\pi^2uy^2}e^{-2\pi i\beta y}\, dy}_{W(\beta,u)}\Big)\, du \\
&= \frac{2}{\pi}\int_0^\infty e^{-u}\Big(\frac{1}{2}\sqrt{\frac{\pi}{u}}\, e^{-\beta^2/(4u)}\Big)\, du
\end{aligned}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \, e^{-\beta^2/(4u)} \, du \, .$$

The proof is complete. □

Proof of Theorem 3.29. (a) By a change of variables formula it suffices to prove the formula for $t = 1$. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2\pi|x|} e^{-2\pi i x\cdot\xi} \, dx &= \int_{\mathbb{R}^n} \Big(\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-4\pi^2|x|^2/(4u)} \, du \Big) \, e^{-2\pi i x\cdot\xi} \, dx \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \Big(\underbrace{\int_{\mathbb{R}^n} e^{-4\pi^2|x|^2/(4u)} e^{-2\pi i x\cdot\xi} \, dx}_{W(\xi,(4u)^{-1})} \Big) \, du \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left(\sqrt{\frac{u}{\pi}} \right)^n e^{-u|\xi|^2} \, du \\
&= \frac{1}{\pi^{(n+1)/2}} \int_0^\infty e^{-u(1+|\xi|^2)} u^{(n-1)/2} \, du \\
&= \frac{1}{\pi^{(n+1)/2}} \frac{1}{(1+|\xi|^2)^{(n+1)/2}} \underbrace{\int_0^\infty e^{-s} s^{(n-1)/2} \, ds}_{\Gamma[(n+1)/2]} \, .
\end{aligned}$$

(b) is obvious.

(c) Because of the scaling property (b) it suffices to consider $t = 1$. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} P(x,1) \, dx &= c_n \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+1)/2}} \\
&\overset{\text{polar}}{=} c_n \int_0^\infty \left(\int_{S^{n-1}(0,1)} \frac{d\sigma}{(1+r^2)^{(n+1)/2}} \right) r^{n-1} \, dr \\
&= c_n \, n \, \omega_n \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{(n+1)/2}} \, dr \\
&\overset{r=\tan\theta}{=} c_n \, n \, \omega_n \int_0^{\pi/2} \sin^{n-1}\theta \, d\theta \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} \, n \, \frac{2\pi^{n/2}}{n \, \Gamma\left(\frac{n}{2}\right)} \, \frac{\pi^{1/2}\Gamma\left(\frac{n}{2}\right)}{(n-1)\Gamma\left(\frac{n-1}{2}\right)} \\
&= 1 \, .
\end{aligned}$$

The proof is complete. □

Corollary 3.26 applied to $f(x) = e^{-4\pi^2 t|x|^2}$ and to $f(x) = e^{-2\pi t|x|}$ yields

Corollary 3.39.

$$\int_{\mathbb{R}^n} W(\xi,t) e^{2\pi i x\cdot\xi} \, d\xi = e^{-4\pi^2 t|x|^2} \quad \text{and} \quad \int_{\mathbb{R}^n} P(\xi,t) e^{2\pi i x\cdot\xi} \, d\xi = e^{-2\pi|x|t}$$

for all $x \in \mathbb{R}^n$.

The Weierstrass and Poisson kernels have the following semigroup property.

Corollary 3.40. *If* $W_t(x) = W(x,t)$ *and* $P_t(x) = P(x,t)$, *then for* $t_1, t_2 > 0$

$$(W_{t_1} * W_{t_2})(x) = W_{t_1+t_2}(x) \quad \textit{and} \quad (P_{t_1} * P_{t_2})(x) = P_{t_1+t_2}(x)$$

for all $x \in \mathbb{R}^n$.

Proof. It follows from Corollary 3.39 with x replaced by $-x$ that

$$\hat{W}_t(x) = e^{-4\pi^2 t|x|^2} \quad \text{and} \quad \hat{P}_t(x) = e^{-2\pi|x|t}.$$

Hence Theorem 3.2(b) yields

$$(W_{t_1} * W_{t_2})\hat{\ }(x) = \hat{W}_{t_1}(x)\hat{W}_{t_2}(x) = \hat{W}_{t_1+t_2}(x)$$

$$(P_{t_1} * P_{t_2})\hat{\ }(x) = \hat{P}_{t_1}(x)\hat{P}_{t_2}(x) = \hat{P}_{t_1+t_2}(x)$$

and the result follows from Corollary 3.28. □

3.4. **The Schwarz class and the Plancherel theorem.**

Definition 3.41. We say that f belongs to the *Schwarz class* $\mathscr{S}(\mathbb{R}^n) = \mathscr{S}_n$ if $f \in C^\infty(\mathbb{R}^n)$ and for all multiindices α, β

$$\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| = p_{\alpha,\beta}(f) < \infty.$$

That means all derivatives of f rapidly decrease to zero as $|x| \to \infty$, faster than the inverse of any polynomial.

Clearly $C_0^\infty(\mathbb{R}^n) \subset \mathscr{S}_n$, but also $e^{-|x|^2} \in \mathscr{S}_n$ so functions in the Schwarz class need not be compactly supported.

$\{p_{\alpha,\beta}\}$ is a countable family of norms in $\mathscr{S}_n$ and we can use it to define a topology in $\mathscr{S}_n$.

Definition 3.42. We say that a sequence (f_k) *converges to* f *in* $\mathscr{S}_n$ if

$$\lim_{k\to\infty} p_{\alpha,\beta}(f_k - f) = 0 \quad \text{for all multiindices } \alpha, \beta.$$

This convergence is metrizable. Indeed, $d_{\alpha,\beta}(f,g) = p_{\alpha,\beta}(f-g)$ is a metric and if we arrange all these metrics in a sequence $d_1', d_2', \ldots$, then

$$d(f,g) = \sum_{k=1}^\infty 2^{-k} \frac{d_k'(f,g)}{1 + d_k'(f,g)}$$

defines a metric in $\mathscr{S}_n$ such that $f_n \to f$ in $\mathscr{S}_n$ if and only if $f_n \to f$ in the metric d.

Proposition 3.43. *The space* $\mathscr{S}_n$ *has the following properties.*

(a) $\mathscr{S}_n$ *equipped with the metric* d *is a complete metric space.*
(b) $C_0^\infty(\mathbb{R}^n)$ *is dense in* $\mathscr{S}_n$.
(c) *If* $\varphi \in \mathscr{S}_n$, *then* $\tau_h\varphi \to \varphi$ *in* $\mathscr{S}_n$ *as* $h \to 0$, *where* $\tau_h\varphi(x) = \varphi(x+h)$.

(d) *The mapping*

$$\mathscr{S}_n \ni \varphi \mapsto x^\alpha D^\beta \varphi(x) \in \mathscr{S}_n$$

is continuous.

(e) *If* $\varphi \in \mathscr{S}_n$, *then*

$$\frac{\varphi(x+he_k)-\varphi(x)}{h} \to \frac{\partial \varphi}{\partial x_k}(x) \quad \text{as } h \to 0.$$

in the topology of $\mathscr{S}_n$.

(f) *If* $\varphi, \psi \in \mathscr{S}_n$, *then* $\varphi * \psi \in \mathscr{S}_n$ *and*

$$D^\alpha(\varphi * \psi) = (D^\alpha \varphi) * \psi = \varphi * (D^\alpha \psi)$$

for any multiindex α.

We leave the proof as an exercise.

Theorem 3.44. *The Fourier transform is a continuous, one-to-one mapping of* $\mathscr{S}_n$ *onto* $\mathscr{S}_n$ *such that*

(a)

$$\left(\frac{\partial f}{\partial x_j}\right)^\wedge(\xi) = 2\pi i \xi_j \hat{f}(\xi),$$

(b)

$$(-2\pi i x_j f)\hat{}\,(\xi) = \frac{\partial \hat{f}}{\partial \xi_j}(\xi)\,,$$

(c)

$$(f*g)\hat{} = \hat{f}\hat{g},$$

(d)

$$\int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^n} \hat{f}(x) g(x)\,dx\,,$$

(e)

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\,d\xi\,.$$

Proof. We already proved (a), (b), (c) and (d). Now we will prove that the Fourier transform is a continuous mapping from $\mathscr{S}_n$ into $\mathscr{S}_n$. Formulas (a) and (b) imply that

$$\xi^\alpha D^\beta \hat{f}(\xi) = C(D^\alpha(x^\beta f))\hat{}\,(\xi).$$

Since $\|\hat{g}\|_\infty \leq \|g\|_1$ we get

$$p_{\alpha,\beta}(\hat{f}) = \|\xi^\alpha D^\beta \hat{f}(\xi)\|_\infty \leq C\|D^\alpha(x^\beta f)\|_1\,.$$

An application of the Leibnitz rule[28] implies that $D^\alpha(x^\beta f)$ equals a finite sum of expressions of the form $x^{\beta_i} D^{\alpha_i} f$. Since

$$\|x^{\beta_i} D^{\alpha_i} f\|_1 \;=\; \int_{\mathbb{R}^n} |(1+|x|^2)^n x^{\beta_i} D^{\alpha_i} f(x)|(1+|x|^2)^{-n}\,dx$$

[28] Product rule.

$$\leq \quad C(n) \sup_{x\in\mathbb{R}^n} |(1+|x|^2)^n x^{\beta_i} D^{\alpha_i} f(x)| < \infty$$

it follows that $\hat{f} \in \mathscr{S}_n$. One can also easily deduce that the mapping $\hat{}: \mathscr{S}_n \to \mathscr{S}_n$ is continuous.

If $f \in \mathscr{S}_n$ than $\hat{f} \in \mathscr{S}_n$ and hence both f and $\hat{f}$ are integrable, so (e) follows from the inversion formula. This formula also shows that the Fourier transform applied four times is an identity on $\mathscr{S}_n$ and hence the Fourier transform is a bijection on $\mathscr{S}_n$. □

Theorem 3.45 (Plancherel)**.** *The Fourier transform is an L^2 isometry on a dense subset $\mathscr{S}_n$ of L^2*

$$\|\hat{f}\|_2 = \|f\|_2, \quad f \in \mathscr{S}_n,$$

and hence it uniquely extends to an isometry of L^2

$$\|\hat{f}\|_2 = \|f\|_2, \quad f \in L^2(\mathbb{R}^n)\,.$$

Moreover for $f \in L^2(\mathbb{R}^n)$

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

in the L^2 sense, i.e.

$$\left\| \hat{f} - \int_{|x|<R} f(x) e^{-2\pi i x\cdot\xi}\, dx \right\|_2 \to 0 \quad \textit{as } R\to\infty$$

and similarly

$$f(x) = \lim_{R\to\infty} \int_{|\xi|<R} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$$

in the L^2 sense.

Proof. Given $f \in \mathscr{S}_n$ let $g = \bar{\hat{f}}$, so $\hat{g} = \bar{f}$. Indeed,

$$\hat{g}(\xi) = \int_{\mathbb{R}^n} \bar{\hat{f}}(x) e^{-2\pi i x\cdot\xi}\, dx = \overline{\int_{\mathbb{R}^n} \hat{f}(x) e^{2\pi i x\cdot\xi}\, dx} = \bar{f}(x)\,.$$

Hence Theorem 3.44(d) gives

$$\|f\|_2 = \int_{\mathbb{R}^n} f\bar{f} = \int_{\mathbb{R}^n} f\hat{g} = \int_{\mathbb{R}^n} \hat{f} g = \int_{\mathbb{R}^n} \hat{f}\bar{\hat{f}} = \|\hat{f}\|_2\,.$$

Thus the Fourier transform is an L^2 isometry on $\mathscr{S}_n$. Since $\mathscr{S}_n$ is a dense subset of L^2 it uniquely extends to an isometry of L^2. Now for $f \in L^2(\mathbb{R}^n)$ we have

$$L^1 \ni f\chi_{B(0,R)} \xrightarrow{L^2} f \qquad \text{as } R\to\infty$$

and hence

$$(f\chi_{B(0,R)})\hat{}(\xi) = \int_{|x|\leq R} f(x) e^{-2\pi i x\cdot\xi}\, dx \xrightarrow{L^2} \hat{f}(\xi)$$

as $R\to\infty$. Similarly

$$\int_{|\xi|<R} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi \xrightarrow{L^2} f(x) \qquad \text{as } R\to\infty.$$

□

Proposition 3.46. *If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \hat{f}(x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx\,.$$

Proof. Approximate f and g in L^2 by functions in $\mathscr{S}_n$, apply Theorem 3.44(d) and pass to the limit. □

Consider the class $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ consisting of functions of the form $f = f_1 + f_2$, $f_1 \in L^1$, $f_2 \in L^2$. Then we define

$$\hat{f} = \hat{f}_1 + \hat{f}_2.$$

In order to show that the Fourier transform is well defined in the class $L^1 + L^2$ we need to show that it does not depend on the particular choice of the representation $f = f_1 + f_2$. Indeed, if we also have $f = g_1 + g_2$, $g_1 \in L^1$, $g_2 \in L^2$, then $f_1 - g_1 = g_2 - f_2 \in L^1 \cap L^2$ and hence

$$\hat{f}_1 - \hat{g}_1 = (f_1 - g_1)\hat{} = (g_2 - f_2)\hat{} = \hat{g}_2 - \hat{f}_2,$$
$$\hat{f}_1 + \hat{f}_2 = \hat{g}_1 + \hat{g}_2\,.$$

It is an easy exercise to show that

$$L^p(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n), \quad \text{for } 1 \leq p \leq 2,$$

and hence the Fourier transform is well defined on $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$ and

$$\hat{}\colon L^p(\mathbb{R}^n) \to L^2(\mathbb{R}^n) + C_0(\mathbb{R}^n), \quad \text{for } 1 \leq p \leq 2.$$

Later we will prove the Hausdorff-Young Inequality which implies that

$$\hat{}\colon L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n), \quad \text{for } 1 \leq p \leq 2,$$

where p' is the Hölder conjugate to p.

4. Miscellaneous results about the Fourier transform

In this chapter we will show some interesting results about the Fourier transform. Some of them, but not all, will be used later.

4.1. **The Poisson summation formula.**

Theorem 4.1 (Poisson summation formula)**.** *If* $f \in C^1(\mathbb{R})$ *and*

$$|f(x)| + |f'(x)| \leq \frac{C}{1+x^2} \quad \textit{for } x \in \mathbb{R}, \tag{4.1}$$

then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Remark 4.2. Here $\hat{f}(n)$ stand for the Fourier transform of f evaluated at n – do not get confused with the Fourier series.

Note that it follows from the growth condition (4.1) that the series $\sum_n f(n)$ converges and that any function $f \in \mathscr{S}_n$ satisfies (4.1).

Proof. It follows from (4.1) that

$$g(x) = \sum_{k=-\infty}^{\infty} f(x+k) \tag{4.2}$$

is a periodic function of class C^1 with period 1. Therefore g is can be represented as a Fourier series[29]

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{2\pi i n x}$$

where

$$\begin{aligned} \hat{g}(n) &= \int_0^1 g(x) e^{-2\pi i n x}\, dx = \sum_{k=-\infty}^{\infty} \int_0^1 f(x+k) e^{-2\pi i n x}\, dx = \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(x) e^{-2\pi i n x}\, dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x}\, dx = \hat{f}(n) \end{aligned}$$

[29]Now $\hat{g}(n)$ are Fourier series coefficients – do not get confused with the Fourier transform!

so

$$\sum_{k=-\infty}^{\infty} f(k) = g(0) = \sum_{n=-\infty}^{\infty} \hat{g}(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

□

As an application we will prove the Jacobi identity.

Definition 4.3. The Jacobi 'theta' function is defined by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, \qquad t > 0.$$

Clearly $\vartheta \in C^\infty(\mathbb{R})$.

Corollary 4.4 (Jacobi identity)**.** *For $t > 0$ we have $\vartheta(t) = t^{-\frac{1}{2}}\vartheta(1/t)$.*

Proof. The function $f(x) = t^{-1/2}e^{-\pi x^2/t}$, $t > 0$ can be written as

$$f(x) = t^{-1/2}u(xt^{-1/2}) \quad \text{where} \quad u(x) = e^{-\pi x^2}.$$

Hence[30]

$$\hat{f}(\xi) = t^{-1/2}(\hat{u})_{t^{-1/2}}(\xi) = t^{-1/2}u_{t^{-1/2}}(\xi) = u(\xi t^{1/2}) = e^{-\pi\xi^2 t}.$$

Thus the Poisson summation formula yields

$$t^{-1/2}\sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \quad \text{i.e.,} \quad t^{-1/2}\vartheta(1/t) = \vartheta(t).$$

□

Corollary 4.5. *For $t > 0$ we have*

$$\sum_{n=-\infty}^{\infty} \frac{1}{t^2+n^2} = \frac{\pi}{t}\,\frac{1+e^{-2\pi t}}{1-e^{-2\pi t}}\,.$$

Remark 4.6. It is easy to show by taking the limit as $t \to 0^+$ in the above identity that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

Proof. In Theorem 3.29 we showed that the Fourier transform of $f(x) = e^{-2\pi|x|t}$ equals[31]

$$\hat{f}(x) = \frac{t}{\pi(t^2+x^2)}$$

[30] We use here two facts: $(\varphi(sx))^\wedge(\xi) = (\hat{\varphi})_s(\xi) = s^{-n}\hat{\varphi}(\xi/s)$ with $n = 1$ and $\hat{u}(\xi) = u(\xi)$.

[31] We are in dimension $n = 1$ and $c_1 = 1$.

so the Poisson summation formula gives[32]

$$\frac{t}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{t^2+n^2}=\sum_{n=-\infty}^{\infty}e^{-2\pi|n|t}=\frac{1+e^{-2\pi t}}{1-e^{-2\pi t}}$$

from which the theorem follows. □

4.2. **The Heisenberg inequality.** Celebrated *Heisenberg's uncertainty principle* asserts that position and momentum of a particle cannot be measured at the same time and it can be written as an inequality

$$\sigma_x\sigma_p\geq\frac{h}{4\pi}$$

where σ_x and σ_p are standard deviations of position and momentum and h is the Plank constant. Translating it into our language the above inequality can be formulated as

Theorem 4.7 (Heisenberg's inequality)**.** *For any $f\in L^2(\mathbb{R})$ and $a,b\in\mathbb{R}$,*

$$\sqrt{\int_{-\infty}^{\infty}(x-a)^2|f(x)|^2\,dx}\,\sqrt{\int_{-\infty}^{\infty}(\xi-b)^2|\hat{f}(\xi)|^2\,d\xi}\geq\frac{\|f\|_2^2}{4\pi}\,.$$

Moreover the equality holds if and only if $f(x)=Ce^{2\pi ibx}e^{-k(x-a)^2}$ for some $C\in\mathbb{C}$ and $k>0$.

Proof. We will prove the result under the assumption that $f\in\mathscr{S}_1$. The case of general $f\in L^2$ is left to the reader as an exercise. Let $f(x)=e^{2\pi ibx}g(x-a)$. Since

$$\hat{f}(\xi)=\hat{g}(\xi-b)e^{-2\pi a(\xi-b)}$$

we easily obtain that the Heisenberg inequality is equivalent to

$$\left(\int_{-\infty}^{\infty}|x|^2|g(x)|^2\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty}|\xi|^2|\hat{g}(\xi)|^2\,d\xi\right)^{1/2}\geq\frac{\|g\|_2^2}{4\pi}\,. \tag{4.3}$$

Note that $(|g|^2)'=(g\overline{g})'=2\,\mathrm{re}\,g'\,\overline{g}$ so integration by parts gives

$$\int_\alpha^\beta|g(x)|^2\,dx=\int_\alpha^\beta x'|g(x)|^2\,dx=x|g(x)|^2\Big|_\alpha^\beta-2\,\mathrm{re}\int_\alpha^\beta xg'(x)\overline{g}(x)\,dx.$$

Letting $\alpha\to-\infty$, $\beta\to+\infty$ and using the fact that $g\in\mathscr{S}_1$ yields

$$\begin{aligned}\int_{-\infty}^{\infty}|g(x)|^2\,dx &= -2\,\mathrm{re}\int_{-\infty}^{\infty}xg'(x)\overline{g}(x)\,dx\\ &\leq 2\int_{-\infty}^{\infty}|x|\,|g(x)|\,|g'(x)|\,dx \qquad (4.4)\\ &\leq 2\left(\int_{-\infty}^{\infty}|x|^2|g(x)|^2\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty}|g'(x)|^2\,dx\right)^{1/2} \qquad (4.5)\end{aligned}$$

[32]In the Poisson summation formula we assumed that $f\in C^1(\mathbb{R})$ and it satisfies (4.1). However, our function f is not C^1. The condition (4.1) was required for the function (4.2) to be represented as a Fourier series. Although the condition (4.1) is not satisfied any longer, it is easy to check that the function defined by (4.2) is periodic and Lipschitz continuous so it is still equal to its Fourier series and the proof carries on.

$$= \; 2\left(\int_{-\infty}^{\infty}|x|^2|g(x)|^2\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty}|\widehat{g'}(\xi)|^2\,d\xi\right)^{1/2}$$
$$= \; 2\left(\int_{-\infty}^{\infty}|x|^2|g(x)|^2\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty}|2\pi i\xi\hat{g}(\xi)|^2\,d\xi\right)^{1/2}$$

from which (4.3) follows. Finally, let us investigate the case of equality. For the equality in the Schwarz inequality (4.5) we need $|g'(x)|$ to be proportional to $|xg(x)|$, say $|g'(x)| = k|xg(x)|$, for some $k>0$ so $g'(x) = ke^{i\varphi(x)}xg(x)$ for some real valued function $\varphi(x)$. But then equality in (4.4) implies that

$$-\mathrm{re}\, e^{-\varphi(x)} = 1 \quad \text{so} \quad e^{i\varphi(x)} = -1$$

and we have $g'(x) = -kxg(x)$. Solving this differential equation yields $g(x) = Ce^{-kx^2}$ and hence $f(x) = Ce^{2\pi ibx}e^{-k(x-a)^2}$. □

4.3. **Eigenfunctions of the Fourier transform.** The Fourier transform applied four times is identity on $L^2(\mathbb{R}^n)$ so if f is an eigenfunction of the Fourier transform,

$$\hat{f} = \lambda f, \quad \text{then } f = |\lambda|^4 f \quad \text{so } \lambda\in\{-1,1,-i,i\}.$$

We know already one eigenfunction with the eigenvalue 1, namely $f(x) = e^{-\pi x^2}$ and now we will find now *all* eigenfunctions in dimension $n=1$.

It is natural to search among functions of the form $p(x)e^{-\pi x^2}$, where $p(x)$ is a polynomial.[33]

Definition 4.8. *Hermite functions* are defined by the formula

$$h_n(x) = \frac{(-1)^n}{n!}e^{\pi x^2}\left(\frac{d}{dx}\right)^n e^{-2\pi x^2}, \quad \text{for all integers } n\geq 0.$$

Note that $h_0(x) = e^{-\pi x^2}$.

Lemma 4.9. $h_n(x) = p_n(x)e^{-\pi x^2}$ *for a polynomial* $p_n(x)$ *of order* n *of the form*

$$p_n(x) = \frac{(4\pi)^n}{n!}\,x^n \; + \; \textit{lower order terms.}$$

In particular $h_n\in\mathscr{S}_1$.

This lemma easily follows from the definition of h_n.

Lemma 4.10. $h_n'(x) - 2\pi x h_n(x) = -(n+1)h_{n+1}(x)$.

Proof. Applying the product rule to

$$h_n(x) = \left[e^{\pi x^2}\right]\cdot\left[\frac{(-1)^n}{n!}\left(\frac{d}{dx}\right)^n e^{-2\pi x^2}\right]$$

[33]Note that evaluation of the Fourier transform of $p(x)e^{-\pi x^2}$ is straightforward: since $\mathcal{F}(e^{-\pi x^2})(\xi) = e^{-\pi\xi^2}$ it suffices to apply (3.1).

gives

$$h'_m(x) = 2\pi x h_n(x) + \Big[e^{\pi x}\Big]\cdot\Big[\underbrace{\frac{(-1)^n}{n!}}_{-(n+1)\frac{(-1)^{n+1}}{(n+1)!}}\Big(\frac{d}{dx}\Big)^{n+1}e^{-2\pi x^2}\Big] = 2\pi x h_n(x) - (n+1)h_{n+1}(x).$$

□

Lemma 4.11. *If $f \in C^\infty(\mathbb{R})$ and n is a positive integer, then*

$$(4\pi x)\Big(\frac{d}{dx}\Big)^n f(x) - \Big(\frac{d}{dx}\Big)^n (4\pi x f(x)) = -4\pi n\Big(\frac{d}{dx}\Big)^{n-1} f(x) \quad \textit{for all } x\in\mathbb{R}.$$

The result follows from a simple induction.

Lemma 4.12. *$h'_n(x) + 2\pi x h_n(x) = 4\pi h_{n-1}(x)$, where $h_{-1}(x) = 0$.*

Proof. For $n = 0$ we check the equality directly. For $n \geq 1$ Lemma 4.10 gives

$$\begin{aligned} h'_n(x) + 2\pi x h_n(x) &= 4\pi x h_n(x) - (n+1)h_{n+1}(x) \\ &= 4\pi x\frac{(-1)^n}{n!}e^{\pi x^2}\Big(\frac{d}{dx}\Big)^n e^{-2\pi x^2} \underbrace{-(n+1)\frac{(-1)^{n+1}}{(n+1)!}}_{+(-1)^n/n!}\, e^{\pi x^2}\underbrace{\Big(\frac{d}{dx}\Big)^{n+1}e^{-2\pi x^2}}_{(\frac{d}{dx})^n(-4\pi x e^{-2\pi x^2})} \\ &= \frac{(-1)^n}{n!}e^{\pi x^2}\underbrace{\Big[4\pi x\Big(\frac{d}{dx}\Big)^n e^{-2\pi x^2} - \Big(\frac{d}{dx}\Big)^n(4\pi x e^{-2\pi x^2})\Big]}_{-4\pi n(\frac{d}{dx})^{n-1}e^{-2\pi x^2}\text{ by Lemma 4.11}} \\ &= 4\pi\,\frac{(-1)^{n-1}}{(n-1)!}e^{\pi x^2}\Big(\frac{d}{dx}\Big)^{n-1}e^{-2\pi x^2} = 4\pi h_{n-1}(x). \end{aligned}$$

□

Lemma 4.13. *$\hat{h}_n = (-i)^n h_n$ and $\check{h}_n = i^n h_n$.*

Remark 4.14. The lemma yields

$$\hat{h}_{4n} = h_{4n}, \quad \hat{h}_{4n+2} = -h_{4n+2}, \quad \hat{h}_{4n+3} = i h_{4n+3}, \quad \hat{h}_{4n+1} = -i h_{4n+1}$$

Thus the lemma provides examples of eigenfunctions corresponding to all possible eigenvalues $1, -1, i, -i$.

Proof. Since $h_n \in \mathscr{S}_1$ we can compute the Fourier transform directly. Lemma 4.10 gives

$$h'_n - 2\pi x h_n = -(n+1)h_{n+1}, \qquad \widehat{h'_n} - \widehat{2\pi x h_n} = -(n+1)\hat{h}_{n+1}.$$

Since $(-2\pi i x f)^\wedge = (\hat{f})'$ and $\widehat{f'} = 2\pi i\xi \hat{f}(\xi)$ we have

$$2\pi i\xi\hat{h}_n(\xi) - i(\hat{h}_n)'(\xi) = -(n+1)\hat{h}_{n+1}(\xi) \tag{4.6}$$

We will prove the equality $\hat{h}_n = (-i)^n h_n$ by induction.

For $n = 0$, $h_0 = e^{-\pi x^2}$ so $\hat{h}_0 = h_0 = (-i)^0 h_0$.

Suppose now that the equality holds for n and we will prove it for $n+1$. The equality (4.6) yields.

$$\begin{aligned} 2\pi i\xi(-i)^n h_n(\xi) - i((-i)^n h_n(\xi))' &= -(n+1)\hat{h}_{n+1}(\xi), \\ -(-i)^{n+1}2\pi\xi h_n(\xi) + (-i)^{n+1}h_n'(\xi) &= -(n+1)\hat{h}_{n+1}(\xi), \\ (-i)^{n+1} \underbrace{\left[h_n'(\xi) - 2\pi\xi h_n(\xi)\right]}_{-(n+1)h_{n+1}(\xi) \text{ by Lemma 4.10}} &= -(n+1)\hat{h}_{n+1}(\xi). \end{aligned}$$

This proves that $\hat{h}_{n+1} = (-i)^{n+1}h_{n+1}(\xi)$ which completes the proof of the claim that $\hat{h}_n = (-i)^n h_n(\xi)$ fro all n. Finally

$$h_n = (\hat{h}_n)^\vee = ((-i)^n h_n)^\vee = (-i)^n \check{h}_n, \qquad \check{h}_n = i^n h_n.$$

□

Let $\mathcal{K}f = f'' - 4\pi^2 x^2 f$.

Lemma 4.15. $\mathcal{K}h_n = -4\pi(n+\frac{1}{2})h_n$

Proof. Lemmas 4.10 and 4.12 yield

$$\begin{aligned} h_n' - 2\pi x h_n &= -(n+1)h_{n+1}, \\ h_n'' - 2\pi h_n - 2\pi x h_n' &= -(n+1)h_{n+1}', \\ h_n'' &= 2\pi h_n + 2\pi x h_n' - (n+1)h_{n+1}'. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{K}h_n &= h_n'' - 4\pi^2x^2h_n = 2\pi h_n + 2\pi x h_n' - (n+1)h_{n+1}' - 4\pi^2x^2h_n \\ &= 2\pi h_n - (n+1)h_{n+1}' + 2\pi x\underbrace{(h_n' - 2\pi x h_n)}_{-(n+1)h_{n+1}} \\ &= 2\pi h_n - (n+1)(h_{n+1}' + 2\pi x h_{n+1}) \\ &= -4\pi\Big(n+\frac{1}{2}\Big)h_n. \end{aligned}$$

□

Lemma 4.16. *For real valued functions $f, g \in \mathscr{S}_1$ we have*[34]

$$\langle \mathcal{K}f, g\rangle = \langle f, \mathcal{K}g\rangle$$

Proof. The integration by parts yields

$$\begin{aligned} \langle \mathcal{K}f, g\rangle &= \int_{\mathbb{R}}(f'' - 4\pi^2x^2f)g = \int_{\mathbb{R}} f''g - \int_{\mathbb{R}} 4\pi^2x^2fg \\ &= \int_{\mathbb{R}} fg'' - \int_{\mathbb{R}} 4\pi^2x^2fg = \int f(g'' - 4\pi^2x^2g) = \langle f, \mathcal{K}g\rangle. \end{aligned}$$

□

Lemma 4.17. *The Hermite functions h_n are orthogonal,*

$$\langle h_n, h_m\rangle = 0, \quad \textit{for } n \neq m.$$

[34]Here $\langle\cdot,\cdot\rangle$ denotes the L^2 scalar product.

Proof. We have

$$-4\pi\Big(n+\frac{1}{2}\Big)\langle h_n, h_m\rangle = \langle \mathcal{K}h_n, h_m\rangle = \langle h_n, \mathcal{K}h_m\rangle = -4\pi\Big(m+\frac{1}{2}\Big)\langle h_n, h_m\rangle$$

and the result easily follows. □

Lemma 4.18.

$$\|h_n\|_2^2 = \frac{(4\pi)^n}{\sqrt{2}n!}.$$

Proof. Since by Lemma 4.10 we have $(n+1)h_{n+1} = 2\pi x h_n - h_n'$, integration by parts yields

$$\begin{aligned}(n+1)\|h_{n+1}\|_2^2 &= 2\pi\int_{\mathbb{R}} x h_n h_{n+1} - \int_{\mathbb{R}} h_n' h_{n+1} = 2\pi\int_{\mathbb{R}} x h_n h_{n+1} + \int_{\mathbb{R}} h_n h_{n+1}' \\ &= \int_{\mathbb{R}} h_n \underbrace{(h_{n+1}' + 2\pi x h_{n+1})}_{4\pi h_n} = 4\pi\|h_n\|_2^2.\end{aligned}$$

Replacing n by $n-1$ yields

$$\|h_n\|_2^2 = \frac{4\pi}{n}\,\|h_{n-1}\|_2^2$$

and a simple induction argument yields

$$\|h_n\|_2^2 = \frac{(4\pi)^n}{n!}\,\|h_0\|_2^2.$$

Now the result follows from a simple observation that

$$\|h_0\|_2^2 = \int_{\mathbb{R}} (e^{-\pi x^2})^2\,dx = \int_{\mathbb{R}} e^{-2\pi x^2}\,dx = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{-\pi y^2}\,dy = \frac{1}{\sqrt{2}}.$$

□

The last two lemmas show that the *scaled Hermite functions*

$$e_n = \Big(\frac{(4\pi)^n}{\sqrt{2}n!}\Big)^{-1/2} h_n$$

form an orthonormal family in $L^2(\mathbb{R})$.

Theorem 4.19. *The fmaily $\{e_n\}_{n=0}^\infty$ is an orthonormal basis of $L^2(\mathbb{R}^n)$.*

Proof. It suffices to prove that if $f\in L^2(\mathbb{R})$ and $\int_{\mathbb{R}} fh_n = 0$ for all $n\geq 0$, then $f=0$ a.e. Thus suppose that $\int_{\mathbb{R}} fh_n = 0$ for all $n\geq 0$. Since

$$h_n(x) = \Big(\frac{(4\pi)^2}{n!}\,x^n + \dots\Big)\,e^{-\pi x^2}\,,$$

it easily follows that the functions $x^n e^{-\pi x^2}$, $n\geq 0$, are linear combinations of the functions $h_0, h_1, \dots, h_n$. Hence

$$\int_{\mathbb{R}} f(x)x^n e^{-\pi x^2}\,dx = 0 \quad \text{for all } n\geq 0.$$

Since by the Schwarz inequality $f(x)e^{-\pi x^2} \in L^1$ we can compute the Fourier transform directly

$$\begin{aligned}\left(f(x)e^{-\pi x^2}\right)^\wedge &= \int_{\mathbb{R}} f(x)e^{-\pi x^2} e^{-2\pi i x\xi}\, dx = \int_{\mathbb{R}} f(x)e^{-\pi x^2} \sum_{n=0}^{\infty} \frac{(-2\pi i x\xi)^n}{n!}\, dx \\ &= \sum_{n=0}^{\infty} \frac{(-2\pi i\xi)^n}{n!} \int_{\mathbb{R}} f(x)x^n e^{-\pi x^2}\, dx = 0.\end{aligned}$$

Vanishing of the Fourier transform of $f(x)e^{-\pi x^2}$ implies that $f(x)e^{-\pi x^2} = 0$ a.e. and hence $f(x) = 0$ a.e. □

Since $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{R})$ any $f \in L^2$ can be written as

$$f = \sum_{n=0}^{\infty} \langle f, e_n\rangle\, e_n$$

This and the fact that $\hat{e}_n = (-i)^n e_n$ yield

Theorem 4.20 (Wiener's definition of the Fourier transform)**.** *For $f \in L^2(\mathbb{R})$ we have*

$$\hat{f} = \sum_{n=0}^{\infty} \langle f, e_n\rangle (-i)^n e_n.$$

Let H_k, $k = 0, 1, 2, 3$ be subspaces of $L^2(\mathbb{R})$ defined by

$$H_k = \overline{\text{span}\,\{e_{4n+k}\}_{n=0}^{\infty}}.$$

The subspaces H_k are orthogonal and

$$L^2(\mathbb{R}) = H_0 \oplus H_1 \oplus H_2 \oplus H_3.$$

Clearly, the subspaces H_k are eigenspaces of the Fourier transform:

$$\hat{f} = (-i)^k f \quad \text{for } f \in H_k \text{ and } k = 0, 1, 2, 3.$$

4.4. **The Bochner-Hecke formula.** As was pointed out in Section 4.3 computing the Fourier transform of $P(x)e^{-\pi|x|^2}$ is straightforward due to formula (3.1) and Theorem 3.7, but the computations are often algebraically complicated and it is not always obvious how to write the answer in a compact and nice form. We have see examples of such computations in Section 4.3 when we found eigenfunctions of the Fourier transform and now we will see another example of this type. The results of this section will be used later in Section 6.3

Definition 4.21. We say that $P_k(x)$ is a *homogeneous harmonic polynomial of degree k* if

$$P_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha \quad \text{and} \quad \Delta P_k = 0\,.$$

Theorem 4.22 (Bochner-Hecke)**.** *If $P_k(x)$ is a homogeneous harmonic polynomial of degree k, then*

$$\mathcal{F}\left(P_k(x)e^{-\pi|x|^2}\right)(\xi) = (-i)^k P_k(\xi)e^{-\pi|\xi|^2}\,.$$

Proof. Since the polynomial P_k is homogeneous of degree k,

$$P_k(-2\pi ix)=\sum_{|\alpha|=k}a_\alpha(2\pi ix)^\alpha=(-2\pi i)^kP_k(x).$$

Hence (3.1) applied to $f(x)=e^{-\pi|x|^2}$ yields[35]

$$\begin{aligned}\int_{\mathbb{R}^n}P_k(x)e^{-\pi|x|^2}e^{-2\pi ix\cdot\xi}\,dx &= \mathcal{F}\Big(P_k(x)e^{-\pi|x|^2}\Big)(\xi)\\ &= (-2\pi i)^{-k}P_k(D)e^{-\pi|\xi|^2}=Q(\xi)e^{-\pi|\xi|^2}\end{aligned}\tag{4.7}$$

for some polynomial Q and it remains to prove that $Q(\xi)=P_k(-i\xi)$.

Lemma 4.23. *If $P(x)$ is a polynomial[36] in $\mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n}P(x)e^{-\pi\sum_j(x_j+i\xi_j)^2}\,dx=\int_{\mathbb{R}^n}P(x-i\xi)e^{-\pi|x|^2}\,dx\,.\tag{4.8}$$

Proof. By the same argument involving the same contour integration as in the proof of Theorem 3.7 for *any* polynomial P of one variable we have

$$\int_{-\infty}^{\infty}P(x)e^{-\pi(x+i\xi)^2}\,dx=\int_{-\infty}^{\infty}P(x-i\xi)e^{-\pi x^2}\,dx\,.\tag{4.9}$$

If P is a polynomial in n variables, then (4.8) follows from (4.9) and the Fubini theorem. □

Multiplying both sides of (4.7) by $e^{\pi|\xi|^2}$ and applying (4.8) we obtain

$$Q(\xi)=\int_{\mathbb{R}^n}P_k(x)e^{-\pi\sum_j(x_j+i\xi_j)^2}\,dx=\int_{\mathbb{R}^n}P_k(x-i\xi)e^{-\pi|x|^2}\,dx\,.$$

We can write

$$P_k(x-\eta)=\sum_\alpha\eta^\alpha P_\alpha(x)$$

and then

$$\int_{\mathbb{R}^n}P_k(x-\eta)e^{-\pi|x|^2}\,dx=\sum_\alpha\eta^\alpha\int_{\mathbb{R}^n}P_\alpha(x)e^{-\pi|x|^2}\,dx\,,$$

so clearly the integral is a polynomial is η. With this notation we obtain

$$Q(\xi)=\sum_\alpha(i\xi)^\alpha\int_{\mathbb{R}^n}P_\alpha(x)e^{-\pi|x|^2}\,dx$$

and hence

$$Q(\xi/i)=\sum_\alpha\xi^\alpha\int_{\mathbb{R}^n}P_\alpha(x)e^{-\pi|x|^2}\,dx=\int_{\mathbb{R}^n}P_k(x-\xi)e^{-\pi|x|^2}\,dx\,.$$

[35]Instead of using (3.1) we could apply the differential operator $P_k(D_\xi)$ to both sides of the equality

$$\int_{\mathbb{R}^n}e^{-\pi|x|^2}e^{-2\pi ix\cdot\xi}\,dx=e^{-\pi|\xi|^2}$$

and differentiate under the sign of the integral.

[36]Any polynomial, not necessarily harmonic or homogeneous.

Since P_k is a harmonic function, it has the mean value property[37]

$$\int_{S^{n-1}} P_k(s\theta-\xi)\, d\sigma(\theta) = |S^{n-1}|P_k(-\xi)\,.$$

Thus integration in polar coordinates gives

$$\begin{aligned} Q(\xi/i) &= \int_0^\infty s^{n-1}\left(\int_{S^{n-1}} P_k(s\theta-\xi)\, d\sigma(\theta)\right) e^{-\pi s^2}\, ds \\ &= P_k(-\xi)\int_0^\infty s^{n-1}|S^{n-1}|e^{-\pi s^2}\, ds \\ &= P_k(-\xi)\int_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx = P_k(-\xi) \end{aligned}$$

and hence $Q(\xi)=P_k(-i\xi)$. The proof is complete. □

Using homogeneity of P_k and the second part of Theorem 3.2(e) one can easily deduce from the Bochner-Hecke formula the folloiwing result.

Corollary 4.24. *If P_k is a homogeneous harmonic polynomial of degree k, then for any $t>0$ we have*

$$\mathcal{F}\left(P_k(x)e^{-\pi t|x|^2}\right)(\xi) = t^{-k-\frac{n}{2}}(-i)^k P_k(\xi)e^{-\pi|\xi|^2/t}\,.$$

We leave details as an easy exercise.

4.5. **Symmetry of the Fourier transform.** If f is radially symmetric i.e., $f=f\circ\rho$ for all $\rho\in O(n)$, then $\hat{f}\circ\rho=\widehat{f\circ\rho}=\hat{f}$ for all $\rho\in O(n)$ so $\hat{f}$ is radially symmetric too. Now we will prove a more delicate result about the symmetry of the Fourier transform.

Proposition 4.25. *Let $f\in L^1(\mathbb{R}^n)$ be a real valued function of the form*

$$f(x)=\frac{x_j}{|x|}g(|x|).$$

Then there is a continuous function $h:[0,\infty)\to\mathbb{R}$, $h(0)=0$, $h(t)\to 0$ as $t\to\infty$ such that

$$\hat{f}(\xi)=i\,\frac{\xi_j}{|\xi|}h(|\xi|), \quad \xi\in\mathbb{R}^n.$$

While the function f is not radially symmetric a more symmetric object is the map

$$F(x)=\frac{x}{|x|}\,g(|x|)\in L^1(\mathbb{R}^n,\mathbb{R}^n), \quad x\in\mathbb{R}^n$$

so f is the jth component of F. We will prove that

$$\hat{F}(\xi)=i\,\frac{\xi}{|\xi|}h(|\xi|), \quad \xi\in\mathbb{R}^n$$

from which Proposition 4.25 will readily follow.

In the proof we will use the following result which is interesting on its own. It ill be used later one more time in a proof of Theorem 6.2.

[37] We integrate here P_k over the sphere centered at $-\xi$ and of radius s.

Lemma 4.26. *If $m : \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function that is homogeneous of degree 0, i.e. $m(tx) = m(x)$ for $t > 0$, and commutes with the orthogonal transformations, i.e.*

$$m(\rho(x)) = \rho(m(x)) \tag{4.10}$$

for all $x \in \mathbb{R}^n$ and $\rho \in O(n)$, then there is a constant c such that

$$m(x) = c\,\frac{x}{|x|} \quad \text{for all } x \neq 0. \tag{4.11}$$

Proof. Let $e_1, e_2, \ldots, e_n$ be the standard orthogonal basis of $\mathbb{R}^n$. If $[\rho_{jk}]$ is the matrix representation of $\rho \in O(n)$, then the condition (4.10) reads as

$$m_j(\rho(x)) = \sum_{k=1}^n \rho_{jk} m_k(x), \quad j = 1, 2, \ldots, n, \tag{4.12}$$

where $m(x) = (m_1(x), \ldots, m_n(x))$.

Let $m_1(e_1) = c$. Consider all $\rho \in O(n)$ such that $\rho(e_1) = e_1$. This condition means that the first column of the matrix $[\rho_{jk}]$ equals e_1, i.e. $\rho_{11} = 1$, $\rho_{j1} = 0$, for $j > 1$. Since columns are orthogonal, for $k > 1$ we have

$$0 = \sum_{j=1}^n \rho_{j1}\rho_{jk} = \rho_{1k}\,.$$

Thus

$$\rho = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & \rho_{22} & \ldots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \rho_{n2} & \ldots & \rho_{nn} \end{bmatrix},$$

where $[\rho_{jk}]_{j,k=2}^n$ is the matrix of an arbitrary orthogonal transformation in the $(n-1)$-dimensional subspace orthogonal to e_1.

For $x = e_1 = \rho(e_1) = \rho(x)$ and $j \geq 2$ identity (4.12) yields

$$m_j(e_1) = \sum_{k=1}^n \rho_{jk} m_k(e_1) = \sum_{k=2}^n \rho_{jk} m_k(e_1)\,,$$

and hence

$$\begin{bmatrix} m_2(e_1) \\ \vdots \\ m_n(e_1) \end{bmatrix} = \begin{bmatrix} \rho_{22} & \ldots & \rho_{2n} \\ \vdots & \ddots & \vdots \\ \rho_{n2} & \ldots & \rho_{nn} \end{bmatrix} \begin{bmatrix} m_2(e_1) \\ \vdots \\ m_n(e_1) \end{bmatrix}.$$

That means the vector $[m_2(e_1), \ldots, m_n(e_1)]^T$ is fixed under an arbitrary orthogonal transformation of $\mathbb{R}^{n-1}$, so it must be a zero vector, i.e.

$$m_2(e_1) = \ldots = m_n(e_1) = 0\,.$$

Now formula (4.12) for any $\rho \in O(n)$ and $x = e_1$, takes the form

$$m_j(\rho(e_1)) = \rho_{j1} m_1(e_1) = c\rho_{j1}\,.$$

By homogeneity it suffices to prove (4.11) for $|x| = 1$. Let $\rho \in O(n)$ be such that $\rho(e_1) = x$. Then $\rho_{j1} = x_j$, $j = 1, 2, \ldots, n$ and hence

$$m_j(x) = c\rho_{j1} = cx_j = c\,\frac{x_j}{|x|}\,.$$

This completes the proof of the lemma. □

Proof of Proposition 4.25. Since the function F is odd

$$i\hat{F}(\xi) = \int_{\mathbb{R}^n} iF(x)(\cos(2\pi x\cdot\xi) - i\sin(2\pi x\cdot\xi))\,dx = \int_{\mathbb{R}^n} \sin(2\pi x\cdot\xi)F(x)\,dx$$

so the mapping $\xi \mapsto i\hat{F}(\xi)$ takes values in $\mathbb{R}^n$. Fix $k > 0$ and define $m_k : S^{n-1}(0,k) \to \mathbb{R}^n$ by $m_k(\xi) = i\hat{F}(\xi)$ for $|\xi| = k$. Extend m_k to $m_k : \mathbb{R}^n\setminus\{0\} \to \mathbb{R}^n$ as a function homogeneous of degree 0, i.e.

$$m_k(\xi) = i\,\hat{F}\Big(\frac{k\xi}{|\xi|}\Big) \quad \text{for } \xi \neq 0.$$

We claim that

$$m_k(\rho(\xi)) = \rho(m_k(\xi)) \quad \text{for } \rho \in O(n) \text{ and } \xi \neq 0. \tag{4.13}$$

Since m_k is homogeneous of degree 0, it suffices to check (4.13) for $|\xi| = k$. We have[38]

$$\begin{aligned} m_k(\rho(\xi)) &= \int_{\mathbb{R}^n} \sin(2\pi x\cdot\rho(\xi))\frac{x}{|x|}g(|x|)\,dx = \int_{\mathbb{R}^n} \sin(2\pi\rho^{-1}(x)\cdot\xi)\frac{x}{|x|}g(|x|)\,dx \\ &= \int_{\mathbb{R}^n} \sin(2\pi x\cdot\xi)\frac{\rho(x)}{|\rho(x)|}g(|\rho(x)|)\,dx = \int_{\mathbb{R}^n} \sin(2\pi x\cdot\xi)\frac{\rho(x)}{|x|}g(|x|)\,dx \\ &= \rho\Big(\int_{\mathbb{R}^n} \sin(2\pi x\cdot\xi)\frac{x}{|x|}g(|x|)\,dx\Big) = \rho(m_k(\xi)). \end{aligned}$$

According to Lemma 4.26 there is a constant $h(k) \in \mathbb{R}$ such that $m_k(\xi) = -h(k)\xi/|\xi|$. In particular for $|\xi| = k$ we have

$$i\hat{F}(\xi) = m_k(\xi) = -\frac{\xi}{|\xi|}h(|\xi|).$$

Clearly h is continuous, $h(0) = 0$ and $h(t) \to 0$ as $t \to \infty$, because $i\hat{F} \in C_0(\mathbb{R}^n, \mathbb{R}^n)$. □

[38]First we use equality $x\cdot\rho(\xi) = \rho^{-1}(x)\cdot\xi$ which follows from the fact that the orthogonal transformation ρ^{-1} preserves the scalar product and then we use the change of variables $x = \rho(y)$ whose absolute value of the Jacobian equals 1.

5. Tempered distributions

5.1. **Basic constructions.**

Definition 5.1. The space $\mathscr{S}'_n$ of all continuous linear functionals on $\mathscr{S}_n$ is called the space of *tempered distributions*. The evaluation of $u \in \mathscr{S}'_n$ on $\varphi \in \mathscr{S}_n$ will usually be denoted by $u[\varphi]$. The space $\mathscr{S}'_n$ is equipped with the *weak-$*$ convergence* (usually called *weak convergence* or just *convergence*). Namely $u_k \to u$ in $\mathscr{S}'_n$ if $u_k[\psi] \to u[\psi]$ for every $\psi \in \mathscr{S}_n$.

Here are examples:

1. If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then

$$L_f[\varphi] = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx \quad \text{defines } L_f \in \mathscr{S}'_n.$$

2. If μ is a measure of finite total variation, then

$$L_\mu[\varphi] = \int_{\mathbb{R}^n} \varphi \, d\mu \quad \text{defines } L_\mu \in \mathscr{S}'_n.$$

3. We say that a function f is a *tempered L^p function* if $f(x)(1+|x|^2)^{-k} \in L^p(\mathbb{R}^n)$ for some nonnegative integer k. If $p = \infty$ we call f a *slowly increasing function*. Then

$$L_f[\varphi] = \int_{\mathbb{R}^n} f(x)\varphi(x)\, d\mu \quad \text{defines } L_f \in \mathscr{S}'_n \text{ for all } 1 \leq p \leq \infty.$$

Note that slowly increasing functions are exactly measurable functions bounded by polynomials.

4. A *tempered measure* is a Borel measure μ such that

$$\int_{\mathbb{R}^n} (1 + |x|^2)^{-k}\, d\mu < \infty$$

for some integer $k \geq 0$. As before $L_\mu \in \mathscr{S}'_n$.

5. $L[\varphi] = D^\alpha \varphi(x_0)$ is a tempered distribution $L \in \mathscr{S}'_n$.

The distributions generated by a function or by a measure will often be denoted by $L_f[\varphi] = f[\varphi]$, $L_\mu[\varphi] = \mu[\varphi]$.

Suppose that $u \in \mathscr{S}_n'$. If there is a tempered L^p function f such that $u[\varphi] = f[\varphi]$ for $\varphi \in \mathscr{S}_n$, then we can identify u with the function f and simply write $u = f$. The identification is possible, because the function f is uniquely defined (up to a.e. equivalence). This follows from a well known result.

Lemma 5.2. *If $\Omega \subset \mathbb{R}^n$ is open and $f \in L^1_{\rm loc}(\Omega)$ satisfies $\int_\Omega f\varphi = 0$ for all $\varphi \in C_0^\infty(\Omega)$, then $f = 0$ a.e.*

Note that not every function $f \in C^\infty(\mathbb{R}^n)$ defines a tempered distribution, because it may happen that form some $\varphi \in \mathscr{S}_n$ the function $f\varphi$ is not integrable and hence the integral $f[\varphi] = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx$ does not make sense.

Theorem 5.3. *A linear functional L on $\mathscr{S}_n$ is a tempered distribution if and only if there is a constant $C > 0$ and a positive integer m such that*

$$|L(\varphi)| \leq C \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\varphi) \quad \textit{for all } \varphi \in \mathscr{S}_n.$$

Proof. If a linear functional L satisfies the given estimate, then clearly it is continuous on $\mathscr{S}_n$, so it remains to prove the converse implication. Let $L \in \mathscr{S}_n'$. We claim that there is a positive integer m such that $|L(\varphi)| \leq 1$ for all

$$\varphi \in \Big\{\varphi \in \mathscr{S}_n : \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\varphi) \leq \frac{1}{m}\Big\} := N_m\,.$$

To the contrary suppose that there is a sequence $\varphi_k \in \mathscr{S}_n$ such that $|L(\varphi_k)| > 1$ and

$$\sum_{|\alpha|,|\beta|\leq k} p_{\alpha,\beta}(\varphi_k) \leq \frac{1}{k}, \quad k = 1, 2, 3, \dots \tag{5.1}$$

Note that (5.1) implies that $\varphi_k \to 0$ in $\mathscr{S}_n$, so the inequality $|L(\varphi_k)| > 1$ contradicts continuity of L. This proves the claim. Denote

$$\|\varphi\| = \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\varphi)\,.$$

Observe that $\|\cdot\|$ is a norm. For an arbitrary $0 \neq \varphi \in \mathscr{S}_n$, $\tilde{\varphi} = \varphi/(m\|\varphi\|)$ satisfies $\|\tilde{\varphi}\| \leq 1/m$, so $\tilde{\varphi} \in N_m$ and hence

$$|L(\varphi)| = m\|\varphi\||L(\tilde{\varphi})| \leq m\|\varphi\|$$

which proves the theorem. □

For any function g on $\mathbb{R}^n$ we define $\tilde{g}(x) = g(-x)$. Then it easily follows from the Fubini theorem that for $u, \varphi, \psi \in \mathscr{S}_n$

$$\int_{\mathbb{R}^n} (u * \varphi)(x)\psi(x)\, dx = \int_{\mathbb{R}^n} u(x)(\tilde{\varphi} * \psi)(x)\, dx\,.$$

Regarding the functions u and $u * \varphi$ as distributions we can rewrite this equality as

$$(u * \varphi)[\psi] = u[\tilde{\varphi} * \psi]$$

If $u \in \mathscr{S}_n'$ and $\varphi \in \mathscr{S}_n$, then $\psi \mapsto u[\tilde{\varphi} * \psi]$ is a tempered distribution so it motivates the following definition.

Definition 5.4. If $u \in \mathscr{S}_n'$ and $\varphi \in \mathscr{S}_n$, then the *convolution of* u *and* φ is a tempered distribution defined by the formula

$$(u * \varphi)[\psi] := u[\tilde{\varphi} * \psi]\,.$$

The following two results are left as an easy exercise.

Proposition 5.5. *If* $u \in \mathscr{S}_n'$, $\varphi, \psi \in \mathscr{S}_n$, *then*

$$(u * \varphi) * \psi = u * (\varphi * \psi)\,.$$

Proposition 5.6. *If* $\mu \in \mathcal{B}(\mathbb{R}^n)$ *and* $\varphi \in \mathscr{S}_n$, *then* $\mu * \varphi = \varphi * \mu$ *where* $\mu * \varphi$ *is the convolution of a distribution* $\mu \in \mathscr{S}_n'$ *with a function* $\varphi \in \mathscr{S}_n$ *and* $\varphi * \mu$ *is the convolution of a function with a measure.*

Theorem 5.7. *If* $u \in \mathscr{S}_n'$ *and* $\varphi \in \mathscr{S}_n$, *then the tempered distribution* $u*\varphi$ *can be identified with a slowly increasing function* f *defined by*

$$f(x) = u[\tau_{-x}\tilde{\varphi}] = u[\varphi(x-\cdot)] \quad \text{for all } x \in \mathbb{R}^n.$$

Moreover $f \in C^\infty(\mathbb{R}^n)$ *and all its derivatives are slowly increasing.*

Proof. Note that the equality $u[\tau_{-x}\tilde{\varphi}] = u[\varphi(x-\cdot)]$ is obvious. First we will prove that the function $f(x) = u[\tau_{-x}\tilde{\varphi}] = u[\varphi(x-\cdot)]$ is C^∞ and f, as well as all its derivatives, are slowly increasing.

It follows from Theorem 3.43(c) that f is continuous. Observe that for a fixed $x \in \mathbb{R}^n$

$$\frac{f(x+he_k)-f(x)}{h} = u\left[\frac{\varphi(x+he_k-\cdot)-\varphi(x-\cdot)}{h}\right] \to u[(\partial_k\varphi)(x-\cdot)] \quad \text{as } h \to 0$$

because it is easy to check using Theorem 3.43(e) that for every $x \in \mathbb{R}^n$

$$\frac{\varphi(x+he_k-\cdot)-\varphi(x-\cdot)}{h} \to (\partial_k\varphi)(x-\cdot) \quad \text{as } h \to 0$$

in the topology of $\mathscr{S}_n$. Thus the partial derivatives

$$\partial_k f(x) = u[(\partial_k\varphi)(x-\cdot)]$$

exist and are continuous by Theorem 3.43(c). Since $\partial_k\varphi \in \mathscr{S}_n$, iterating the above process for any multiindex α we obtain

$$D^\alpha f(x) = u[(D^\alpha\varphi)(x-\cdot)]\,. \tag{5.2}$$

This implies that $f \in C^\infty(\mathbb{R}^n)$. Since u is a tempered distribution, Theorem 5.3 gives the estimate

$$|f(x)| = |u[\tau_{-x}\tilde{\varphi}]| \leq C \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\tau_{-x}\tilde{\varphi}) \leq C'(1+|x|)^m\,. \tag{5.3}$$

Indeed,

$$\begin{aligned} p_{\alpha,\beta}(\tau_{-x}\tilde{\varphi}) &= \sup_{z\in\mathbb{R}^n} |z^\alpha (D^\beta\tilde{\varphi})(z-x)| = \sup_{z\in\mathbb{R}^n} |(z+x)^\alpha (D^\beta\tilde{\varphi})(z)| \\ &\leq C\big(1+|x|^{|\alpha|}\big) \sup_{z\in\mathbb{R}^n} \big(1+|z|^{|\alpha|}\big)\, |D^\beta\tilde{\varphi}(z)| \leq C'(1+|x|)^m\,. \end{aligned}$$

Hence f is slowly increasing, and so it defines a tempered distribution. Since the derivatives of f satisfy (5.2), which is an expression of the same type as the one in the definition of f, we conclude that all derivatives $D^\alpha f$ are slowly increasing.

It remains to prove that $f = u * \varphi$ in the sense of tempered distributions i.e.,

$$(u * \varphi)[\psi] = f[\psi] \quad \text{for all } \psi \in \mathscr{S}_n. \tag{5.4}$$

Using $f(y) = u[\varphi(y - \cdot)]$, (5.4) reads as

$$u\Big[\int_{\mathbb{R}^n} \varphi(y-\cdot)\psi(y)\,dy\Big] = \int_{\mathbb{R}^n} u[\varphi(y-\cdot)]\psi(y)\,dy.$$

The idea of proving this equality is to approximate the integral on on the left hand side by Riemann sums, use linearity of u to go under the sign of the sum and to pass to the limit.

However, the details of this approximation argument are not obvious. They are elementary and tedious, but not obvious and so in most of the textbooks they are summarized as 'easy to see'. For the sake of completeness we decided to include details.

First of all observe that is suffices to prove (5.4) under the assumption that $\varphi, \psi \in C_0^\infty$. Indeed, suppose we know (5.4) for compactly supported functions, but now $\varphi, \psi \in \mathscr{S}_n$. Let[39] $C_0^\infty \ni \varphi_k \to \varphi$ in $\mathscr{S}_n$ and $C_0^\infty \ni \psi_\ell \to \psi$ in $\mathscr{S}_n$. Observe that $f_k(x) = u[\tau_{-x}\tilde{\varphi}_k] \to f(x)$ pointwise. It follows from the proof of (5.3) that all the functions f_k have a common estimate independent of k

$$|f_k(x)| \leq C(1+|x|)^m$$

and hence $f_k \to f$ in $\mathscr{S}_n'$ by the Dominated Convergence Theorem. It is also clear from the definition of the convolution that $u * \varphi_k \to u * \varphi$ in $\mathscr{S}_n'$, so

$$(u*\varphi)[\psi_\ell] = \lim_{k\to\infty}(u*\varphi_k)[\psi_\ell] = \lim_{k\to\infty} f_k[\psi_\ell] = f[\psi_\ell]\,.$$

The second equality follows from assuming (5.4) for all $\varphi_k, \psi_\ell \in C_0^\infty$. Now letting $\ell \to \infty$ yields (5.4).

If the support of ψ is contained in a cube Q with integer edge-length we divide Q into cubes $\{Q_{ki}\}_i$ of edge-length 2^{-k} and centes y_{ki}. Then the Riemann sums converge in the topology of $\mathscr{S}_n$ (as a function of x) to the integral

$$\sum_i \varphi(y_{ki}-x)\psi(y_{ki})|Q_{ki}| \to \int_{\mathbb{R}^n} \varphi(y-x)\psi(y)\,dy$$

and hence

$$u\Big[\int_{\mathbb{R}^n}\varphi(y-\cdot)\psi(y)\,dy\Big] = \lim_{k\to\infty}\sum_i u[\varphi(y_{ki}-\cdot)]\psi(y_{ki})|Q_{ki}| = \int_{\mathbb{R}^n} u[\varphi(y-\cdot)]\psi(y)\,dy.$$

Boring (but elementary) details of the convergence of the Riemann sums in the topology of $\mathscr{S}_n$ are moved to Appendix 5.7.

Note that formula (5.2) implies

Theorem 5.8. *If $u \in \mathscr{S}_n'$ and $\varphi \in \mathscr{S}_n$, then for any multiindex α we have*

$$D^\alpha(u*\varphi)(x) = (u*(D^\alpha\varphi))(x)\,.$$

[39] See Theorem 3.43(b).

The next result will require the use of the Banach-Steinhaus theorem.

Theorem 5.9. *If $u_k \to u$ is $\mathscr{S}_n'$ and $\varphi_k \to \varphi$ in $\mathscr{S}_n$, then $u_k[\varphi_k] \to u[\varphi]$.*

Proof. In order to be able to use triangle inequality to prove the convergence we need to know uniform estimates for the sequence u_k. Convergence $u_k \to u$ in $\mathscr{S}_n'$ means pointwise convergence on a complete metric space $\mathscr{S}_n$ so it is natural to use the Banach-Steinhaus Theorem to get uniform estimates for u_k. We state it as a lemma.

Lemma 5.10 (Banach-Steinhaus)**.** *Let X be a complete metric space and let $\{f_i\}_{i\in I}$ be a family of continuous real-valued functions on X. If the functions in the family are pointwise bounded, i.e.*

$$\sup_{i\in I} |f_i(x)| < \infty \quad \textit{for } x \in X,$$

then there is an open set $U \subset X$ and a constant $M > 0$ such that

$$\sup_{i\in I} |f_i(x)| \leq M \quad \textit{for all } x \in U.$$

It easily follows from the linearity that it suffices to consider the case $u_k \to 0$ and $\varphi_k \to 0$. The functions $\varphi \mapsto |u_k[\varphi]|$ are continuous on the complete metric space $(\mathscr{S}_n, d)$. They are pointwise bounded on $\mathscr{S}_n$ since $|u_k[\varphi]| \to 0$ for every $\varphi \in \mathscr{S}_n$. Thus Lemma 5.10 implies that

$$\sup_k |u_k[\varphi]| \leq M \quad \text{for all } \varphi \in B(\varphi_0, r_0).$$

Fix $\varepsilon > 0$. There is k_1 such that $|u_k[\varphi_0]| < \varepsilon$ for $k \geq k_1$. Since $\varphi_k \to 0$ in $\mathscr{S}_n$, then also $\varepsilon^{-1}\varphi_k \to 0$ and hence

$$d(\varphi_0 + \varepsilon^{-1}\varphi_k, \varphi_0) = d(\varepsilon^{-1}\varphi_k, 0) \to 0\,,$$

so there is k_2 such that for $k \geq k_2$, $\varphi_0 + \varepsilon^{-1}\varphi_k \in B(\varphi_0, r_0)$. Hence for $k \geq \max\{k_1, k_2\}$ we have

$$|u_k[\varepsilon^{-1}\varphi_k]| \leq |u_k[\varphi_0 + \varepsilon^{-1}\varphi_k]| + |u_k[\varphi_0]| \leq M + \varepsilon\,, \quad \text{so} \quad |u_k[\varphi_k]| \leq \varepsilon(M+\varepsilon).$$

□

Theorem 5.11. *If $u_k \to u$ in $\mathscr{S}_n'$, then there is a constant $C > 0$ and a positive integer m such that*

$$\sup_k |u_k[\varphi]| \leq C \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\varphi) \quad \textit{for all } \varphi \in \mathscr{S}_n.$$

Proof. Let $v_k = u_k - u$. Clearly $v_k \to 0$ in $\mathscr{S}_n'$. It suffices to prove that there is $C > 0$ and a positive integer m such that

$$\sup_k |v_k[\varphi]| \leq C \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\varphi) \quad \text{for all } \varphi \in \mathscr{S}_n.$$

Indeed, this estimate, the inequality

$$\sup_k |u_k[\varphi]| \leq |u[\varphi]| + \sup_k |v_k[\varphi]|$$

and Theorem 5.3 readily yield the result.

As in the proof of Theorem 5.3 it suffices to prove that there is a positive integer m such that $\sup_k |v_k[\varphi]| \leq 1$ for

$$\varphi \in N_m := \Big\{\varphi \in \mathscr{S}_n : \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\varphi) \leq \frac{1}{m}\Big\}.$$

To the contrary suppose that for any positive integer i we can find $\varphi_i \in N_i$ such that $\sup_k |v_k[\varphi_i]| > 1$, i.e.

$$|v_{k_i}[\varphi_i]| > 1 \quad \text{for some } k_i. \tag{5.5}$$

Clearly $\varphi_i \to 0$ in $\mathscr{S}_n$. If the sequence k_i is bounded, (5.5) contradicts continuity of functionals v_k. If k_i is unbounded, then there is a subsequence $k_{i_j} \to \infty$ and again (5.5) contradicts Theorem 5.9. □

Theorem 5.12. *$C_0^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}'_n$, i.e. if $u \in \mathscr{S}'_n$, then there is a sequence $w_k \in C_0^\infty(\mathbb{R}^n)$ such that*

$$w_k[\psi] \to u(\psi) \quad \textit{for all } \psi \in \mathscr{S}_n.$$

Proof. Let $u \in \mathscr{S}'_n$. If $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \varphi = 1$, then for every $\psi \in \mathscr{S}_n$, $\tilde{\varphi}_\varepsilon * \psi \to \psi$ in $\mathscr{S}_n$[40] so

$$(u * \varphi_\varepsilon)[\psi] = u[\tilde{\varphi}_\varepsilon * \psi] \to u[\psi].$$

Since $u_\varepsilon = u * \varphi_\varepsilon \in C^\infty$ we obtain a sequence of smooth functions that converge to u in $\mathscr{S}'_n$. If we take a cut-off function $\eta \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \eta \leq 1$, $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$, then one can easily prove that for $\psi \in \mathscr{S}_n$

$$\eta(x/k)\psi(x) \to \psi(x) \quad \text{in } \mathscr{S}_n \text{ as } k \to \infty.$$

Using this fact and Theorem 5.9[41] we obtain that $w_k(x) = \eta(x/k)(u * \varphi_{k^{-1}}) \in C_0^\infty$ converges to u in $\mathscr{S}'_n$. Indeed,

$$w_k[\psi] = (u * \varphi_{k^{-1}})[\eta(\cdot/k)\psi(\cdot)] \to u[\psi].$$

□

If $\varphi, \psi \in \mathscr{S}_n$, then the integration by parts gives

$$\int_{\mathbb{R}^n} D^\alpha\varphi(x)\psi(x)\,dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n}\varphi(x)D^\alpha\psi(x)\,dx, \quad D^\alpha\varphi[\psi] = \varphi[(-1)^\alpha\psi].$$

For $h \in \mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n}(\tau_h\varphi)(x)\psi(x)\,dx = \int_{\mathbb{R}^n}\varphi(x)(\tau_{-h}\psi)(x)\,dx, \quad \tau_h\varphi[\psi] = \varphi[\tau_{-h}\psi].$$

Moreover

$$\int_{\mathbb{R}^n}\tilde{\varphi}(x)\psi(x)\,dx = \int_{\mathbb{R}^n}\varphi(x)\tilde{\psi}(x)\,dx, \quad \tilde{\varphi}[\psi] = \varphi[\tilde{\psi}],$$

$$\int_{\mathbb{R}^n}\hat{\varphi}(x)\psi(x)\,dx = \int_{\mathbb{R}^n}\varphi(x)\hat{\psi}(x)\,dx, \quad \hat{\varphi}[\psi] = \varphi[\hat{\psi}],$$

$$\int_{\mathbb{R}^n}\check{\varphi}(x)\psi(x)\,dx = \int_{\mathbb{R}^n}\varphi(x)\check{\psi}(x)\,dx, \quad \check{\varphi}[\psi] = \varphi[\check{\psi}].$$

[40] Why?

[41] Instead of using Theorem 5.9 which is somewhat not constructive as it is based on the Banach-Steinhaus theorem, we could prove convergence by a brute force estimates.

If $\eta \in C^\infty$ is slowly increasing and all derivatives of η are slowly increasing, i.e. every derivative $D^\alpha\eta$ is bounded by a polynomial, then for $\psi \in \mathscr{S}_n$, $\eta\psi \in \mathscr{S}_n$ and the mapping $\psi \mapsto \eta\psi$ is continuous in $\mathscr{S}_n$. In particular $x^\alpha\psi(x) \in \mathscr{S}_n$. Moreover

$$(\eta\varphi)[\psi] = \int_{\mathbb{R}^n} \big(\eta(x)\varphi(x)\big)\psi(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\big(\eta(x)\psi(x)\big)\,dx = \varphi[\eta\psi].$$

This motivates the following definition.

Definition 5.13. For $u \in \mathscr{S}'_n$ we define

- The *distributional partial derivative* $D^\alpha u$ is a tempered distribution defined by the formula
$$D^\alpha u[\psi] = u[(-1)^{|\alpha|}D^\alpha\psi].$$
- The translation $\tau_h u \in \mathscr{S}'_n$ is defined by
$$(\tau_h u)[\psi] = u[\tau_{-h}\psi].$$
- The reflection $\tilde{u} \in \mathscr{S}'_n$ is defined by
$$\tilde{u}[\psi] = u[\tilde{\psi}].$$
- The Fourier transform $\mathcal{F}(u) = \hat{u} \in \mathscr{S}'_n$ and the inverse Fourier transform $\check{u} \in \mathscr{S}'_n$ are
$$\hat{u}[\psi] = u[\hat{\psi}] \quad \text{and} \quad \check{u}[\psi] = u[\check{\psi}].$$
- If $\eta \in C^\infty$ is slowly increasing and all derivatives of η are slowly increasing, then we define
$$(\eta u)[\psi] = u[\eta\psi]. \quad \text{In particular} \quad (x^\alpha u)[\psi] = u[x^\alpha\psi].$$

The formulas preceding the definition show that on the subclass $\mathscr{S}_n \subset \mathscr{S}'_n$ the partial derivative, the translation, the reflection, the Fourier transform and the multiplication by a function defined in the distributional sense coincide with those defined in the classical way.

If $f \in L^1(\mathbb{R}^n)+L^2(\mathbb{R}^n)$, in particular, if $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then the classical Fourier transform coincides with the distributional one. That easily follows from Theorem 3.2(c) and Proposition 3.46.

The basic properties of the Fourier transform, distributional derivative and convolution in $\mathscr{S}'_n$ are collected in the next result whose easy proof is left to the reader.

Theorem 5.14. *The Fourier transform in a homeomorphism of $\mathscr{S}'_n$ onto itself.*[42] *Moreover for $u \in \mathscr{S}'_n$ and $\varphi \in \mathscr{S}_n$ we have*

(a) $(\hat{u})^\vee = u$,
(b) $(u * \varphi)\hat{} = \hat{\varphi}\hat{u}$,
(c) $D^\alpha(u*\varphi) = D^\alpha u * \varphi = u * D^\alpha\varphi$,
(d) $(D^\alpha u)\hat{} = (2\pi i x)^\alpha\hat{u}$,
(e) $((-2\pi i x)^\alpha u)\hat{} = D^\alpha\hat{u}$.

[42]With respect to the weak convergence in $\mathscr{S}'_n$.

Just for the illustration we will prove the property (d). For any $\psi \in \mathscr{S}_n$ we have

$$\begin{aligned}(D^\alpha u)^\wedge[\psi] &= u[(-1)^{|\alpha|}D^\alpha\hat{\psi}] = u[(-1)^{|\alpha|}((-2\pi ix)^\alpha\psi)^\wedge] \\ &= u[((2\pi ix)^\alpha\psi)^\wedge] = \big((2\pi ix)^\alpha\hat{u}\big)[\psi].\end{aligned}$$

Note that in the case (c), $u*\varphi \in \mathscr{S}_n'$ and $D^\alpha(u*\varphi)$ is understood in the distributional sense. On the other hand $u*\varphi \in C^\infty$ and Theorem 5.8 shows that the distributional derivative of $u*\varphi$ coincides with the classical one.

Example 5.15. The Dirac measure δ_a defines a tempered distribution so

$$\hat{\delta}_a[\psi] = \delta_a[\hat{\psi}] = \hat{\psi}(a) = \int_{\mathbb{R}^n} \psi(x)e^{-2\pi ix\cdot a}\,dx$$

and hence $\hat{\delta}_a = e^{-2\pi ix\cdot a}$.

Example 5.16. Let μ be a measure on $\mathbb{R}$ defined by $\mu(E) = \#(E\cap\mathbb{Z})$, where $\#A$ stands for the cardinality of a set A. In other words μ is the counting measure on $\mathbb{Z}$ trivially extended to $\mathbb{R}$. Clearly, μ is a tempered measure so it defines a tempered distribution. Moreover $\hat{\mu} = \mu$ in $\mathscr{S}_1'$. Indeed, this is a simple consequence of the Poisson summation formula Theorem 4.1 that

$$\hat{\mu}[\varphi] = \mu[\hat{\varphi}] = \sum_{n=-\infty}^{\infty}\hat{\varphi}(n) = \sum_{n=-\infty}^{\infty}\varphi(n) = \mu[\varphi].$$

5.2. **Tempered distributions as derivatives of functions.** Using the notion of distributional derivative we can provide a new class of examples of a distributions in $\mathscr{S}_n'$. If f_α, $|\alpha| \leq m$ are slowly increasing functions and $a_\alpha \in \mathbb{C}$ for $|\alpha| \leq m$, then

$$u = \sum_{|\alpha|\leq m} a_\alpha D^\alpha f_\alpha \in \mathscr{S}_n'. \tag{5.6}$$

where the derivative $D^\alpha f_\alpha$ is understood in the distributional sense.

The aim of this section is to prove a surprising result that every distribution in $\mathscr{S}_n'$ can be represented in the form (5.6), see Theorem 5.21.

Proposition 5.17. *Let N be a positive integer, then the operators*[43]

$$(I-\Delta)^N : \mathscr{S}_n \to \mathscr{S}_n \quad \textit{and} \quad (I-\Delta)^N : \mathscr{S}_n' \to \mathscr{S}_n'$$

are homeomorphisms. The inverse operator in both cases is given by the operator

$$(I-\Delta)^{-N} := \mathcal{F}^{-1}\big((1+4\pi^2|\xi|^2)^{-N}\mathcal{F}\big).$$

Remark 5.18. That means for every $v \in \mathscr{S}_n$ ($v \in \mathscr{S}_n'$) the equation $(I-\Delta)^N u = v$ has a unique solution $u \in \mathscr{S}_n$ ($u \in \mathscr{S}_n'$) given by

$$u = (I-\Delta)^N v = \mathcal{F}^{-1}\big((1+4\pi^2|\xi|^2)^{-N}\mathcal{F}(v)\big).$$

[43] $I-\Delta$ stands for the identity minus the Laplace operator.

Proof. Suppose that $u, v \in \mathscr{S}_n$ satisfy $(I-\Delta)^N u = v$. Taking the Fourier transform yields

$$(1+4\pi^2|\xi|^2)^N \mathcal{F}(u) = \mathcal{F}(v)$$

and hence

$$\mathcal{F}(u) = (1+4\pi^2|\xi|^2)^{-N}\mathcal{F}(v).$$

Since $\mathcal{F}(u) \in \mathscr{S}_n$, the right hand side is also in $\mathscr{S}_n$ so we can take the inverse Fourier transform and we obtain

$$u = \mathcal{F}^{-1}\big((1+4\pi^2|\xi|^2)^{-N}\mathcal{F}(v)\big). \tag{5.7}$$

Thus if a solution $u \in \mathscr{S}_n$ exists, it is unique and given by (5.7). To see that the formula indeed, defines a solution $u \in \mathscr{S}_n$ observe that the function $(1+4\pi^2|\xi|^2)^{-N}$ and all its derivatives are slowly increasing so if $f \in \mathscr{S}_n$, then $(1+4\pi^2|\xi|^2)^{-N} f \in \mathscr{S}_n$. Hence for any $v \in \mathscr{S}_n$, the right hand side of (5.7) defines a function in $\mathscr{S}_n$ and backward computations shows that $(I-\Delta)^N u = v$.

Similarly, if $u, v \in \mathscr{S}_n'$, then $(I-\Delta)^N u = v$ if and only if the Fourier transforms are equal i.e.,

$$(1+4\pi^2|\xi|^2)^N \mathcal{F}(u) = \mathcal{F}(v)$$

which is equivalent to

$$\mathcal{F}(u) = (1+4\pi^2|\xi|^2)^{-N}\mathcal{F}(v), \quad u = \mathcal{F}^{-1}\big((1+4\pi^2|\xi|^2)^{-N}\mathcal{F}(v)\big).$$

Note that we were allowed to multiply both sides by $(1+4\pi^2|\xi|^2)^{-N}$ because this function and all its derivatives are slowly increasing. □

The operator $(I-\Delta)^N$ maps functions of class C^k to functions of class C^{k-2N} so it lowers regularity of functions. That means the inverse operator $(I-\Delta)^{-N}$ increases regularity. Hence we should expect that it also increases regularity of distributions. Since every tempered distribution has finite order in a sense that it can be estimated by a finite sum as in Theorem 5.3, one could expect that for $u \in \mathscr{S}_n'$ and sufficiently large N, the distribution $(I-\Delta)^{-N}u$ is actually a regular function. As we will see this intuition is correct.

Remark 5.19. The operator $(I-\Delta)^{-N}$ is called a *Bessel potential* and one can prove that it can be represented as an integral operator. In Section **??** we will study Bessel potentials in detail. We will find an explicit integral formula for the Bessel potentials and we will show how use them to characterize Sobolev spaces.

Definition 5.20. By $C^{k,1}_{\rm loc}$ we will denote the class of k times continuously differentiable function whose derivatives of order k are locally Lipschitz continuous. In particular $C^{0,1}_{\rm loc}$ denotes the class of locally Lipschitz continuous functions.

Theorem 5.21. *Suppose $u \in \mathscr{S}_n'$ satisfies the estimate*[44]

$$|u(\varphi)| \leq C \sum_{|\alpha|,|\beta|\leq m} p_{\alpha,\beta}(\varphi)\,.$$

If k is a nonnegative integer and $N > (n+m+k)/2$, then the tempered distribution

$$v = (I-\Delta)^{-N}u$$

has the following properties:

[44]See Theorem 5.3.

(a) *If $k = 0$, then v is a slowly increasing function.*
(b) *If $k \geq 1$, then v is a slowly increasing function of the class $C^{k-1,1}_{\rm loc}$. Also all derivatives of v or order $|\alpha| \leq k$ are slowly increasing.*

Proof. We will need

Lemma 5.22. *If $P(x)$ is a polynomial in $\mathbb{R}^n$ of degree p and $N > (n+p)/2$, then all derivatives of*

$$f(x) = \frac{P(x)}{(1+|x|^2)^N}$$

belong to $L^1(\mathbb{R}^n)$.

Proof. Since $2N - p > n$, $f \in L^1(\mathbb{R}^n)$. We have

$$\frac{\partial f}{\partial x_i} = \frac{Q(x)}{(1+|x|^2)^{2N}}, \quad \deg Q = 2N + p - 1.$$

The function on the right hand side is of the same form as f. Since

$$2N > \frac{n + (2N+p-1)}{2}$$

we conclude that $\partial f/\partial x_i \in L^1$. The integrability of higher order derivatives follows by induction. □

We will prove that the distributional derivatives of v of orders $|\gamma| \leq k$ are slowly increasing functions. We have

$$(-1)^{|\gamma|} D^\gamma v[\psi] = u\big[\mathcal{F}\big((1+4\pi^2|\xi|^2)^{-N}\mathcal{F}^{-1}(D^\gamma\psi)\big)\big].$$

Hence

$$\begin{aligned} |D^\gamma v[\psi]| &\leq C \sum_{|\alpha|,|\beta|\leq m} \sup_{x\in\mathbb{R}^n} \Big| x^\alpha D^\beta \Big(\mathcal{F}\big((1+4\pi^2|\xi|^2)^{-N}\mathcal{F}^{-1}(D^\gamma\psi)\big)\Big)\Big| \\ &= C \sum_{|\alpha|,|\beta|\leq m} C_{\alpha,\beta,\gamma} \sup_{x\in\mathbb{R}^n} \left| \mathcal{F}\left(D^\alpha\left(\frac{\xi^{\beta+\gamma}}{(1+4\pi^2|\xi|^2)^N}\,\mathcal{F}^{-1}(\psi)\right)\right)\right| \\ &\leq C' \sum_{|\alpha|,|\beta|\leq m} \left\| D^\alpha\left(\frac{x^{\beta+\gamma}}{(1+4\pi^2|x|^2)^N}\,\mathcal{F}^{-1}(\psi)\right)\right\|_1 . \end{aligned}$$

Note that

$$\begin{aligned} &D^\alpha\left(\frac{x^{\beta+\gamma}}{(1+4\pi^2|x|^2)^N}\,\mathcal{F}^{-1}(\psi)\right) \\ &= \sum_{\alpha_i+\beta_i=\alpha} \frac{\alpha!}{\alpha_1!\beta_i!} D^{\alpha_i}\left(\frac{x^{\beta+\gamma}}{(1+4\pi^2|x|^2)^N}\right) D^{\beta_i}(\mathcal{F}^{-1}(\psi)). \end{aligned}$$

Since $\deg x^{\beta+\gamma} \leq m+k$ and $N > (n+m+k)/2$, Lemma 5.22 gives

$$D^{\alpha_i}\left(\frac{x^{\beta+\gamma}}{(1+4\pi^2|x|^2)^N}\right) \in L^1(\mathbb{R}^n),$$

so

$$\begin{aligned} |D^\gamma v[\psi]| &\leq C\sum_i \|\mathcal{F}^{-1}(x^{\beta_i}\psi)\|_\infty \\ &\leq C\sum_i \|x^{\beta_i}\psi\|_1 \\ &\leq C'\|(1+|x|^2)^{m/2}\psi(x)\|_1 \,. \end{aligned}$$

This proves that for $|\gamma|\leq k$ the functional

$$\psi\mapsto D^\gamma v[\psi]$$

is bounded on $L^1((1+|x|^2)^{m/2}\,dx)$. Thus there are functions $g_\gamma\in L^\infty$ such that

$$D^\gamma v[\psi]=\int_{\mathbb{R}^n}\psi(x)g_\gamma(x)(1+|x|^2)^{m/2}\,dx\,,$$

i.e.

$$D^\gamma v=g_\gamma(x)(1+|x|^2)^{m/2}$$

in the distributional sense. In particular, if $\gamma=0$, $v(x)=g_0(x)(1+|x|^2)^{m/2}$ is slowly increasing which proves (a). The part (b) follows from the following result whose proof is postponed to Section **??**, see Theorem **??**

Proposition 5.23. *If a function u is slowly increasing and its distributional derivatives of order less than or equal k, $k\geq 1$, are slowly increasing functions, then $u\in C^{k-1,1}_{\rm loc}$.*

5.3. **Tempered distributions with compact support.** We will investigate now properties of the Fourier transform of tempered distributions with compact support.

Lemma 5.24. *Let $\Omega\subset\mathbb{R}^n$ be open and $u\in\mathscr{S}'_n$. If $u[\varphi]=0$ for all $\varphi\in C_0^\infty(\Omega)$, then $u[\varphi]=0$ for all $\varphi\in\mathscr{S}_n$ such that* $\operatorname{supp}\varphi\subset\Omega$.

Proof. Let $\varphi\in\mathscr{S}_n$, $\operatorname{supp}\varphi\subset\Omega$ and let η be a cut-off function, i.e. $\eta\in C_0^\infty(\mathbb{R}^n)$, $0\leq\eta\leq 1$, $\eta(x)=1$ for $|x|\leq 1$ and $\eta(x)=0$ for $|x|\geq 2$. One can easily prove that $\eta(x/R)\varphi(x)\to\varphi(x)$ in $\mathscr{S}_n$ as $R\to\infty$. Since $\eta(x/R)\varphi(x)\in C_0^\infty(\Omega)$, the lemma follows. □

Definition 5.25. Let $u\in\mathscr{S}'_n$. The *support of u* ($\operatorname{supp}u$) is the intersection of all closed sets $E\subset\mathbb{R}^n$ such that

$$\varphi\in C_0^\infty(\mathbb{R}^n\setminus E)\quad\Rightarrow\quad u[\varphi]=0.$$

Proposition 5.26. *If $u\in\mathscr{S}'_n$ and $\varphi\in C_0^\infty(\mathbb{R}^n\setminus\operatorname{supp}u)$, then $u[\varphi]=0$.*

Proof. Indeed, complement of $\operatorname{supp}u$ is the union of all open sets U_i such that if $\varphi\in C_0^\infty(U_i)$, then $u[\varphi]=0$, and we need to prove that if $\varphi\in C_0^\infty(\bigcup_i U_i)$, then still $u[\varphi]=0$. Since the support of φ is compact we can select a finite subfamily of open sets U_i covering the support and the result follows from a simple argument involving partition of unity. □

Thus the support of u is the smallest closed set such that the distribution vanishes on C_0^∞ functions supported outside that set.

The lemma shows that we can replace $\varphi\in C_0^\infty(\mathbb{R}^n\setminus E)$ in the above definition by $\varphi\in\mathscr{S}_n$ with $\operatorname{supp}\varphi\subset\mathbb{R}^n\setminus E$.

Before we state the next result we need some facts about analytic and holomorphic functions in several variables.

Definition 5.27. We say that a function $f : \Omega \to \mathbb{C}$ defined in an open set $\Omega \subset \mathbb{R}^n$ is $\mathbb{R}$-*analytic*, if in a neighborhood on any point $x_0 \in \Omega$ it can be expanded as a convergent power series

$$f(x) = \sum_{\alpha} a_\alpha (x - x_0)^\alpha, \quad a_\alpha \in \mathbb{C}, \tag{5.8}$$

i.e. if in a neighborhood of any point f equals to its Taylor series.

We say that a function $f : \Omega \to \mathbb{C}$ defined in an open set $\Omega \subset \mathbb{C}^n$ is $\mathbb{C}$-*analytic* if in a neighborhood of any point $z_0 \in \Omega$ it can be expanded as a convergent power series

$$f(z) = \sum_{\alpha} a_\alpha (z - z_0)^\alpha .$$

$\mathbb{R}^n$ has a natural embedding into $\mathbb{C}^n$, just like $\mathbb{R}$ into $\mathbb{C}$.

$$\mathbb{R}^n \ni x = (x_1, \ldots, x_n) \mapsto (x_1 + i \cdot 0, \ldots, x_n + i \cdot 0) = x + i \cdot 0 \in \mathbb{C}^n .$$

It is easy to see that an $\mathbb{R}$-analytic function f in $\Omega \subset \mathbb{R}^n$ extends to a $\mathbb{C}$-analytic function $\tilde{f}$ in an open set $\tilde{\Omega} \subset \mathbb{C}^n$, $\Omega \subset \tilde{\Omega}$. Namely, if f satisfies (5.8), we set

$$\tilde{f}(z) = \sum_{\alpha} a_\alpha (z - z_0)^\alpha, \quad z_0 = x_0 + i \cdot 0.$$

On the other hand, if $\tilde{f}$ is $\mathbb{C}$-analytic in $\tilde{\Omega} \subset \mathbb{C}^n$, then the restriction f of $\tilde{f}$ to $\Omega = \tilde{\Omega} \cap \mathbb{R}^n$ is $\mathbb{R}$-analytic.

For example for any $\xi \in \mathbb{R}^n$, $f(x) = e^{x \cdot \xi}$ is $\mathbb{R}$-analytic and $\tilde{f}(z) = e^{z \cdot \xi}$ is its $\mathbb{C}$-analytic extension.

Definition 5.28. We say that a continuous function $f : \Omega \to \mathbb{C}$ defined in an open set $\Omega \subset \mathbb{C}^n$ is *holomorphic* if

$$\frac{\partial f}{\partial \overline{z}_i} = 0 \quad \text{for } i = 1, 2, \ldots, n.$$

It is easy to see that $\mathbb{C}$-analytic function are holomorphic, but the converse implication is also true.

Lemma 5.29 (Cauchy)**.** *If f is holomorphic in*[45]

$$D^n(w, r) = D^1(w_1, r_1) \times \ldots \times D^1(w_n, r_n) \subset \mathbb{C}^n$$

and continuous in the closure $\overline{D^n}(x, r)$, then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D^1(w_1, r_1)} \cdots \int_{\partial D^1(w_n, r_n)} \frac{f(\xi)\, d\xi_1 \ldots d\xi_n}{(\xi_1 - z_1) \ldots (\xi_n - z_n)} \tag{5.9}$$

for all $z \in D^n(w, r)$.

[45]Product of one dimensional discs.

Proof. The function f is holomorphic in each variable separately, so (5.9) follows from one dimensional Cauchy formulas and the Fubini theorem. $\square$

Just like in the case of holomorphic functions of one variable one can prove that the integral on the right hand side of (5.9) can be expanded as a power series and hence defines a $\mathbb{C}$-analytic function. We proved

Theorem 5.30. *A function f is holomorphic in $\Omega \subset \mathbb{C}^n$ if and only if it is $\mathbb{C}$-analytic.*

Now we can state an important result about compactly supported distributions.

Theorem 5.31. *If $u \in \mathscr{S}_n'$ has compact support, then $\hat{u}$ is a slowly increasing C^∞ function and all derivatives of $\hat{u}$ are slowly increasing. Moreover $\hat{u}$ is $\mathbb{R}$-analytic on $\mathbb{R}^n$ and has a holomorphic extension to $\mathbb{C}^n$.*

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 1$ in a neighborhood of $\operatorname{supp} u$. Then $u = \eta u$ in $\mathscr{S}_n'$ and hence for $\psi \in \mathscr{S}_n$ we have

$$\begin{aligned} \hat{u}[\psi] &= (\eta u)\hat{}\,[\psi] = u\left[\int_{\mathbb{R}^n} \eta(\cdot)\psi(x)e^{-2\pi i x\cdot(\cdot)}\, dx\right] \\ &= \int_{\mathbb{R}^n} u\left[\eta(\cdot)e^{-2\pi i x\cdot(\cdot)}\right]\psi(x)\, dx\, . \end{aligned}$$

We could pass with u under the sign of the integral, because of an argument with approximation of the integral by Riemann sums.[46]

Note that the function

$$F(x_1,\ldots,x_n) = u\big[\eta(\cdot)e^{-2\pi i x\cdot(\cdot)}\big]$$

is C^∞ smooth and

$$D^\alpha F(x_1,\ldots,x_n) = u\big[(-2\pi i(\cdot))^\alpha \eta(\cdot)e^{-2\pi i x\cdot(\cdot)}\big]\, .$$

Indeed, we could differentiate under the sign of u because the corresponding difference quotients converge in the topology of $\mathscr{S}_n$. It also easily follows from Theorem 5.3 that F and all its derivatives are slowly increasing. Thus we may identify $\hat{u}$ with F, so $\hat{u} \in C^\infty$.

Moreover F has a holomorphic extension to $\mathbb{C}^n$ by the formula

$$F(z_1,\ldots,z_n) = u\big[\eta(\cdot)e^{-2\pi i z\cdot(\cdot)}\big]$$

so in particular $F(x_1,\ldots,x_n)$ is $\mathbb{R}$-analytic. $\square$

Remark 5.32. Note that if $u \in \mathscr{S}_n'$ has compact support, then we can reasonably define $u[e^{-2\pi i x\cdot(\cdot)}]$ by the formula

$$u\big[e^{-2\pi i x\cdot(\cdot)}\big] := u\big[\eta(\cdot)e^{-2\pi i x\cdot(\cdot)}\big]\, ,$$

where $\eta \in C_0^\infty(\mathbb{R}^n)$ is such that $\eta = 1$ in a neighborhood of $\operatorname{supp} u$. Note that this construction does not depend on the choice of η. In the same way we can define $u[f]$ for any $f \in C^\infty(\mathbb{R}^n)$

[46] Compare with the proof of Theorem 5.7.

Theorem 5.33. *If* $u \in \mathscr{S}_n'$ *and* $\operatorname{supp} u = \{x_0\}$*, then there is an integer* $m \geq 0$ *and complex numbers* a_α*,* $|\alpha| \leq m$ *such that*

$$u = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \delta_{x_0} \,.$$

Proof. Without loss of generality we may assume that $x_0 = 0$. According to Theorem 5.3 the distribution u satisfies the estimate

$$|u[\varphi]| \leq C \sum_{|\alpha|,|\beta| \leq m} p_{\alpha,\beta}(\varphi) \,.$$

First we will prove that for $\varphi \in \mathscr{S}_n$ we have

$$D^\alpha \varphi[0] = 0 \quad \text{for } |\alpha| \leq m \qquad \Longrightarrow \qquad u[\varphi] = 0 \,. \tag{5.10}$$

Indeed, it follows from Taylor's formula that

$$\varphi(x) = O(|x|^{m+1}) \quad \text{as } x \to 0$$

and hence also

$$D^\beta \varphi(x) = O(|x|^{m+1-|\beta|}) \quad \text{as } x \to 0 \text{ for all } |\beta| \leq m. \tag{5.11}$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function, i.e. $0 \leq \eta \leq 1$, $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$ and define $\eta_\varepsilon(x) = \eta(x/\varepsilon)$. The estimate (5.11) easily implies that

$$p_{\alpha,\beta}(\eta_\varepsilon \varphi) \to 0 \quad \text{as } \varepsilon \to 0$$

for all $|\alpha|, |\beta| \leq m$.[47] Note that $\varphi - \eta_\varepsilon \varphi = 0$ in a neighborhood of 0, so $u[\varphi - \eta_\varepsilon \varphi] = 0$ and hence

$$|u[\varphi]| \leq |u[\varphi - \eta_\varepsilon \varphi]| + |u[\eta_\varepsilon \varphi]| \leq 0 + \sum_{|\alpha|,|\beta| \leq m} p_{\alpha,\beta}(\eta_\varepsilon \varphi) \to 0 \quad \text{as } \varepsilon \to 0.$$

This completes the proof of (5.10).

Let now $\psi \in \mathscr{S}_n$ be arbitrary and let

$$h(x) = \psi(x) - \sum_{|\alpha| \leq m} \frac{D^\alpha \psi(0)}{\alpha!} \, x^\alpha \,.$$

Clearly

$$D^\alpha h(0) = 0 \quad \text{for } |\alpha| \leq m. \tag{5.12}$$

We have

$$\psi(x) = \eta(x) \left(\sum_{|\alpha| \leq m} \frac{D^\alpha \psi(0)}{\alpha!} \, x^\alpha \right) + \eta(x) h(x) + (1 - \eta(x)) \psi(x) \,.$$

Since $(1-\eta)\psi$ vanishes in a neighborhood of 0 we have $u[(1-\eta)\psi] = 0$. The equality (5.12) implies that $\varphi = \eta h \in C_0^\infty(\mathbb{R}^n)$ satisfies the assumptions of (5.10), so $u[\eta h] = 0$. Hence

$$u[\psi] \;\; = \;\; u\Big[\eta(x) \Big(\sum_{|\alpha| \leq m} \frac{D^\alpha \psi(0)}{\alpha!} \, x^\alpha \Big) \Big]$$

[47] Check it!

$$= \sum_{|\alpha|\leq m} \underbrace{(-1)^{|\alpha|} u\left[\frac{\eta(x)x^\alpha}{\alpha!}\right]}_{a_\alpha} \underbrace{(-1)^{|\alpha|}D^\alpha\psi(0)}_{D^\alpha\delta_0[\psi]}\, .$$

The proof is complete. □

Corollary 5.34. *Let $u \in \mathscr{S}'_n$. If $\operatorname{supp}\hat{u} = \{\xi_0\}$, then $u = P(x)e^{2\pi i x\cdot\xi_0}$ for some polynomial $P(x)$ in $\mathbb{R}^n$. In particular if $\operatorname{supp}\hat{u} = \{0\}$, then u is a polynomial.*

We leave the proof as an exercise.

Corollary 5.35. *If $u \in \mathscr{S}'_n$ satisfies $\Delta u = 0$, then u is a polynomial.*

Proof. We have

$$-4\pi^2|\xi|^2\hat{u} = (\Delta u)\hat{} = 0 \quad \text{in } \mathscr{S}'_n.$$

This implies that $\operatorname{supp}\hat{u} = \{0\}$, so u is a polynomial by Corollary 5.34. □

5.4. **Homogeneous distributions.** A very important class of distributions is provided by so called *homogeneous distributions* that we will study now.

Definition 5.36. A function $f \in C(\mathbb{R}\setminus\{0\})$ is *homogeneous of degree $a \in \mathbb{R}$* if for all $x \neq 0$ and all $t > 0$ we have $f(tx) = t^a f(x)$. More generally a measurable function on $\mathbb{R}^n$ is *homogeneous of degree $a \in \mathbb{R}$* if

$$f(tx) = t^a f(x) \quad \text{for almost all } (x,t) \in \mathbb{R}^n \times (0,\infty).$$

If $f \in C(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $a > -n$, then it defines a tempered distribution. Indeed,

$$|f(x)| = \left| |x|^a f\left(\frac{x}{|x|}\right)\right| \leq C|x|^a$$

and $|x|^a$ is integrable in a neighborhood of the origin (because $a > -n$).

If f is a measurable homogeneous function of degree a we can assume (by changing f on a set of measure zero) that f is continuous on almost all open rays going out of 0. Then on the remaining rays (which form a set of measure zero) we can make f homogeneous. This is to say that we can find a representative of f (in class of functions that are equal a.e.) such that

$$f(tx) = t^a f(x) \quad \text{for } \textit{all } x \neq 0 \text{ and } t > 0.$$

If a measurable function that is not a.e. equal to zero is homogeneous of degree $a \leq -n$, then the integration in the spherical coordinate system shows that it is not integrable in any neighborhood of 0 and hence it cannot define a tempered distribution.

Lemma 5.37. *Suppose that $f \in L^1_{\rm loc}(\mathbb{R}^n)$ defines a tempered distribution. Then the function f is homogeneous of degree $a \in \mathbb{R}$ if and only if* [48]

$$f[\varphi_t] = t^a f[\varphi] \quad \textit{for all } \varphi \in \mathscr{S}_n \textit{ and all } t > 0. \tag{5.13}$$

Remark 5.38. If f is not equal to zero a.e., then necessarily $a > -n$.

[48] As always $\varphi_t(x) = t^{-n}\varphi(x/t)$.

Proof. For $\varphi \in \mathscr{S}_n$ we have

$$f[\varphi_t] = \int_{\mathbb{R}^n} f(x)\varphi_t(x)\,dx = \int_{\mathbb{R}^n} f(tx)\varphi(x)\,dx \quad \text{and} \quad t^a f[\varphi] = \int_{\mathbb{R}^n} t^a f(x)\varphi(x)\,dx.$$

Hence condition (5.13) is equivalent to the equality

$$\int_{\mathbb{R}^n} f(tx)\varphi(x)\,dx = \int_{\mathbb{R}^n} t^a f(x)\varphi(x)\,dx \quad \text{for all } \varphi \in \mathscr{S}_n \text{ and all } t > 0 \tag{5.14}$$

which, in turn, is equivalent to the fact that for all $t > 0$, equality $f(tx) = t^a f(x)$ holds a.e. □

The above result motivates the following definition.

Definition 5.39. We say that a distribution $u \in \mathscr{S}'_n$ is *homogeneous of degree* $a \in \mathbb{R}$ if [49]

$$u[\varphi_t] = t^a u[\varphi] \quad \text{for all } \varphi \in \mathscr{S}_n \text{ and all } t > 0.$$

Proposition 5.40. *$u \in \mathscr{S}'_n$ is homogeneous of degree $a \in \mathbb{R}$ if and only if $\hat{u} \in \mathscr{S}'_n$ is homogeneous of degree $-n - a$.*

Proof. Let $\varphi \in \mathscr{S}_n$. Then

$$\widehat{\varphi}_t(\xi) = \hat{\varphi}(t\xi) = t^{-n}(t^{-1})^{-n}\hat{\varphi}(\xi/t^{-1}) = t^{-n}(\hat{\varphi})_{t^{-1}}(\xi)$$

and hence

$$\hat{u}[\varphi_t] = u[\widehat{\varphi}_t] = t^{-n}u[(\hat{\varphi})_{t^{-1}}] = t^{-n-a}u[\hat{\varphi}] = t^{-n-a}\hat{u}[\varphi].$$

If $u \in \mathscr{S}'_n$ is homogeneous of degree a, the above equality shows that $\hat{u}$ is homogeneous of degree $-n - a$. Now if $\hat{u}$ is homogeneous of degree $-n - a$, then its Fourier transform i.e., $\tilde{u}$ is homogeneous of degree $-n - (-n - a) = a$ and hence also u is homogeneous of degree a. □

Example 5.41. $u(x) = |x|^a$, $a \geq 0$, is homogeneous of degree a and it defines a tempered distribution. However, $\hat{u}$ is homogeneous of degree $-n - a \leq -n$ and hence it cannot be represented as a function.

The notion of a homogeneous distribution will play a significant role in the proof of the next result.

Theorem 5.42. *For $0 < a < n$,*

$$\mathcal{F}\left(|x|^{-a}\right)(\xi) = \frac{\pi^{a-\frac{n}{2}}\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}\,|\xi|^{a-n}\,.$$

Remark 5.43. $|x|^{-a}$ is homogeneous of degree $-a > -n$ and $|\xi|^{a-n}$ is homogeneous of degree $a - n > -n$ so both $|x|^{-a}$ and $|\xi|^{a-n}$ are tempered distributions represented as functions.

[49] Now a can be any real number and we do not impose restriction $a > -n$ that was necessary in the case of functions.

Proof. First we will prove the theorem under the assumption that $n/2 < a < n$. In that case

$$|x|^{-a} = |x|^{-a}\chi_{\{|x|\leq 1\}} + |x|^{-a}\chi_{\{|x|>1\}} \in L^1 + L^2\,.$$

Thus the Fourier transform of $|x|^{-a}$ is a function in $C_0 + L^2$.

The Fourier transform commutes with orthogonal transformations

$$\widehat{f\circ\rho} = \hat{f}\circ\rho, \quad \text{for } \rho\in O(n) \text{ and } f\in L^1+L^2.$$

If f is radially symmetric i.e., $f = f\circ\rho$ for all $\rho\in O(n)$, then $\hat{f}\circ\rho = \widehat{f\circ\rho} = \hat{f}$ for all $\rho\in O(n)$ so $\hat{f}$ is radially symmetric too. Thus the Fourier transform of $|x|^{-a}\in L^1+L^2$ is a radially symmetric function in C_0+L^2, homogeneous of degree $a-n$ as a distribution. Hence Lemma 5.37 yields

$$\mathcal{F}\left(|x|^{-a}\right)(\xi) = c_{a,n}|\xi|^{a-n}$$

and it remains to compute the coefficient $c_{a,n}$. Employing the fact that $\varphi(x) = e^{-\pi|x|^2}$ is a fixed point of the Fourier transform we have

$$c_{a,n}|x|^{a-n}[\varphi] = \mathcal{F}(|x|^{-a})[\varphi] = |x|^{-a}[\hat{\varphi}] = |x|^{-a}[\varphi],$$

i.e.,

$$c_{a,n}\int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^{a-n}\,dx = \int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^{-a}\,dx. \tag{5.15}$$

The integrals in this identity are easy to compute.

Lemma 5.44. *For $\gamma > -n$ we have*

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^{\gamma}\,dx = \frac{\Gamma\left(\frac{n+\gamma}{2}\right)}{\pi^{\frac{\gamma}{2}}\Gamma\left(\frac{n}{2}\right)}\,.$$

Proof. For $\gamma > -n$ we have[50]

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^{\gamma}\,dx &= n\omega_n\int_0^\infty e^{-\pi s^2}s^{\gamma+n-1}\,ds \\
&\overset{t=\pi s^2}{=} \frac{n\omega_n}{2\pi^{\frac{n+\gamma}{2}}}\int_0^\infty e^{-t}t^{\frac{n+\gamma}{2}-1}\,dt \\
&= \frac{n\omega_n}{2\pi^{\frac{n+\gamma}{2}}}\Gamma\left(\frac{n+\gamma}{2}\right) = \frac{\Gamma\left(\frac{n+\gamma}{2}\right)}{\pi^{\frac{\gamma}{2}}\Gamma\left(\frac{n}{2}\right)}\,.
\end{aligned}$$

□

Applying the lemma to both sides of (5.15) yields

$$c_{a,n}\frac{\Gamma\left(\frac{a}{2}\right)}{\pi^{\frac{a-n}{2}}\Gamma\left(\frac{n}{2}\right)} = \frac{\Gamma\left(\frac{n-a}{2}\right)}{\pi^{-\frac{a}{2}}\Gamma\left(\frac{n}{2}\right)} \quad \text{so} \quad c_{a,n} = \pi^{a-\frac{n}{2}}\frac{\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}\,.$$

This completes the proof when $n/2 < a < n$.

[50]Recall that the volume of the unit sphere in $\mathbb{R}^n$ equals $n\omega_n$ so the first equality is just integration in the spherical coordinates. In the last equality we use $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$, see (3.9).

Suppose now that $0 < a < n/2$. Since $n/2 < n - a < n$ we have

$$\mathcal{F}\left(|x|^{-(n-a)}\right)(\xi) = \frac{\pi^{\frac{n}{2}-a}\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}|\xi|^{-a}$$

and applying the Fourier transform to this equality yields

$$|-x|^{a-n} = \frac{\pi^{\frac{n}{2}-a}\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}\mathcal{F}\left(|\xi|^{-a}\right)(x).$$

Replacing x by ξ and ξ by x yields the result when $0 < a < n/2$.

We are left with the case of $a = n/2$. Take a sequence $n/2 < a_k < n$, $a_k \to a = n/2$. Since the result is true for a_k we have

$$\int_{\mathbb{R}^n} |x|^{-a_k}\hat{\varphi}(x)\,dx = \frac{\pi^{a_k-\frac{n}{2}}\Gamma\left(\frac{n-a_k}{2}\right)}{\Gamma\left(\frac{a_k}{2}\right)}\int_{\mathbb{R}^n} |x|^{a_k-n}\varphi(x)\,dx.$$

Passing to the limit $a_k \to a = n/2$ yields the result for $a = n/2$. □

Remark 5.45. Clearly $|\xi|^{a_k-n} \to |\xi|^{a-n}$ and $|x|^{-a_k} \to |x|^{-a}$ in $\mathscr{S}_n'$. The last convergence and the continuity of the Fourier transform yields that $\mathcal{F}(|x|^{-a_k}) \to \mathcal{F}(|x|^{-a})$ in $\mathscr{S}_n'$. Hence the result for $a = n/2$ follows by passing to the limit in the equality

$$\mathcal{F}\left(|x|^{-a_k}\right)(\xi) = \frac{\pi^{a_k-\frac{n}{2}}\Gamma\left(\frac{n-a_k}{2}\right)}{\Gamma\left(\frac{a_k}{2}\right)}|\xi|^{a_k-n} \quad \text{in } \mathscr{S}_n'.$$

Remark 5.46. In the case $a = n/2$ we obtain a particularly simple formula

$$\mathcal{F}(|x|^{-\frac{n}{2}}) = |x|^{-\frac{n}{2}}, \quad \text{i.e.,} \quad \int_{\mathbb{R}^n} |x|^{-\frac{n}{2}}\hat{\varphi}(x)\,dx = \int_{\mathbb{R}^n} |x|^{-\frac{n}{2}}\varphi(x)\,dx$$

for all $\varphi \in \mathscr{S}_n$. This formula can be regarded as a version of the Poisson summation formula.[51]

5.5. **The principal value.**

Proposition 5.47. *If $u \in \mathscr{S}_n'$ is homogeneous of degree a, then $D^\alpha u$ is homogeneous of degree $a - |\alpha|$.*

We leave the proof as an (easy) exercise.

In particular if $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $1 - n$, then it defines a distribution, but its first order distributional partial derivatives are homogeneous of degree $-n$ and hence they cannot be represented as functions. It is however, still possible to represent these distributional derivatives as improper integrals, the so called *principal value* of the integral.

Definition 5.48. If for any $\varepsilon > 0$, a function f is integrable in $\mathbb{R}^n \setminus B(0,\varepsilon)$, then we define the *principal value of the integral* as

$$\text{p.v.} \int_{\mathbb{R}^n} f(x)\,dx = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} f(x)\,dx$$

provided the limit exists.

[51] cf. Theorem 4.1.

If a function f is slowly increasing, but has singularity at 0, then we can *try* to define a distribution p.v. f by

$$(\text{p.v.}\, f)[\varphi] = \text{p.v.} \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} f(x)\varphi(x)\, dx \quad \text{for } \varphi\in\mathscr{S}_n.$$

In many interesting cases p.v. f defines a tempered distribution, but not always.

Theorem 5.49. *Let*

$$K_\Omega(x) = \frac{\Omega(x/|x|)}{|x|^n}, \quad x\neq 0,$$

where $\Omega\in L^1(S^{n-1})$ *satisfies*

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0.$$

Then

$$\text{p.v.}\, K_\Omega \in \mathscr{S}'_n$$

and

$$|(\text{p.v.}\, K_\Omega)[\phi]| \leq C(n)\|\Omega\|_{L^1(S^{n-1})}\big(\|\nabla\varphi\|_\infty + \||x|\varphi(x)\|_\infty\big) \quad \textit{for all } \varphi\in\mathscr{S}_n.$$

Remark 5.50. Distributions p.v. $K_\Omega\in\mathscr{S}'_n$ described in the above result will be called *distributions of the p.v. type* or simply *p.v. distributions*.

Remark 5.51. Note that the function K_Ω is homogeneous of degree $-n$ so the function K_Ω does not define a tempered distribution.

Remark 5.52. It is not difficult to verify that if $\int_{S^{n-1}}\Omega(\theta)\, d\sigma(\theta)\neq 0$, then p.v. K_Ω does not define a tempered distribution because p.v. $K_\Omega[\varphi]$ will diverge for some functions $\varphi\in\mathscr{S}_n$.

Proof. For $\varphi\in\mathscr{S}_n$ we have

$$\begin{aligned}\big|(\text{p.v.}\, K_\Omega)[\varphi]\big| &= \left|\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x|\leq 1}\frac{\Omega(x/|x|)}{|x|^n}\,(\varphi(x)-\varphi(0)) + \int_{|x|\geq 1}\frac{\Omega(x/|x|)}{|x|^n}\,\varphi(x)\, dx\right| \\ &\leq \|\nabla\varphi\|_\infty\int_{|x|\leq 1}\frac{|\Omega(x/|x|)|}{|x|^{n-1}}\, dx + \|\varphi(y)|y|\|_\infty\int_{|x|\geq 1}\frac{|\Omega(x/|x|)|}{|x|^{n+1}}\, dx\end{aligned}$$

and the result follows upon integration in spherical coordinates. We could subtract $\varphi(0)$ in the first integral because $\int_{\varepsilon\leq|x|\leq 1}\frac{\Omega(x/|x|)}{|x|^n}\, dx = 0$. In the last step used the following estimates

$$|\varphi(x)-\varphi(0)| = \left|\int_0^1 \frac{d}{dt}\varphi(tx)\, dt\right| = \left|\int_0^1 \nabla\varphi(tx)\cdot x\, dt\right| \leq |x|\,\|\nabla\varphi\|_\infty$$

and

$$|\varphi(x)| = \frac{|\varphi(x)|\,|x|}{|x|} \leq \frac{\|\varphi(y)|y|\|_\infty}{|x|}.$$

□

The next result illustrates a simple situation when the principal value appears in a natural way. Note that the function $x_j/|x|^{n+1}$ satisfies the assumptions of Theorem 5.49. However, the proof of Proposition 5.53 will be straightforward and we will not use Theorem 5.49.

Proposition 5.53. *For $n \geq 2$, the distributional partial derivatives of $|x|^{1-n} \in \mathscr{S}_n'$ satisfy*

$$\frac{\partial}{\partial x_j}|x|^{1-n} = (1-n)\,\text{p.v.}\left(\frac{x_j}{|x|^{n+1}}\right) \in \mathscr{S}_n',$$

i.e.,

$$\left(\frac{\partial}{\partial x_j}|x|^{1-n}\right)[\varphi] = (1-n)\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{x_j}{|x|^{n+1}}\,\varphi(x)\,dx. \quad \textit{for } \varphi\in\mathscr{S}_n. \tag{5.16}$$

Before we prove the proposition let us recall the integration by parts formula for functions defined in a domain in $\mathbb{R}^n$. If $\Omega\subset\mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary and $f,g\in C^1(\overline{\Omega})$, then

$$\int_\Omega (\nabla f(x)g(x) + f(x)\nabla g(x))\,dx = \int_{\partial\Omega} fg\vec{\nu}\,d\sigma\,, \tag{5.17}$$

where $\vec{\nu}=(\nu_1,\ldots,\nu_n)$ is the unit outer normal vector to $\partial\Omega$. Comparing jth components on both sides of (5.17) we have

$$\int_\Omega \frac{\partial f}{\partial x_j}(x)\,g(x)\,dx = -\int_\Omega f(x)\,\frac{\partial g}{\partial x_j}(x)\,dx + \int_{\partial\Omega} fg\,\nu_j\,d\sigma\,. \tag{5.18}$$

Proof of Proposition 5.53. Let $A(\varepsilon,R)=\{x:\,\varepsilon\leq|x|\leq R\}$. We have

$$\begin{aligned}
\left(\frac{\partial}{\partial x_j}|x|^{1-n}\right)[\varphi] &= -\int_{\mathbb{R}^n}\frac{\partial\varphi}{\partial x_j}\,|x|^{1-n}\,dx\\
&= \lim_{\varepsilon\to0}\left(\lim_{R\to\infty} -\int_{\varepsilon\leq|x|\leq R}\frac{\partial\varphi}{\partial x_j}\,|x|^{1-n}\,dx\right)\\
&= \lim_{\varepsilon\to0}\Bigg(\lim_{R\to\infty}\Bigg(\int_{\varepsilon\leq|x|\leq R}\varphi(x)\underbrace{\frac{\partial}{\partial x_j}\,|x|^{1-n}}_{(1-n)x_j/|x|^{n+1}}\,dx - \int_{\partial A(\varepsilon,R)}\varphi(x)|x|^{1-n}\,\nu_j\,d\sigma\Bigg)\Bigg)\\
&= \lim_{\varepsilon\to0}\left((1-n)\int_{|x|\geq\varepsilon}\varphi(x)\,\frac{x_j}{|x|^{n+1}} - \int_{|x|=\varepsilon}\varphi(x)|x|^{1-n}\,\nu_j\,d\sigma\right)
\end{aligned}$$

Indeed, we could pass to the limit as $R\to\infty$, because the part of the integral over $\partial A(\varepsilon,R)$ corresponding to $\{|x|=R\}$ converges to zero. Since the integral of the function $|x|^{1-n}\,\nu_j$ over the sphere $|x|=\varepsilon$ equals zero we have

$$\left|\int_{|x|=\varepsilon}\varphi(x)|x|^{1-n}\,\nu_j\,d\sigma\right| = \left|\int_{|x|=\varepsilon}(\varphi(x)-\varphi(0))|x|^{1-n}\,\nu_j\,d\sigma\right| \leq C\varepsilon\to 0 \quad \text{as } \varepsilon\to0,$$

because

$$\big|\big(\varphi(x)-\varphi(0)\big)|x|^{1-n}\,\nu_j\big| \leq C\varepsilon\cdot\varepsilon^{1-n} = \varepsilon^{2-n}$$

and the surface area of the sphere $\{|x|=\varepsilon\}$ equals $C\varepsilon^{n-1}$. □

Remark 5.54. We proved that p.v. $(x_j/|x|^{n+1})$ defines a tempered distribution somewhat indirectly by showing that for $\varphi\in\mathscr{S}_n$ equality (5.16) is satisfied. Since $\partial|x|^{1-n}/\partial x_j\in\mathscr{S}_n'$, equality (5.16) implies that p.v. $(x_j/|x|^{n+1})\in\mathscr{S}_n'$.

Remark 5.55. When $n=1$ the counterpart of Theorem 5.53 is that

$$\frac{d}{dx}(\log|x|)=\text{p.v.}\,\frac{x}{|x|^2}=\text{p.v.}\,\frac{1}{x}\quad\text{in }\mathscr{S}_n'.$$

We leave the proof as an exercise.

We plan to generalize Proposition 5.53 to a class of more general functions. Let's start with the following elementary result.

Theorem 5.56. *Suppose that $K\in C^1(\mathbb{R}^n\setminus\{0\})$ is such that both K and $|\nabla K|$ have polynomial growth*[52] *for $|x|\geq 1$ and there are constants $C,\alpha>0$ such that*

$$|K(x)|\leq\frac{C}{|x|^{n-1-\alpha}}\quad\textit{for } 0<|x|<1,$$

$$|\nabla K(x)|\leq\frac{C}{|x|^{n-\alpha}}\quad\textit{for } 0<|x|<1.$$

Then $K\in\mathscr{S}_n'$ and the distributional partial derivatives $\partial K/\partial x_j$, $1\leq j\leq n$, coincide with pointwise derivatives, i.e. for $\varphi\in\mathscr{S}_n$

$$\frac{\partial K}{\partial x_j}[\varphi]:=-\int_{\mathbb{R}^n}K(x)\,\frac{\partial\varphi}{\partial x_j}(x)\,dx=\int_{\mathbb{R}^n}\frac{\partial K}{\partial x_j}(x)\,\varphi(x)\,dx\,.$$

Proof. Let $A(\varepsilon)=\{x:\,\varepsilon\leq|x|\leq\varepsilon^{-1}\}$. From (5.18) we have

$$\begin{aligned}\frac{\partial K}{\partial x_j}[\varphi] &= \lim_{\varepsilon\to 0}-\int_{\varepsilon\leq|x|\leq\varepsilon^{-1}}K(x)\,\frac{\partial\varphi}{\partial x_j}(x)\,dx\\ &= \lim_{\varepsilon\to 0}\left(\int_{\varepsilon\leq|x|\leq\varepsilon^{-1}}\frac{\partial K}{\partial x_j}(x)\,\varphi(x)\,dx-\int_{\partial A(\varepsilon)}K(x)\varphi(x)\,\nu_j\,d\sigma(x)\right)\,.\end{aligned}$$

Since

$$\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x|\leq\varepsilon^{-1}}\frac{\partial K}{\partial x_j}(x)\,\varphi(x)\,dx=\int_{\mathbb{R}^n}\frac{\partial K}{\partial x_j}(x)\,\varphi(x)\,dx$$

it remains to show that

$$\lim_{\varepsilon\to 0}\int_{\partial A(\varepsilon)}K(x)\varphi(x)\,\nu_j\,d\sigma=0\,.$$

The integral on the outer sphere $|x|=\varepsilon^{-1}$ converges to 0 since K has polynomial growth and φ rapidly converges to 0 as $|x|\to\infty$ and on the inner sphere $|x|=\varepsilon$ we have

$$\left|\int_{|x|=\varepsilon}K(x)\varphi(x)\,\nu_j\,d\sigma\right|\leq\frac{C}{\varepsilon^{n-1-\alpha}}\,\varepsilon^{n-1}=C\varepsilon^{\alpha}\to 0\quad\text{as }\varepsilon\to 0.$$

The proof is complete. □

[52] i.e., $|K(x)|+|\nabla K(x)|\leq C(1+|x|)^m$ for some $C>0$, $m\geq 1$ and all $|x|\geq 1$.

An interesting problem is the case $\alpha = 0$, i.e. when K and ∇K satisfy the estimates

$$|K(x)| \leq \frac{C}{|x|^{n-1}}\,, \quad |\nabla K(x)| \leq \frac{C}{|x|^{n}} \quad \text{for } x \neq 0. \tag{5.19}$$

One such situation was described in Proposition 5.53.

Here we make an additional assumption about K. We assume that $K \in C^1(\mathbb{R}^n \setminus \{0\})$ is homogenelus of degree $1-n$, i.e.

$$K(x) = \frac{K(x/|x|)}{|x|^{n-1}} \quad \text{for } x \neq 0.$$

Since K is bounded in $\{|x| \geq 1\}$ and integrable in $\{|x| \leq 1\}$ we have $K \in \mathscr{S}'_n$ and there is no need to interpret K through the principal value of the integral. The first estimate at (5.19) is satisfied. To see that the second estimate if satisfied too we observe that ∇K is homogeneous of degree $-n$. Indeed, for $1 \leq j \leq n$ and $t > 0$

$$\frac{\partial K}{\partial x_j}(tx)t = \frac{\partial}{\partial x_j}\,(K(tx)) = t^{1-n}\frac{\partial}{\partial x_j}\,K(x)$$

and hence

$$(\nabla K)(tx) = t^{-n}\nabla K(x)\,.$$

Thus

$$\nabla K(x) = \frac{(\nabla K)(x/|x|)}{|x|^n}, \quad x \neq 0$$

from which the second estimate at (5.19) follows.

Observe that $\nabla K(x)$ is not integrable at any neighborhood of 0, but we may try to consider the principal value of $\nabla K(x)$, i.e. the principal value of each of the partial derivatives $\partial K/\partial x_j$.

Theorem 5.57. *Suppose that $K \in C^1(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $1-n$. Then $\nabla K(x)$ is homogeneous of degree $-n$. Moreover*

$$\int_{S^{n-1}} \nabla K(\theta)\, d\sigma(\theta) = 0 \tag{5.20}$$

and

$$\text{p.v.}\, \nabla K(x) \in \mathscr{S}'_n$$

is a well defined tempered distribution, i.e. for each $1 \leq j \leq n$

$$\text{p.v.}\, \frac{\partial K}{\partial x_j}(x) \in \mathscr{S}'_n\,.$$

Finally the distributional gradient ∇K satisfies

$$\underbrace{\nabla K}_{dist.} = c\delta_0 + \text{p.v.}\, \underbrace{\nabla K(x)}_{pointwise}\,, \tag{5.21}$$

where

$$c = \int_{S^{n-1}} K(x)\frac{x}{|x|}\, d\sigma(x)\,.$$

In other words for $\varphi\in\mathscr{S}_n$ and $1\leq j\leq n$ we have

$$\frac{\partial K}{\partial x_j}[\varphi]:=-\int_{\mathbb{R}^n}K(x)\,\frac{\partial\varphi}{\partial x_j}(x)\,dx=c_j\varphi(0)+\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\partial K}{\partial x_j}(x)\varphi(x)\,dx\,,$$

where

$$c_j=\int_{S^{n-1}}K(x)\,\frac{x_j}{|x|}\,d\sigma(x)\,.$$

Remark 5.58. Note that Proposition 5.53 follows from this result: since $K(x)=|x|^{1-n}$, the constant $c=\int_{S^{n-1}}|x|^{-n}x\,d\sigma(x)=0$.

Remark 5.59. The fact that p.v. $\nabla K\in\mathscr{S}_n'$ is a straightforward consequence of (5.20) and Theorem 5.49. However, our argument will be direct without referring to Theorem 5.49.

Proof. We already checked that $\nabla K(x)$ is homogeneous of degree $-n$. For $r>1$ let $A(1,r)=\{x:\,1\leq|x|\leq r\}$. From the integration by parts formula (5.17) we have

$$\begin{aligned}\int_{1\leq|x|\leq r}\nabla K(x)\,dx &= \int_{\partial A(1,r)}K(x)\,\vec{\nu}(x)\,d\sigma(x)\\ &= -\int_{|x|=1}K(x)\,\frac{x}{|x|}\,d\sigma(x)+\int_{|x|=r}K(x)\,\frac{x}{|x|}\,d\sigma(x)=0\,.\end{aligned}$$

Indeed, the last two integrals are equal by a simple change of variables and homogeneity of K. Thus the integral on the left hand side equals 0 independently of r. Hence its derivative with respect to r is also equal zero.

$$0=\frac{d}{dr}\Big|_{r=1^+}\int_{1\leq|x|\leq r}\nabla K(x)\,dx=\int_{|x|=1}\nabla K(\theta)\,d\sigma(\theta)\,.$$

This proves (5.20). As we will see (5.20) implies that p.v. $\nabla K(x)\in\mathscr{S}_n'$ is a well defined tempered distribution as well as that the distributional gradient ∇K satisfies (5.21). Let $\varphi\in\mathscr{S}_n$. We have

$$\begin{aligned}\nabla K[\varphi] &:= -\int_{\mathbb{R}^n}K(x)\nabla\varphi(x)\,dx\\ &= \lim_{\varepsilon\to 0}\lim_{R\to\infty}-\int_{\varepsilon\leq|x|\leq R}K(x)\nabla\varphi(x)\,dx\\ &= \lim_{\varepsilon\to 0}\lim_{R\to\infty}\left(\int_{\varepsilon\leq|x|\leq R}\nabla K(x)\,\varphi(x)\,dx-\int_{\partial A(\varepsilon,R)}K(x)\varphi(x)\,\vec{\nu}(x)\,d\sigma(x)\right)\\ &= \lim_{\varepsilon\to 0}\left(\int_{|x|\geq\varepsilon}\nabla K(x)\,\varphi(x)\,dx+\int_{|x|=\varepsilon}K(x)\varphi(x)\,\frac{x}{|x|}\,d\sigma(x)\right)\,.\end{aligned}$$

It remains to prove that

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon}K(x)\varphi(x)\,\frac{x}{|x|}\,d\sigma(x)=\varphi(0)\int_{|x|=1}K(x)\,\frac{x}{|x|}\,d\sigma(x)\,.$$

Let

$$c=\int_{|x|=1}K(x)\,\frac{x}{|x|}\,d\sigma(x)=\int_{|x|=\varepsilon}K(x)\,\frac{x}{|x|}\,d\sigma(x)\,.$$

The last equality follows from a simple change of variables and homogeneity of K. We have

$$\begin{aligned} \int_{|x|=\varepsilon} K(x)\varphi(x)\,\frac{x}{|x|}\,d\sigma(x) &= c\varphi(0) + \int_{|x|=\varepsilon} K(x)\big(\varphi(x)-\varphi(0)\big)\,\frac{x}{|x|}\,d\sigma(x) \\ &\to c\varphi(0) \end{aligned} \tag{5.22}$$

as $\varepsilon \to 0$. Indeed, for $|x| = \varepsilon$

$$\left| K(x)\big(\varphi(x)-\varphi(0)\big)\,\frac{x}{|x|}\right| \leq C\varepsilon^{1-n}\varepsilon = C\varepsilon^{2-n}\,.$$

Since the surface area of the sphere $\{|x| = \varepsilon\}$ is $n\omega_n\varepsilon^{n-1}$, the integral on the right hand side of (5.22) converges to 0 as $\varepsilon \to 0$. □

5.6. **Fundamental solution of the Laplace equation.** A straightforward application of Theorem 5.57 gives a well known formula for the *fundamental solution* to the Laplace equation.

Theorem 5.60. *For $n \geq 2$ we have*[53]

$$\Delta\Phi = \delta_0\,,$$

where

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } n = 2, \\ -\frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}} & \text{if } n \geq 3. \end{cases}$$

Proof. We will prove the theorem for $n \geq 3$, but a similar argument works for $n = 2$. According to Theorem 5.56[54]

$$\nabla\Phi(x) = \frac{1}{n\omega_n}\,\frac{x}{|x|^n} \quad \text{in } \mathscr{S}'_n. \tag{5.23}$$

Note that the function $\Phi(x)$ is harmonic in $\mathbb{R}^n \setminus \{0\}$ and hence

$$0 = \Delta\Phi(x) = \operatorname{div}\nabla\Phi(x) = \frac{1}{n\omega_n}\sum_{j=1}^n \frac{\partial}{\partial x_j}\,\frac{x_j}{|x|^n} \quad \text{for } x \neq 0.$$

Now Theorem 5.57 gives a formula for the distributional Laplacian

$$\Delta\Phi = \operatorname{div}\nabla\Phi = c\delta_0 + \frac{1}{n\omega_n}\,\text{p.v.}\underbrace{\sum_{j=1}^n \frac{\partial}{\partial x_j}\,\frac{x_j}{|x|^n}}_{0} = c\delta_0\,,$$

where

$$c = \sum_{j=1}^n \int_{|x|=1} \frac{1}{n\omega_n}\,\frac{x_j}{|x|^n}\,\frac{x_j}{|x|}\,d\sigma(x) = \frac{1}{n\omega_n}\int_{|x|=1} d\sigma(x) = 1\,.$$

The proof is complete. □

[53]Note that when $n \geq 3$, then $\Phi * f = -I_2 f$, where I_2 is the Riesz potential, see Definition 2.14.

[54]Formula (5.23) is also true for $n = 2$.

For $\varphi \in \mathscr{S}_n$ let $u(x) = (\Phi * \varphi)(x)$. Then[55] $u \in C^\infty(\mathbb{R}^n)$ and

$$\Delta u(x) = \Delta(\Phi * \varphi)(x) = \big((\Delta\Phi) * \varphi\big)(x) = (\delta_0 * \varphi)(x) = \varphi(x)\,.$$

Hence convolution with the fundamental solution of the Laplace operator provides an explicit solution of the *Poisson equation*

$$\Delta u = \varphi\,.$$

This explains the importance of the fundamental solution in partial differential equations.

Observe that the above calculation gives also

$$\begin{aligned}
\varphi(x) &= \Delta(\Phi * \varphi)(x) \\
&= \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\,(\Phi * \varphi)(x) \\
&= \sum_{j=1}^{n} \left(\frac{\partial \Phi}{\partial x_j} * \frac{\partial \varphi}{\partial x_j}\right)(x) \\
&= \int_{\mathbb{R}^n} \nabla\Phi(x-y)\cdot\nabla\varphi(y)\,dy \\
&= \frac{1}{n\omega_n}\int_{\mathbb{R}^n} \frac{(x-y)\cdot\nabla\varphi(y)}{|x-y|^n}\,dy
\end{aligned}$$

for every $x \in \mathbb{R}^n$. In the last equality we employed (5.23). Thus we proved

Theorem 5.61. *For $\varphi \in \mathscr{S}_n$, $n \geq 2$ we have*

$$\varphi(x) = \frac{1}{n\omega_n}\int_{\mathbb{R}^n} \frac{(x-y)\cdot\nabla\varphi(y)}{|x-y|^n}\,dy \quad \text{for all } x \in \mathbb{R}^n.$$

From this theorem we can conclude a similar result for higher order derivatives.

Theorem 5.62. *For $\varphi \in \mathscr{S}_n$, $n \geq 2$ and $m \geq 1$ we have*

$$\varphi(x) = \frac{m}{n\omega_n}\int_{\mathbb{R}^n} \sum_{|\alpha|=m} \frac{D^\alpha\varphi(y)}{\alpha!}\,\frac{(x-y)^\alpha}{|x-y|^n}\,dy \quad \text{for all } x \in \mathbb{R}^n.$$

Proof. Fix $x \in \mathbb{R}^n$ and define

$$\psi(y) = \sum_{|\beta|\leq m-1} D^\beta\varphi(y)\,\frac{(x-y)^\beta}{\beta!}\,.$$

Then $\psi(x) = \varphi(x)$ and[56]

$$\frac{\partial\psi}{\partial y_j}(y) = \sum_{|\beta|=m-1} D^{\beta+\delta_j}\varphi(y)\,\frac{(x-y)^\beta}{\beta!}$$

[55]As a convolution of $\Phi \in \mathscr{S}_n'$ with $\varphi \in \mathscr{S}_n$.

[56]We compute $\partial\psi/\partial y_j$ using the Leibniz rule and observe that the lower order terms cancel out.

where[57] $\delta_j = (0, \ldots, 1, \ldots, 0)$. Hence Theorem 5.60 applied to ψ yields

$$\begin{aligned}\varphi(x) &= \psi(x)\\ &= \frac{1}{n\omega_n}\int_{\mathbb{R}^n}\sum_{j=1}^n \frac{\partial\psi}{\partial y_j}(y)\,\frac{x_j-y_j}{|x-y|^n}\,dy\\ &= \frac{1}{n\omega_n}\sum_{j=1}^n\sum_{|\beta|=m-1}\int_{\mathbb{R}^n} D^{\beta+\delta_j}\varphi(y)\,\frac{(x-y)^\beta}{\beta!}\,\frac{x_j-y_j}{|x-y|^n}\,dy\\ &= \frac{m}{n\omega_n}\sum_{|\alpha|=m}\int_{\mathbb{R}^n}\frac{D^\alpha\varphi(y)}{\alpha!}\,\frac{(x-y)^\alpha}{|x-y|^n}\,dy\,,\end{aligned}$$

because for α with $|\alpha| = m$

$$\sum_{j,\beta:\,\beta+\delta_j=\alpha}\frac{1}{\beta!} = \frac{m}{\alpha!}\,.$$

The proof is complete. □

5.7. **Appendix.** We will present here details of the last step in the proof of Theorem 5.7. Namely we will show that if $u \in \mathscr{S}_n'$ and $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$, then

$$u\Big[\int_{\mathbb{R}^n}\varphi(y-\cdot)\psi(y)\,dy\Big] = \int_{\mathbb{R}^n} u[\varphi(y-\cdot)\psi(y)]\,dy.$$

To approximate the integral $\int_{\mathbb{R}^n}\eta(y)\,dy$, $\eta \in C_0^\infty$ by Riemann sums, we fix a cube $Q = [-N, N]^n$ so large that $\operatorname{supp}\eta \subset Q$ and divide the cube into cubes $\{Q_{ki}\}_{i=1}^{m_k}$ of sidelength 2^{-k}. Denote the centers of these cubes by y_{ki}. Clearly

$$\sum_{i=1}^{m_k}\eta(y_{ki})|Q_{ki}| \to \int_{\mathbb{R}^n}\eta(y)\,dy \quad \text{as } k\to\infty,$$

and

$$\sum_{i=1}^{m_k}\int_{Q_{ki}}|\eta(y)-\eta(y_{ki})|\,dy \to 0 \quad \text{as } k\to\infty. \tag{5.24}$$

Suppose $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$, so $\tilde\varphi * \psi \in C_0^\infty(\mathbb{R}^n)$. Hence for every $x \in \mathbb{R}^n$

$$w_k(x) = \sum_{i=1}^{m_k}\tilde\varphi(x-y_{ki})\psi(y_{ki})|Q_{ki}| \to \int_{\mathbb{R}^n}\tilde\varphi(x-y)\psi(y)\,dy \quad \text{as } k\to\infty.$$

The functions $w_k(x)$ belong to $\mathscr{S}_n$.[58] We will prove that

$$w_k(x) \to \int_{\mathbb{R}^n}\tilde\varphi(x-y)\psi(y)\,dy = (\tilde\varphi * \psi)(x) \quad \text{as } k\to\infty$$

in the topology of $\mathscr{S}_n$. First let us show that

$$\sup_{x\in\mathbb{R}^n}|w_k(x) - (\tilde\varphi*\psi)(x)| \to 0 \quad \text{as } k\to\infty.$$

[57] 1 on jth coordinate, 0 otherwise.

[58] As finite linear combinations of functions from $\mathscr{S}_n$.

We have

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}|w_k(x)-(\tilde{\varphi}*\psi)(x)| &\leq \sum_{i=1}^{m_k}\sup_{x\in\mathbb{R}^n}\int_{Q_{ki}}|\tilde{\varphi}(x-y_{ki})\psi(y_{ki})-\tilde{\varphi}(x-y)\psi(y)|\,dy\\ &\leq \|\tilde{\varphi}\|_\infty\sum_{i=1}^{m_k}\int_{Q_{ki}}|\psi(y_{ki})-\psi(y)|\,dy\\ &+ \sum_{i=1}^{m_k}\sup_{x\in\mathbb{R}^n}\int_{Q_{ki}}|\tilde{\varphi}(x-y_{ki})-\tilde{\varphi}(x-y)|\,|\psi(y)|\,dy\to 0.\end{aligned}$$

The convergence of the first sum follows from (5.24) and the convergence of the second one follows from uniform continuity of $\tilde{\varphi}$. Since

$$D^\beta w_k(x)=\sum_{i=1}^{m_k}D^\beta\tilde{\varphi}(x-y_{ki})\psi(y_{ki})|Q_{ki}|\,,$$

exactly the same argument as above shows that

$$\sup_{x\in\mathbb{R}^n}|D^\beta w_k(x)-\underbrace{(D^\beta\tilde{\varphi}*\psi)(x)}_{D^\beta(\tilde{\varphi}*\psi)(x)}|\to 0\quad\text{as } k\to\infty.$$

Therefore

$$p_{\alpha,\beta}(w_k-\tilde{\varphi}*\psi)\to 0\quad\text{as } k\to\infty,$$

because $|x|^\alpha$ is bounded as the functions have compact support.

This proves that $w_k\to\tilde{\varphi}*\psi$ in $\mathscr{S}_n$.

Since the function $f(y)=u[\tilde{\varphi}(\cdot-y)]$ is smooth, $u[\tilde{\varphi}(\cdot-y)]\psi(y)\in C_0^\infty$ and hence

$$\begin{aligned}u\Big[\int_{\mathbb{R}^n}\varphi(y-\cdot)\psi(y)\,dy\Big] &= u[\tilde{\varphi}*\psi]=\lim_{k\to\infty}u[w_k]=\lim_{k\to\infty}\sum_{i=1}^{m_k}u[\tilde{\varphi}(\cdot-y_{ki})]\psi(y_{ki})|Q_{ki}|\\ &= \int_{\mathbb{R}^n}u[\tilde{\varphi}(\cdot-y)]\psi(y)\,dy=\int_{\mathbb{R}^n}u[\varphi(y-\cdot)]\psi(y)\,dy.\end{aligned}$$

6. Convolution with principal value distributions

In Theorem 5.49 we proved that if $\Omega \in L^1(S^{n-1})$ satisfies

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0,$$

then

$$W_\Omega = \text{p.v.}\, K_\Omega, \quad \text{where} \quad K_\Omega(x) = \frac{\Omega(x/|x|)}{|x|^n}$$

defines a tempered distribution. In this chapter we will study convolution operators of the form

$$T_\Omega \varphi(x) = W_\Omega * \varphi(x) = \lim_{\varepsilon\to 0} \int_{|y|\geq\varepsilon} \frac{\Omega(y/|y|)}{|y|^n}\, \varphi(x-y)\, dy.$$

Since $T_\Omega\varphi = (\hat{\varphi}\widehat{W_\Omega})^\vee$ for $\varphi \in \mathscr{S}_n$, the main objective will be to compute the Fourier transform $\widehat{W_\Omega}$ and conclude properties of T_Ω from those of $\widehat{W_\Omega}$. We will focus on the setting of tempered distributions and L^2 spaces only.

Operators T_Ω are called *singular integrals* and the main objective is to study their boundedness in L^p spaces for $1 < p < \infty$. We will return to this problem in Chapters **??** and **??**.

First we will consider the *Riesz transforms* which are amongst the most important singular integrals, but have a very simple structure. Finally we will consider W_Ω in its general form.

6.1. **Riesz transforms: L^2 theory.** In Proposition 5.53 we proved that for $n \geq 2$

$$\frac{\partial}{\partial x_j}|x|^{1-n} = (1-n)\,\text{p.v.}\left(\frac{x_j}{|x|^{n+1}}\right) \in \mathscr{S}'_n.$$

In this section we will investigate the *Riesz transforms* which are convolutions with the distributions

$$W_j = c_n\,\text{p.v.}\left(\frac{x_j}{|x|^{n+1}}\right), \quad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}, \quad n \geq 2.$$

The choice of the coefficient c_n will be justified in Theorem 6.2; with this choice the Fourier transform of W_j has a particularly simple form.

Note that when $n = 1$, W_j becomes[59]

$$W = \frac{1}{\pi}\,\text{p.v.}\,\frac{x}{|x|^2} = \frac{1}{\pi}\,\text{p.v.}\,\frac{1}{x}$$

[59] cf. Remark 5.55.

and convolution with W is known as the *Hilbert transform*. Thus the Riesz transforms are higher dimensional generalizations of the Hilbert transform. The Hilbert transform will be carefully studied in Section 10. The reader may choose to read Section 10.1 now. This section contains elementary properties of the Hilbert transform and it fits well into the content of Chapter 6.

Definition 6.1. For $1 \leq j \leq n$ the *Riesz transform* R_j of a function $\varphi \in \mathscr{S}_n$ is defined by

$$\begin{aligned}(R_j\varphi)(x) &= (W_j * \varphi)(x)\\ &= c_n\, \text{p.v.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}}\, \varphi(x-y)\, dy\\ &= c_n \lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} \frac{x_j-y_j}{|x-y|^{n+1}}\, \varphi(y)\, dy\,.\end{aligned}$$

Theorem 6.2.

$$\widehat{W}_j(\xi) = -i\,\frac{\xi_j}{|\xi|}\,.$$

Proof. Applying Proposition 5.53 and then Theorem 5.42 we have

$$\begin{aligned}\left(\text{p.v.}\,\frac{x_j}{|x|^{n+1}}\right)^{\wedge} &= \frac{1}{1-n}\left(\frac{\partial}{\partial x_j}\,|x|^{1-n}\right)^{\wedge}\\ &= \frac{2\pi i\xi_j}{1-n}\left(|x|^{1-n}\right)^{\wedge}\\ &= \frac{2\pi i\xi_j}{1-n}\,\pi^{n-1-\frac{n}{2}}\,\frac{\Gamma\left(\frac{n-(n-1)}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\,|\xi|^{-1}\\ &= \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\,\frac{-i\xi_j}{|\xi|}\,.\end{aligned}$$

We used here equalities

$$\Gamma(1/2) = \pi^{1/2} \quad \text{and} \quad \frac{n-1}{2}\Gamma\left(\frac{n-1}{2}\right) = \Gamma\left(\frac{n+1}{2}\right)\,.$$

The proof is complete. □

Corollary 6.3. *For $\varphi \in \mathscr{S}_n$ and $1 \leq j \leq n$*

$$(R_j\varphi)(x) = \left(-\frac{i\xi_j}{|\xi|}\,\hat{\varphi}(\xi)\right)^{\vee}(x)\,. \tag{6.1}$$

$R_j\varphi \in C^\infty$ is slowly increasing and all its derivatives are slowly increasing. Moreover

$$\|R_j\varphi\|_2 \leq \|\varphi\|_2 \quad \textit{for } \varphi \in \mathscr{S}_n.$$

Remark 6.4. One has to be careful here. Although $\varphi \in \mathscr{S}_n$, in general $-i\frac{\xi_j}{|\xi|}\hat{\varphi} \notin \mathscr{S}_n$ so the inverse Fourier transform in (6.1) has to be understood in the distributional or in the L^2 sense.

Proof. Equality (6.1) follows from

$$(R_j\varphi)^\wedge = (W_j * \varphi)^\wedge = \hat{\varphi}\hat{W}_j = -i\,\frac{\xi_j}{|\xi|}\,\hat{\varphi}.$$

Since $R_j\varphi$ is a convolution with a tempered distribution, it is smooth, slowly increasing and all its derivatives are slowly increasing, see Theorem 5.7. Finally the L^2 estimate follows from the fact that the Fourier transform is an isometry on L^2 and the multiplication by the function $-i\xi_j/|\xi|$ is bounded in L^2 with norm 1 (since the supremum of the function is 1).[60] □

The above result allows us to extend the Riesz transforms to bounded operators in L^2.

Definition 6.5. The *Riesz transforms* are bounded operators on $L^2(\mathbb{R}^n)$, $n \geq 2$,

$$\|R_j f\|_2 \leq \|f\|_2 \quad \text{for } f \in L^2(\mathbb{R}^n) \text{ and } j = 1, 2, \ldots, n,$$

defined by the formula

$$(R_j f)(x) = \left(-\frac{i\xi_j}{|\xi|}\,\hat{f}(\xi)\right)^\vee (x) \quad \text{for } f \in L^2(\mathbb{R}^n).$$

Remark 6.6. The Riesz transform on L^2 was defined as an extension of a bounded operator R_j defined on a subspace $\mathscr{S}_n \subset L^2$ by the formula $R_j\varphi = c_n(\text{p.v.}\, x_j/|x|^{n+1}) * \varphi$. An interesting question is whether the L^2 extension can also be defined by the same p.v. integral formula and if yes, whether the limit in the definition of the p.v. integral exists in the L^2 sense or in the a.e. sense. These highly non-trivial questions will be answered in Section **??**.

Corollary 6.7. *The Riesz transforms satisfy*

$$\sum_{j=1}^n R_j^2 = -I \quad \textit{on } L^2(\mathbb{R}^n).$$

Proof. For $f \in L^2$ we have

$$\Big(\sum_{j=1}^n R_j^2 f\Big)^\wedge(\xi) = \sum_{j=1}^n \left(-\frac{i\xi_j}{|\xi|}\right)^2 \hat{f}(\xi) = -\hat{f}(\xi)$$

and the result follows upon taking the inverse Fourier transform. □

The next general and elementary result applies to Riesz transforms.

Theorem 6.8. *If $m \in L^\infty(\mathbb{R}^n)$, then the operator*

$$T_m f = (m\hat{f})^\vee, \quad f \in L^2$$

is bounded in L^2,

$$\|T_m f\|_2 \leq \|m\|_\infty \|f\|_2.$$

Moreover, there is a tempered distribution $u \in \mathscr{S}_n'$ such that

$$T_m\varphi = u * \varphi \quad \textit{for all } \varphi \in \mathscr{S}_n.$$

[60]cf. Theorem 6.8 and estimates (6.3).

Hence $T_m\varphi$ is smooth, slowly increasing, all its derivatives are slowly increasing and

$$D^\alpha(T_m\varphi) = T_m(D^\alpha\varphi) \quad \text{for all multiindices } \alpha. \tag{6.2}$$

If $m_1, m_2 \in L^\infty$, then $T_{m_1}\circ T_{m_2} = T_{m_2}\circ T_{m_1} = T_{m_1 m_2}$.

Proof. Boundedness of T_m on L^2 is an obvious consequence of the Plancherel theorem

$$\|T_m f\|_2 = \|\widehat{T_m f}\|_2 = \|m\hat{f}\|_2 \leq \|m\|_\infty \|\hat{f}\|_2 = \|m\|_\infty \|f\|_2. \tag{6.3}$$

To prove existence of $u \in \mathscr{S}_n'$, observe that $m \in L^\infty$ is a tempered distribution and $u = \check{m} \in \mathscr{S}_n'$ satisfies $\hat{u} = m$. Thus $(u*\varphi)^\wedge = \hat{\varphi}m = (T_m\varphi)^\wedge$ so $T_m\varphi = u*\varphi$.

Smoothness of $T_m\varphi = u*\varphi$ along with equality (6.2) follow from Theorems 5.7 and 5.8.

Finally, the composition formula $T_{m_1}\circ T_{m_2} = T_{m_2}\circ T_{m_1} = T_{m_1 m_2}$ follows directly from the definition of T_m. □

An amazing property of the Riesz transforms is that they allow us to compute mixed partial derivatives $\partial_j\partial_k u$ if we only know Δu. More precisely we have

Proposition 6.9. *If $\varphi \in \mathscr{S}_n$, then for $1 \leq j,k \leq n$ we have*

$$\frac{\partial^2\varphi}{\partial x_j \partial x_k} = -R_j R_k \Delta\varphi(x) = -\Delta(R_j R_k \varphi)(x), .$$

Proof. For $\varphi \in \mathscr{S}_n$ we have

$$\begin{aligned}
(\partial_j\partial_k\varphi)^\wedge(\xi) &= (2\pi i\xi_j)(2\pi i\xi_k)\hat{\varphi}(\xi) \\
&= -\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)(-4\pi^2|\xi|^2)\hat{\varphi}(\xi) \\
&= -\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)\widehat{\Delta\varphi}(\xi) \\
&= (-R_j R_k \Delta\varphi)^\wedge(\xi).
\end{aligned}$$

Thus taking the inverse Fourier transform in L^2 yields

$$\frac{\partial^2\varphi}{\partial x_j \partial x_k} = -R_j R_k \Delta\varphi(x).$$

The composition of the Riesz transforms $R_j R_k$ is understood as composition of bounded operators in L^2 and according to Theorem 6.8 there is $u \in \mathscr{S}_n'$ such that

$$R_j R_k \varphi = \left(\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)\hat{\varphi}\right)^\vee = u*\varphi.$$

so $R_j R_k\varphi \in C^\infty$ is slowly increasing and all its derivatives are slowly increasing. Also

$$\frac{\partial^2\varphi}{\partial x_j \partial x_k} = -R_j R_k\Delta\varphi(x) = -u*\Delta\varphi = -\Delta(u*\varphi) = -\Delta(R_j R_k\varphi).$$

The proof is complete. □

Remark 6.10. One can prove that if $j \neq k$, then R_jR_k can be written as a convolution with a p.v. distribution. However, R_j^2 cannot be written as a convolution with a p.v. distribution. Fortunately, it is still a convolution with a tempered distribution, but not of a p.v. type.[61]

As a direct consequence of Proposition 6.9 and of boundedness of the Riesz transforms in L^2 we obtain

Corollary 6.11.

$$\left\| \frac{\partial \varphi}{\partial x_j \partial x_k} \right\|_2 \leq \|\Delta\varphi\|_2 \quad \textit{for } \varphi \in \mathscr{S}_n.$$

Remark 6.12. The same estimate can be proved directly using integration by parts. Later we will see that a similar estimate is also true for the L^p norm, $1 < p < \infty$. But this is a much harder result and a simple integration by parts is not enough.

Remark 6.13. It is quite convincing that the proof of Proposition 6.9 applied to $u \in \mathscr{S}_n'$ such that $\Delta u = f \in L^2$, gives $\partial_j\partial_k u = -R_jR_kf$. However this is not true. For example if $u = xy$, then, as a slowly increasing function, $u \in \mathscr{S}_2'$. Clearly $\Delta u = 0 \in L^2$, but $\partial_x\partial_y u = 1 \neq 0 = -R_xR_y0$. The problem is of a very delicate nature and it is important to understand it well. Remember that $\hat{u}$ is a tempered distribution, not a function so in the equality

$$(2\pi i\xi_j)(2\pi i\xi_k)\hat{u}(\xi) = -\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)[(-4\pi^2|\xi|^2)\hat{u}(\xi)]$$

the left hand side is well defined since we are allowed to multiply distributions by $(2\pi i\xi_j)(2\pi i\xi_k)$. However, the right hand side is not well defined: $(-4\pi^2|\xi|^2)\hat{u}(\xi)$ is a distribution and we are not allowed to multiply this distribution by a non-smooth function. It is actually still confusing. After all, $(-4\pi^2|\xi|^2)\hat{u} = \widehat{\Delta u} = \hat{f} \in L^2$ so why are we not allowed to multiply it by a bounded function? We are, but if we do, the above equality is no longer true, because we switch from the distributional context to the L^2 context. In fact, the distribution $\hat{u}$ may have a non-trivial part that is supported at 0 such that the multiplication by $(2\pi i\xi_j)(2\pi i\xi_k)$ does not 'kill' this part. However, multiplication by $-4\pi^2|\xi|^2$ does 'kill' it so it may happen that

$$(2\pi i\xi_j)(2\pi i\xi_k)\hat{u}(\xi) = -\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)\hat{f} + v$$

where v is a distribution supported at 0. This will be carefully explained in the next result.

Theorem 6.14. *If $u \in \mathscr{S}_n'$ satisfies*

$$\Delta u = f \in L^2(\mathbb{R}^n)\,, \tag{6.4}$$

then for any $1 \leq j,k \leq n$

$$\frac{\partial^2 u}{\partial x_j \partial x_k} = -R_jR_kf + P_{jk}\,,$$

where P_{jk} is a harmonic polynomial.

[61] c.f. Problems **??** and **??** and Example 6.41.

Proof. We will only prove that P_{jk} are polynomials and we leave the proof that they are harmonic to the reader. According to Corollary 5.34 it suffices to prove that the tempered distribution

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k} + R_j R_k f\right)^{\wedge}$$

has support contained in $\{0\}$. To this end we need to show that if $\varphi \in \mathscr{S}_n$, $\operatorname{supp}\varphi \subset \mathbb{R}^n \setminus \{0\}$, say $\varphi(x) = 0$ for $|x| \le r$, then

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k} + R_j R_k f\right)^{\wedge}[\varphi] = 0, \quad \text{i.e.,} \quad (\partial_j\partial_k u)^\wedge[\varphi] = (-R_jR_kf)^\wedge[\varphi]\,.$$

Let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 0$ for $|x| \le r/2$ and $\eta(x) = 1$ for $|x| \ge r$. The function η and its derivatives are slowly increasing and $\eta\varphi = \varphi$. Hence

$$\begin{aligned}(\partial_j\partial_k u)^\wedge[\varphi] &= \big((2\pi i\xi_i)(2\pi i \xi_j)\hat{u}\big)[\eta\varphi] \\ &= \big(\eta(\xi)(2\pi i\xi_i)(2\pi i \xi_j)\hat{u}\big)[\varphi] = \heartsuit.\end{aligned}$$

Observe that

$$\eta(\xi)(2\pi i\xi_i)(2\pi i\xi_j) = -\underbrace{\eta(\xi)\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)}_{\lambda(\xi)}(-4\pi^2|\xi|^2)$$

and $\lambda \in C^\infty$ and its derivatives are slowly increasing since η vanishes in a neighborhood of $\xi = 0$. Thus[62]

$$\begin{aligned}\heartsuit &= \big(-\lambda(\xi)(-4\pi^2|\xi|^2)\hat{u}\big)[\varphi] \\ &= (-\lambda(\xi)\hat{f}(\xi))[\varphi] \\ &= \left(-\eta(\xi)\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)\hat{f}(\xi)\right)[\varphi] \\ &= \left(-\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)\hat{f}(\xi)\right)[\eta\varphi] \\ &= \left(-\left(\frac{-i\xi_j}{|\xi|}\right)\left(\frac{-i\xi_k}{|\xi|}\right)\hat{f}(\xi)\right)[\varphi] \\ &= (-R_jR_kf)^\wedge[\varphi]\,.\end{aligned} \tag{6.5}$$

The proof is complete. □

6.2. **The Beurling-Ahlfors transform.** We proved that

$$\frac{\partial^2\varphi}{\partial x_j\partial x_k} = -R_jR_k\Delta\varphi = -\Delta(R_jR_k\varphi) \quad \text{for } \varphi \in \mathscr{S}_n$$

[62]Taking the Fourier transform of (6.4) yields $-4\pi^2|\xi|^2\hat{u} = \hat{f}$ in $\mathscr{S}_n'$. We should understand this equality in $\mathscr{S}_n'$ and not in L^2, because despite the fact that $-4\pi^2|\xi|^2\hat{u} = \hat{f} \in L^2$, it may happen that the distribution $\hat{u}$ is not a function. If $\lambda \in C^\infty(\mathbb{R}^n)$ and all its derivatives are slowly increasing, then $-\lambda(\xi)(-4\pi^2|\xi|^2)\hat{u} = -\lambda(\xi)\hat{f}$ in $\mathscr{S}_n'$. Now $-\lambda\hat{f} \in L^2$ and $\hat{f} \in L^2$ are functions. *Functions* that are equal a.e. define the same distribution. Hence equality (6.5) is true because the functions that define distributions on the left hand side and on the right hand side of (6.5) are equal for all $\xi \neq 0$ so they are equal a.e.

so the Riesz transforms, in some sense, can change the direction of partial derivatives. Now we will find an operator S such that

$$S\left(\frac{\partial\varphi}{\partial\bar z}\right)=\frac{\partial\varphi}{\partial z}\quad\text{for }\varphi\in\mathscr{S}(\mathbb{R}^2)=\mathscr{S}(\mathbb{C}).$$

We will be using complex notation for points in $\mathbb{R}^2=\mathbb{C}$

$$z=x+iy\quad\text{and}\quad\xi=\xi_1+i\xi_2,$$

and also for the Lebesgue measure

$$dz=dz+idy,\quad d\bar z=dx-idy\quad\text{so}\quad\frac{d\bar z\wedge dz}{2i}=dx\wedge dy=dx\,dy.$$

Definition 6.15. The operator

$$S=(iR_1+R_2)^2\quad\text{defined on }L^2(\mathbb{C}).$$

is called the *Beurling-Ahlfors transform.*

The next result collects basic properties of the Beurling-Ahlfors transform.

Theorem 6.16. *The Beurling-Ahlfors transform has the following properties*

(1) $S:L^2(\mathbb{C})\to L^2(\mathbb{C})$ *is a surjective isometry,*

$$\|Sf\|_2=\|f\|_2\quad\textit{for }f\in L^2(\mathbb{C}).$$

(2) *The inverse operator is*

$$S^{-1}f=\overline{S\bar f}.$$

(3) *For* $f\in L^2(\mathbb{C})$ *we have*

$$Sf(\xi)=\left(\frac{\bar\xi}{\xi}\hat f(\xi)\right)^{\vee},\qquad S^{-1}f(\xi)=\left(\frac{\xi}{\bar\xi}\hat f(\xi)\right)^{\vee}.$$

(4)

$$S\left(\frac{\partial\varphi}{\partial\bar z}\right)=\frac{\partial}{\partial\bar z}(S\varphi)=\frac{\partial\varphi}{\partial z}\quad\textit{for }\varphi\in\mathscr{S}(\mathbb{R}^2)=\mathscr{S}(\mathbb{C}).$$

Proof. Since

$$\frac{\bar\xi}{\xi}=\left(\frac{\bar\xi}{|\xi|}\right)^2=\left(\frac{\xi_1-i\xi_2}{|\xi|}\right)^2=\left(i\left(-\frac{i\xi_1}{|\xi|}\right)+\left(-\frac{i\xi_2}{|\xi|}\right)\right)^2$$

we conclude that

$$Sf=(iR_1+R_2)^2f=\left(\frac{\bar\xi}{\xi}\,\hat f\right)^{\vee}\quad\text{for }f\in L^2(\mathbb{C}).$$

Clearly

$$\frac{\xi}{\bar\xi}\cdot\frac{\bar\xi}{\xi}=1\quad\text{so}\quad S^{-1}f=\left(\frac{\xi}{\bar\xi}\,\hat f\right)^{\vee}$$

which proves (3). The fact that S is an isometry easily follows from the Plancherel theorem

$$\|Sf\|_2=\|\widehat{Sf}\|_2=\left\|\frac{\bar\xi}{\xi}\hat f\right\|_2=\|\hat f\|_2=\|f\|_2.$$

Since S has the inverse operator, it maps $L^2(\mathbb{C})$ onto $L^2(\mathbb{C})$. This establishes (1). A simple computation based on the fact that

$$\mathcal{F}(\bar{f}(\xi)) = \overline{\mathcal{F}(f)}(-\xi) \quad \text{and} \quad \mathcal{F}^{-1}(\bar{f}(\xi)) = \overline{\mathcal{F}^{-1}(f)}(-\xi)$$

yields

$$\overline{S\bar{f}} = \left(\frac{\xi}{\bar{\xi}}\hat{f}\right)^{\vee} = S^{-1}f$$

which is (2). Equality

$$S\left(\frac{\partial\varphi}{\partial\bar{z}}\right) = \frac{\partial}{\partial\bar{z}}(S\varphi) \quad \text{for } \varphi \in \mathscr{S}(\mathbb{R}^2) = \mathscr{S}(\mathbb{C})$$

follows from Theorem 6.8, see (6.2). Recall that

$$\frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)$$

so

$$\left(\frac{\partial\varphi}{\partial\bar{z}}\right)^{\wedge} = \frac{1}{2}(2\pi i\xi_1 + i(2\pi i\xi_2))\hat{\varphi} = \pi i\xi\hat{\varphi}$$

and

$$\left(\frac{\partial\varphi}{\partial z}\right)^{\wedge} = \frac{1}{2}(2\pi i\xi_1 - i(2\pi i\xi_2))\hat{\varphi} = \pi i\bar{\xi}\hat{\varphi}.$$

Therefore

$$\left(S\left(\frac{\partial\varphi}{\partial\bar{z}}\right)\right)^{\wedge} = \frac{\bar{\xi}}{\xi}\,\pi i\xi\hat{\varphi} = \pi i\bar{\xi}\hat{\varphi} = \left(\frac{\partial\varphi}{\partial z}\right)^{\wedge}.$$

This completes the proof of (4). □

Corollary 6.17. *For $\varphi \in \mathscr{S}(\mathbb{C})$ we have*

$$\left\|\frac{\partial\varphi}{\partial z}\right\|_2 = \left\|\frac{\partial\varphi}{\partial\bar{z}}\right\|_2.$$

Proof. This is an immediate consequence of parts (4) and (1) of Theorem 6.16. □

Similarly as in the case of the Riesz transforms, the Beurling-Ahlfors transform can be represented as a convolution with a principal value distribution. In order to find a formula we need to investigate a connection between the Beurling-Ahlfors transform and the fundamental solution to the Laplace equation. Recall that[63]

$$\Phi(z) = \frac{1}{2\pi}\log|z| = \frac{1}{4\pi}\log|z|^2$$

satisfies

$$\Delta\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi = 4\frac{\partial^2}{\partial\bar{z}\partial z}\,\Phi = \delta_0.$$

Hence

$$\text{(6.6)} \qquad \frac{1}{\pi}\left(\frac{\partial}{\partial z}(4\pi\Phi) * \frac{\partial\varphi}{\partial\bar{z}}\right) = \frac{1}{\pi}\frac{\partial}{\partial\bar{z}}\left(\frac{\partial}{\partial z}(4\pi\Phi) * \varphi\right) = \frac{1}{\pi}\left(\frac{\partial^2}{\partial\bar{z}\partial z}(4\pi\Phi) * \varphi\right) = \varphi.$$

[63]Theorem 5.60.

An argument similar to that used in the proof of Theorem 5.56 allows us to differentiate Φ once in the pointwise sense without necessity of taking the principal value distribution so[64]

$$\frac{\partial}{\partial z}(4\pi\Phi(z)) = \frac{\partial}{\partial z}\log|z|^2 = \frac{1}{|z|^2}\frac{\partial}{\partial z}(z\bar{z}) = \frac{\bar{z}}{|z|^2} = \frac{1}{z} \quad \text{for } z \neq 0 \tag{6.7}$$

and hence

$$\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\pi z}\right) = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}(4\Phi)\right) = \Delta\Phi = \delta_0.$$

We proved

Proposition 6.18. *The function $G(z) = \frac{1}{\pi z}$ is a fundamental solution of the operator $\partial/\partial\bar{z}$, i.e.*

$$\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\pi z}\right) = \delta_0 \quad \textit{in } \mathscr{S}(\mathbb{C}).$$

Also (6.7) allows us to rewrite (6.6) as[65]

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\varphi_{\bar{z}}(\xi)}{z-\xi}\, d\bar{\xi}\wedge d\xi = \frac{1}{2\pi i}\frac{\partial}{\partial \bar{z}}\int_{\mathbb{C}} \frac{\varphi(\xi)}{z-\xi}\, d\bar{\xi}\wedge \xi = \varphi(z). \tag{6.8}$$

This is a representation formula similar to that in Theorem 5.61. It allows us to reconstruct the function φ from its derivative $\varphi_{\bar{z}}$. In paricular it follows immediately that the only holomorphic function in $\mathscr{S}(\mathbb{C})$ is the zero function.

Definition 6.19. The *Cauchy transform* of a function f is defined by

$$\mathcal{C}f(z) = (G * f)(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(\xi)}{z-\xi}\, d\bar{\xi}\wedge \xi.$$

Thus (6.8) reads as

Corollary 6.20.

$$\frac{\partial}{\partial \bar{z}}(\mathcal{C}\varphi) = \mathcal{C}\left(\frac{\partial \varphi}{\partial \bar{z}}\right) = \varphi \quad \textit{for } \varphi \in \mathscr{S}(\mathbb{C}).$$

The next result allows us to identify the Beurling-Ahlfors transform with the convolution with a principal value distribution. A different proof will be presented in Section 6.3, see Example 6.27.

Theorem 6.21. *For $\varphi \in \mathscr{S}(\mathbb{C})$ we have*

$$S(\varphi) = \frac{\partial}{\partial z}(\mathcal{C}\varphi) = \mathcal{C}\left(\frac{\partial \varphi}{\partial z}\right) = -\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{|\xi-z|\geq\varepsilon} \frac{\varphi(\xi)}{(z-\xi)^2}\, d\bar{\xi}\wedge d\xi = -\frac{1}{\pi}\left(\text{p.v.}\,\frac{1}{z^2}\right) * \varphi. \tag{6.9}$$

Proof. First we will prove that for $\varphi \in \mathscr{S}(\mathbb{C})$ we have

$$\frac{\partial}{\partial z}(\mathcal{C}\varphi) = \mathcal{C}\left(\frac{\partial \varphi}{\partial z}\right) = -\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{|\xi-z|\geq\varepsilon} \frac{\varphi(\xi)}{(z-\xi)^2}\, d\bar{\xi}\wedge d\xi = -\frac{1}{\pi}\left(\text{p.v.}\,\frac{1}{z^2}\right) * \varphi. \tag{6.10}$$

[64] If f is holomorphic and g is smooth in the real sense, then $\frac{\partial}{\partial z}(f\circ g)(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z)$.

[65] Recall that $d\xi_1\, d\xi_2 = \frac{1}{2i}\, d\bar{\xi}\wedge d\xi$.

We could actually conclude this equality from Theorem 5.57 but we prefer to provide a direct proof. Note that

$$\mathcal{C}\left(\frac{\partial\varphi}{\partial z}\right) = \frac{\partial}{\partial z}(\mathcal{C}\varphi), \quad \text{for } \varphi\in\mathscr{S}(\mathbb{C}) \tag{6.11}$$

is a consequence of a general rule how we differentiate convolution of a tempered distribution with $\varphi\in\mathscr{S}(\mathbb{C})$.

It is easy to check that in the complex notation Green's theorem can be rewritten as follows

Lemma 6.22 (Green's theorem)**.** *If $\Omega\subset\mathbb{C}$ is a bounded domain with the piecewise C^1 boundary and $f,g\in C^1(\overline{\Omega})$, then*

$$\int_\Omega\left(\frac{\partial f}{\partial z}+\frac{\partial g}{\partial\bar z}\right)d\bar z\wedge dz = \int_{\partial\Omega} g\,dz - f\,d\bar z.$$

Here

$$\int_{\partial\Omega} f\,d\bar z = \int_{\partial\Omega} f\,(dx-idy) = \overline{\int_{\partial\Omega}\bar f\,dz}.$$

In particular

$$\int_\Omega \frac{\partial}{\partial z}(fg)\,d\bar z\wedge dz = -\int_{\partial\Omega} fg\,d\bar z$$

i.e. we have the following version of the integration by parts formula

$$\int_\Omega\frac{\partial f}{\partial z}g\,d\bar z\wedge dz = -\int_\Omega f\frac{\partial g}{\partial z}\,d\bar z\wedge dz - \int_{\partial\Omega} fg\,d\bar z. \tag{6.12}$$

Applying (6.12) to our situation yields[66]

$$\begin{aligned}
2\pi i\,\mathcal{C}\left(\frac{\partial\varphi}{\partial z}\right) &= \int_{\mathbb{C}}\frac{\partial\varphi(\xi)/\partial\xi}{z-\xi}\,d\bar\xi\wedge d\xi\\
&= \lim_{\varepsilon\to 0}\int_{|\xi-z|\geq\varepsilon}\frac{\partial\varphi(\xi)/\partial\xi}{z-\xi}\,d\bar\xi\wedge d\xi\\
&= \lim_{\varepsilon\to 0}-\int_{|\xi-z|\geq\varepsilon}\varphi(\xi)\frac{\partial}{\partial\xi}\left(\frac{1}{z-\xi}\right)d\bar\xi\wedge d\xi + \lim_{\varepsilon\to 0}\int_{|\xi-z|=\varepsilon}\frac{\varphi(\xi)}{z-\xi}\,d\bar\xi = \heartsuit.
\end{aligned}$$

The sign '+' in the last limit is opposite to the sign '-' in the boundary integral in (6.12). This is because the positive orientation of the boundary $|\xi-z|=\varepsilon$ of the exterior domain $|\xi-z|\geq\varepsilon$ is clockwise.

Note that

$$\int_{|\xi-z|=\varepsilon}\frac{d\bar\xi}{z-\xi} = -\int_{|\xi|=\varepsilon}\frac{d\bar\xi}{\xi} = -\overline{\int_{|\xi|=\varepsilon}\frac{d\xi}{\bar\xi}} = -\varepsilon^{-2}\overline{\int_{|\xi|=\varepsilon}\xi\,d\xi} = 0$$

and hence

$$\lim_{\varepsilon\to 0}\int_{|\xi-z|=\varepsilon}\frac{\varphi(\xi)}{z-\xi}\,d\bar\xi = \lim_{\varepsilon\to 0}\int_{|\xi-z|=\varepsilon}\frac{\varphi(\xi)-\varphi(z)}{z-\xi}\,d\bar\xi = 0.$$

[66]In the integration by parts we actually should consider the domain $\varepsilon\leq|\xi-z|\leq R$ and let $\varepsilon\to 0$ and $R\to\infty$ since (6.12) applies to bounded domains. We simply skipped one step and passed to the limit as $R\to\infty$, 'in mind', cf. the proof of Proposition 5.53.

Since

$$\frac{\partial}{\partial \xi}\left(\frac{1}{z-\xi}\right) = \frac{1}{(z-\xi)^2}$$

we obtain

$$\heartsuit = -\lim_{\varepsilon\to 0}\int_{|\xi-z|\geq\varepsilon} \frac{\varphi(\xi)}{(z-\xi)^2}\, d\bar{\xi}\wedge d\xi.$$

This completes the proof of (6.10).

Note that (6.10) was a quite straightforward consequence of Green's formula. It remains to prove that $S(\varphi) = \frac{\partial}{\partial z}(\mathcal{C}\varphi)$ and this is not obvious.[67] Let us start with an interesting remark.

Remark 6.23. The space

$$\mathscr{S}_{\bar{z}} = \{\varphi_{\bar{z}} : \varphi \in \mathscr{S}(\mathbb{C})\}$$

is dense in $L^2(\mathbb{C})$. For if not, we would find $0 \neq f \in L^2$ which is orthogonal to $\mathscr{S}_{\bar{z}}$, i.e.,

$$\int_{\mathbb{C}} \varphi_{\bar{z}}\bar{f}\, d\bar{z}\wedge dz = 0 \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{C}).$$

That means $\bar{f}_{\bar{z}} = 0$ in $\mathscr{S}'(\mathbb{C})$ and hence $\Delta\bar{f} = 4\bar{f}_{\bar{z}z} = 0$ in $\mathscr{S}'(\mathbb{C})$. According to Corollary 5.35, $\bar{f}$ is a polynomial. Since $\bar{f} \in L^2$, we conclude that $f = 0$ which is a contradiction. Observe also that

$$S(\varphi_{\bar{z}}) = \varphi_z \quad \text{and} \quad \frac{\partial}{\partial z}(\mathcal{C}\varphi_{\bar{z}}) = \varphi_z.$$

This shows that equality (6.9) holds on the subspace $\mathscr{S}_{\bar{z}} \subset \mathscr{S}(\mathbb{C})$ which is dense in L^2. Since the operator S is an isometry on L^2, we conclude that $\varphi \mapsto \frac{\partial}{\partial z}\mathcal{C}(\varphi)$ is an isometry on a dense subset $\mathscr{S}_{\bar{z}}$ of L^2. We could conclude the theorem if we could prove that $\varphi \mapsto \frac{\partial}{\partial z}\mathcal{C}(\varphi)$ is bounded in the L^2 norm on $\mathscr{S}(\mathbb{C})$. After all, boundedness of this operator follows from Theorem 6.21, but we cannot use it before we prove Theorem 6.21. Vicious circle! In our proof given below we will follow the idea described in this remark by constructing a suitable L^2 approximation of $\varphi \in \mathscr{S}(\mathbb{C})$ by functions of the class $\mathscr{S}_{\bar{z}}(\mathbb{C})$.

Let's return to the proof. Let $\varphi \in \mathscr{S}(\mathbb{C})$ and let $\psi = \mathcal{C}\varphi$. As we know[68] $\psi_{\bar{z}} = \varphi$. However, in general $\psi \notin \mathscr{S}(\mathbb{C})$.[69] Since the Cauchy transform is bounded by the Riesz potential I_1 and $\varphi \in L^p$ for all p, the Fractional Integration Theorem 2.15 implies that $\psi \in L^p$ for all $2 < p < \infty$, but in general $\psi \notin L^2$.[70]

[67]The function $\bar{\xi}/\xi \in L^\infty$ defines a tempered distribution. If $u = (\bar{\xi}/\xi)^\vee \in \mathscr{S}'(\mathbb{C})$, then $S(\varphi) = u * \varphi$ (c.f. Theorem 6.8) and it remains to prove that u is the p.v. distribution given by (6.10). However, finding the Fourier transform of $\bar{\xi}/\xi$ is not easy, see Example 6.27. On the other hand since $S = (iR_1 + R_2)^2$ is a composition of convolutions with p.v. distributions, S should be easy to represent as a convolution with a p.v. distribution. However, finding a formula for the composition is not easy either. For example R_j^2 cannot be represented as a composition with a p.v. distribution (see Problem **??** and Example 6.41.) We will discuss this topic in detail in Section 6.6.

[68]Corollary 6.20.

[69]cf. Problem **??**

[70]cf. Problem **??**

Let $\eta \in C^\infty(\mathbb{C})$, $\eta(z) = 1$ for $|z| \leq 1$ and $\eta(z) = 0$ for $|z| \geq 2$ be a standard cut-off function. Let $\psi^R(z) = \psi(z)\eta(z/R)$. Since the function has compact support we have $S(\psi^R_{\bar{z}}) = \psi^R_z$. We claim that

$$\psi^R_{\bar{z}} \to \varphi \quad \text{in } L^2 \text{ as } R \to \infty. \tag{6.13}$$

The chain rule gives

$$\psi^R_{\bar{z}} = \psi_{\bar{z}}\eta\left(\frac{z}{R}\right) + \psi(z)\eta_{\bar{z}}\left(\frac{z}{R}\right)\frac{1}{R} = \varphi(z)\eta\left(\frac{z}{R}\right) + \psi(z)\eta_{\bar{z}}\left(\frac{z}{R}\right)\frac{1}{R}\,.$$

Clearly

$$\varphi(z)\eta\left(\frac{z}{R}\right) \to \varphi \quad \text{in } L^2.$$

The function $\eta_{\bar{z}}(z/R)$ is bounded an supported in $\overline{B}(0,2R)$. Hence for any $2 < p < \infty$ we have[71]

$$\begin{aligned} \left\|\psi(z)\eta_{\bar{z}}\left(\frac{z}{R}\right)\frac{1}{R}\right\|_2 &\leq C\Big(⨍_{B(0,2R)}|\psi|^2\Big)^{1/2} \leq C\Big(⨍_{B(0,2R)}|\psi|^p\Big)^{1/p} \\ &\leq CR^{-2/p}\|\psi\|_p \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

This proves (6.13) and thus[72]

$$S(\psi^R_{\bar{z}}) \to S(\varphi) \quad \text{in } L^2. \tag{6.14}$$

On the other hand by the arguing in the same way as in the proof of (6.13) we have

$$S(\psi^R_{\bar{z}}) = \psi^R_z = \psi_z(z)\eta\left(\frac{z}{R}\right) + \psi(z)\eta_z\left(\frac{z}{R}\right)\frac{1}{R} \to \psi_z = \frac{\partial}{\partial z}(\mathcal{C}\varphi) \quad \text{in } L^2 \text{ as } R \to \infty. \tag{6.15}$$

Comparing (6.14) and (6.15) yields $S(\varphi) = \frac{\partial}{\partial z}(\mathcal{C}\varphi)$. The proof is complete. □

6.3. **Higher Riesz transforms.**

Definition 6.24. *Higher Riesz transforms of degree k* are defined as a convolution with

$$\text{p.v.}\,\frac{P_k(x)}{|x|^{n+k}}\,, \tag{6.16}$$

where $P_k(x)$ is a homogeneous harmonic polynomial of degree $k \geq 1$.

If $k = 1$ and $P_1(x) = c_n x_j$, we obtain the Riesz transform R_j.

If P_k is a homogeneous harmonic polynomial of degree $k \geq 1$ we can write

$$\frac{P_k(x)}{|x|^{n+k}} = \frac{P_k(x)|x|^{-k}}{|x|^n} = \frac{P_k(x/|x|)}{|x|^n} = \frac{\Omega(x/|x|)}{|x|^n}$$

and in order to prove that (6.16) defines a tempered distribution we need to prove that

$$\int_{S^{n-1}}\Omega(\theta)\,d\sigma(\theta) = \int_{S^{n-1}}P_k(\theta)\,d\sigma(\theta) = 0\,. \tag{6.17}$$

[71]In the first inequality we use the fact that the area of the disc $B(0,2R)$ equals CR^2. We also use here Hölder's inequality and the fact that $\psi \in L^p$.

[72]S is bounded in L^2.

If $\vec{\nu}$ is an outward normal vector to the unit sphere, then

$$\frac{\partial P_k}{\partial \vec{\nu}} = \frac{d}{dt}\Big|_{t=1} P_k(tx) = kt^{k-1}P_k(x)\Big|_{t=1} = kP_k(x)$$

and hence Green's formula[73] yields

$$k\int_{S^{n-1}} P_k(\theta)\, d\sigma(\theta) = \int_{S^{n-1}} \frac{\partial P_k}{\partial \vec{\nu}}(\theta)\, d\sigma(\theta) = \int_{B^n} \Delta P_k\, dx = 0\,.$$

Thus according to Theorem 5.49

$$W_\Omega[\varphi] = \text{p.v.} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{n+k}}\,\varphi(x)\, dx$$

is a well defined tempered distribution.

Our aim is to prove the following result.

Theorem 6.25. *If P_k is a homogeneous harmonic polynomial of degree $k \geq 1$, then*

$$\left(\text{p.v.} \frac{P_k(x)}{|x|^{n+k}}\right)^\wedge (\xi) = \gamma_k\, \frac{P_k(\xi)}{|\xi|^k}\,,$$

where

$$\gamma_k = (-i)^k \pi^{n/2}\, \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+n}{2}\right)}\,.$$

Example 6.26. This result immediately implies Theorem 6.2.

Example 6.27. We will show now that the formula for the Beurling-Ahlfors transform

$$S(\varphi) = -\frac{1}{\pi}\left(\text{p.v.} \frac{1}{z^2}\right) * \varphi \quad \text{for } \varphi \in \mathscr{S}(\mathbb{C}) \tag{6.18}$$

originally proved in Theorem 6.21 is a straightforward consequence of Theorem 6.25.

Let $n = 2$ and let[74] $P_2(\xi) = \bar{\xi}^2 = \xi_1^2 - 2i\xi_1\xi_2 - \xi_2^2$. Clearly P_2 is a homogeneous harmonic polynomial of degree 2. Since $\gamma_2 = -\pi$ we obtain

$$\left(\text{p.v.} \frac{1}{z^2}\right)^\wedge (\xi) = \left(\text{p.v.} \frac{P_2(z)}{|z|^4}\right)^\wedge (\xi) = -\pi\, \frac{P_2(\xi)}{|\xi|^2} = -\pi\, \frac{\bar{\xi}}{\xi}$$

so

$$\left(-\frac{1}{\pi}\left(\text{p.v.} \frac{1}{z^2}\right) * \varphi\right)^\wedge = \frac{\bar{\xi}}{\xi}\,\hat{\varphi} = \widehat{S(\varphi)}$$

which proves (6.18).

We will deduce Theorem 6.25 from the following result.

Theorem 6.28. *If P_k is a homogeneous harmonic polynomial of degree k and $0 < \alpha < n$, then*

$$\left(\frac{P_k(x)}{|x|^{k+n-\alpha}}\right)^\wedge (\xi) = \gamma_{k,\alpha}\, \frac{P_k(\xi)}{|\xi|^{k+\alpha}}\,,$$

---

[73] $\int_\Omega (f\Delta g - g\Delta f)\, dx = \int_{\partial\Omega} (f\frac{\partial g}{\partial \vec{\nu}} - g\frac{\partial f}{\partial \vec{\nu}})\, d\sigma$ where $\vec{\nu}$ is the outward normal.

[74] We use complex notation in $\mathbb{R}^2 = \mathbb{C}$.

where

$$\gamma_{k,\alpha} = (-i)^k \pi^{\frac{n}{2}-\alpha} \, \frac{\Gamma\left(\frac{k+\alpha}{2}\right)}{\Gamma\left(\frac{k+n-\alpha}{2}\right)} \, .$$

Remark 6.29. Note that the function

$$\frac{P_k(x)}{|x|^{k+n-\alpha}}$$

is a tempered L^1 function, so it defines a tempered distribution without necessity of taking the principal value of the integral.

Proof. For any $t > 0$ and $\varphi \in \mathscr{S}_n$, Corollary 4.24 gives[75]

$$\int_{\mathbb{R}^n} P_k(x) e^{-\pi t|x|^2} \hat{\varphi}(x)\, dx = (-i)^k \int_{\mathbb{R}^n} P_k(x) e^{-\pi|x|^2/t} t^{-k-\frac{n}{2}} \varphi(x)\, dx \, .$$

Now we multiply both sides by

$$t^{\beta-1}, \quad \text{where} \quad \beta = \frac{k+n-\alpha}{2} > 0$$

and integrate with respect to $0 < t < \infty$. Since

$$\int_0^\infty e^{-\pi t|x|^2} t^{\beta-1}\, dt = (\pi|x|^2)^{-\beta} \Gamma(\beta)$$

the integral on the left hand side will be equal to

$$\frac{\Gamma(\beta)}{\pi^\beta} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{2\beta}}\, \hat{\varphi}(x)\, dx = \frac{\Gamma\left(\frac{k+n-\alpha}{2}\right)}{\pi^{\frac{k+n-\alpha}{2}}} \left(\frac{P_k(\cdot)}{|\cdot|^{k+n-\alpha}} \right)^{\wedge} [\varphi] \, . \tag{6.19}$$

Similarly

$$\begin{aligned} \int_0^\infty e^{-\pi|x|^2/t} t^{-k-\frac{n}{2}} t^{\beta-1}\, dt &= \int_0^\infty e^{-\pi|x|^2/t} t^{-\frac{k+\alpha}{2}-1}\, dt \\ &\overset{s=\pi|x|^2/t}{=} (\pi|x|^2)^{-\frac{k+\alpha}{2}} \int_0^\infty e^{-s} s^{\frac{k+\alpha}{2}-1}\, ds \\ &= (\pi|x|^2)^{-\frac{k+\alpha}{2}} \Gamma\left(\frac{k+\alpha}{2}\right) . \end{aligned}$$

Thus the integral on the right hand side equals

$$(-i)^k \frac{\Gamma\left(\frac{k+\alpha}{2}\right)}{\pi^{\frac{k+\alpha}{2}}} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+\alpha}}\, \varphi(x)\, dx \, . \tag{6.20}$$

Since integrals at (6.19) and (6.20) are equal one to another, the theorem follows. □

Proof of Theorem 6.25. For $\varphi \in \mathscr{S}_n$ we take the identity

$$\int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+n-\alpha}}\, \hat{\varphi}(x)\, dx = \gamma_{k,\alpha} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+\alpha}}\, \varphi(x)\, dx$$

[75] $\int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)\, dx = \int_{\mathbb{R}^n} \hat{f}(x)\varphi(x)\, dx$.

and let $\alpha \to 0$. The right hand side converges to

$$(-i)^k \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+n}{2}\right)} \int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^k} \varphi(x)\, dx\,.$$

To compute the limit on the left hand side we observe that the integral of $P_k(x)|x|^{-(k+n-\alpha)}$ on the unit ball equals zero, see (6.17), and hence

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{P_k(x)}{|x|^{k+n-\alpha}} \hat{\varphi}(x)\, dx \\
&= \int_{|x|\leq 1} \frac{P_k(x)}{|x|^{k+n-\alpha}} (\hat{\varphi}(x) - \hat{\varphi}(0))\, dx + \int_{|x|\geq 1} \frac{P_k(x)}{|x|^{k+n-\alpha}} \hat{\varphi}(x)\, dx \\
&\xrightarrow{\alpha\to 0^+} \int_{|x|\leq 1} \frac{P_k(x)}{|x|^{k+n}} (\hat{\varphi}(x) - \hat{\varphi}(0))\, dx + \int_{|x|\geq 1} \frac{P_k(x)}{|x|^{k+n}} \hat{\varphi}(x)\, dx \\
&= \lim_{\varepsilon\to 0} \int_{\varepsilon\leq|x|\leq 1} \frac{P_k(x)}{|x|^{k+n}} (\hat{\varphi}(x) - \hat{\varphi}(0))\, dx + \int_{|x|\geq 1} \frac{P_k(x)}{|x|^{k+n}} \hat{\varphi}(x)\, dx \\
&= \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \frac{P_k(x)}{|x|^{k+n}} \hat{\varphi}(x)\, dx \\
&= \left(\text{p.v.} \frac{P_k(x)}{|x|^{k+n}}\right)^{\wedge} [\varphi]\,.
\end{aligned}$$

Comparing the above limits yields the result. □

6.4. **Another proof of Theorem 6.2.** Two proof presented above[76] are somewhat indirect since they were based on other results: Theorem 5.42 and Theorem 6.25. In this section we will present a different, more straightforward proof. A one dimensional version of this argument will be presented again in Section 10 in the proof of Theorem 10.2.

For $\varphi \in \mathscr{S}_n$ we have

$$\begin{aligned}
\widehat{W_j}[\varphi] &= W_j[\hat{\varphi}] = \lim_{\varepsilon\to 0} c_n \int_{|\xi|\geq\varepsilon} \hat{\varphi}(\xi) \frac{\xi_j}{|\xi|^{n+1}}\, d\xi \\
&= \lim_{\varepsilon\to 0} c_n \int_{\varepsilon\leq|\xi|\leq\varepsilon^{-1}} \left(\int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i x\cdot\xi}\, dx \right) \frac{\xi_j}{|\xi|^{n+1}}\, d\xi \\
&= \lim_{\varepsilon\to 0} c_n \int_{\mathbb{R}^n} \varphi(x) \underbrace{\Big(\int_{\varepsilon\leq|\xi|\leq\varepsilon^{-1}} e^{-2\pi i x\cdot\xi} \frac{\xi_j}{|\xi|^{n+1}}\, d\xi \Big)}_{I} dx = \heartsuit\,.
\end{aligned}$$

Expressing the integral I is the spherical coordinates yields

$$\begin{aligned}
I &= \int_{\varepsilon}^{\varepsilon^{-1}} s^{n-1} \left(\int_{S^{n-1}} e^{-2\pi i x\cdot(s\theta)} \frac{s\theta_j}{s^{n+1}}\, d\sigma(\theta) \right) ds \\
&= \int_{\varepsilon}^{\varepsilon^{-1}} \int_{S^{n-1}} \big(\cos(2\pi s x\cdot\theta) - i\sin(2\pi s x\cdot\theta)\big)\theta_j\, d\sigma(\theta)\, \frac{ds}{s}
\end{aligned}$$

[76]The second proof was given in Remark 6.26.

$$
\begin{aligned}
&= -i\int_\varepsilon^{\varepsilon^{-1}}\int_{S^{n-1}} \sin(2\pi s x\cdot\theta)\theta_j\, d\sigma(\theta)\,\frac{ds}{s}\\
&= -i\int_{S^{n-1}}\left(\int_\varepsilon^{\varepsilon^{-1}} \frac{\sin(2\pi s|x\cdot\theta|)}{s}\,ds\right)\operatorname{sgn}(x\cdot\theta)\theta_j\, d\sigma(\theta)\\
&= -i\int_{S^{n-1}}\left(\int_{2\pi|x\cdot\theta|\varepsilon}^{2\pi|x\cdot\theta|\varepsilon^{-1}} \frac{\sin t}{t}\,dt\right)\operatorname{sgn}(x\cdot\theta)\theta_j\, d\sigma(\theta)\,.
\end{aligned}
$$

Thus

$$\heartsuit = \int_{\mathbb{R}^n}\varphi(x)\left(-ic_n\frac{\pi}{2}\int_{S^{n-1}}\operatorname{sgn}(x\cdot\theta)\theta_j\, d\sigma(\theta)\right)dx\,.$$

Indeed, we could pass to the limit under the sign of the integral using the Dominated Convergence Theorem and we employed the equality[77]

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}\,.$$

Thus it remains to prove that

$$c_n\frac{\pi}{2}\int_{S^{n-1}}\operatorname{sgn}(x\cdot\theta)\theta_j\, d\sigma(\theta) = \frac{x_j}{|x|}\,. \tag{6.21}$$

It is obvious that the function $m:\mathbb{R}^n\to\mathbb{R}^n$ defined as

$$m(x) = \int_{S^{n-1}}\operatorname{sgn}(x\cdot\theta)\theta\, d\sigma(\theta)$$

is homogeneous of degree 0. We will use Lemma 4.26 so we need to check that m commutes with orthogonal transformations. It does because

$$
\begin{aligned}
m(\rho(x)) &= \int_{S^{n-1}}\operatorname{sgn}(\rho(x)\cdot\theta)\theta\, d\sigma(\theta)\\
&= \int_{S^{n-1}}\operatorname{sgn}(x\cdot\rho^{-1}(\theta))\theta\, d\sigma(\theta)\\
&= \int_{S^{n-1}}\operatorname{sgn}(x\cdot\theta)\rho(\theta)\, d\sigma(\theta) &(6.22)\\
&= \rho\left(\int_{S^{n-1}}\operatorname{sgn}(x\cdot\theta)\theta\, d\sigma(\theta)\right) &(6.23)\\
&= \rho(m(x))\,.
\end{aligned}
$$

Note that (6.22) follows from the fact that ρ induces a volume preserving change of variables on S^{n-1}, while (6.23) is a direct consequence of linearity of ρ. Thus the Lemma 4.26 yields

$$\int_{S^{n-1}}\operatorname{sgn}(x\cdot\theta)\theta\, d\sigma(\theta) = c\,\frac{x}{|x|}$$

and hence looking at the jth component we have

$$\int_{S^{n-1}}\operatorname{sgn}(x\cdot\theta)\theta_j\, d\sigma(\theta) = c\,\frac{x_j}{|x|}\,. \tag{6.24}$$

[77]Recall that this is understood as an improper integral which is consistent with our setting since we pass to the limit as $\varepsilon\to 0$.

Thus it remains to prove that

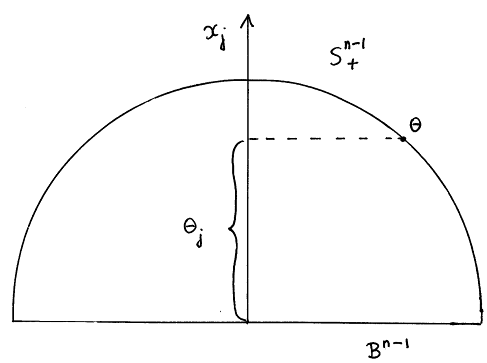
$$c = \left(c_n \frac{\pi}{2}\right)^{-1} = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} = 2\omega_{n-1}\,.$$

Taking $x = e_j$ in (6.24) we have

$$\int_{S^{n-1}} |\theta_j|\, d\sigma(\theta) = c\,.$$

The unit ball B^{n-1} in coordinates perpendicular to x_j split the sphere S^{n-1} into two half spheres $S^{n-1}_{\pm}$. Thus

$$c = 2\int_{S^{n-1}_+} \theta_j\, d\sigma(\theta)\,.$$



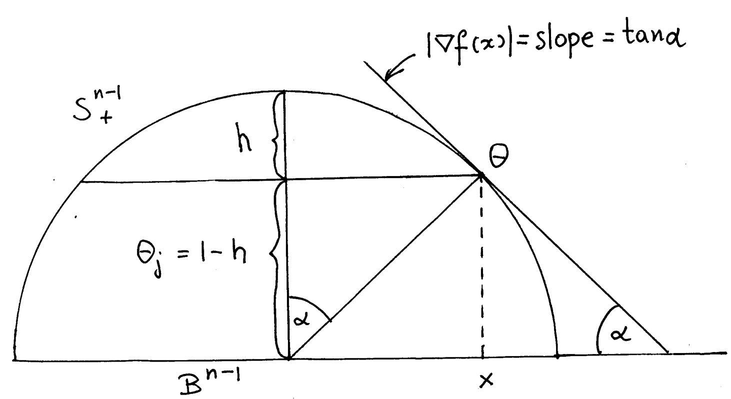
Recall that if $M \subset \mathbb{R}^n$ is a graph of a C^1 function $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^{n-1}$, then for a measurable function g on M we have

$$\int_M g\, d\sigma = \int_\Omega g(x, f(x))\sqrt{1 + |\nabla f(x)|^2}\, dx\,.$$

In our situation we parametrize S^{n-1}_+ as a graph of the function

$$f(x) = \sqrt{1 - |x|^2}\,, \quad x \in B^{n-1}\,.$$

Form the picture



we conclude

$$\begin{aligned}\int_{S^{n-1}_+}\theta_j\,d\sigma(\theta) &= \int_{S^{n-1}_+}(1-h)\,d\sigma(\theta)=\int_{B^{n-1}}(1-h)\sqrt{1+\tan^2\alpha}\,dx\\ &= \int_{B^{n-1}}dx=\omega_{n-1}\,,\end{aligned}$$

because

$$\sqrt{1+\tan^2\alpha}=\frac{1}{\cos\alpha}=\frac{1}{1-h}$$

and the result follows. □

The proof of the next result is based on an argument similar to that used to prove (6.24); we leave details to the reader as an exercise.

Lemma 6.30. *Let K be a function of one variable, then for $n\geq 2$ we have*

$$\int_{S^{n-1}}K(x\cdot\theta)\,d\sigma(\theta)=(n-1)\omega_{n-1}\int_{-1}^{1}K(s|x|)(1-s^2)^{\frac{n-3}{2}}\,ds$$

for all $x\in\mathbb{R}^n\setminus\{0\}$.

6.5. **The general case.** In this section we will consider operators $T_\Omega\varphi=W_\Omega*\varphi$ in the most general form where $\Omega\in L^1(S^{n-1})$, $\int_{S^{n-1}}\Omega(\theta)\,d\sigma(\theta)=0$, and

$$W_\Omega=\text{p.v.}\,K_\Omega,\quad\text{where}\quad K_\Omega(x)=\frac{\Omega(x/|x|)}{|x|^n}\,.$$

The following lemma will play an important role.

Lemma 6.31. *If F is a function homogeneous of degree zero and $F\big|_{S^{n-1}}\in L^1(S^{n-1})$, then F is a tempered L^1 function*

$$\int_{\mathbb{R}^n}|F(x)|(1+|x|^{n+1})^{-1}\,dx<\infty.$$

Hence $F\in\mathscr{S}'_n$. Moreover

$$|F[\varphi]|\leq C(n)\|F\|_{L^1(S^{n-1})}\sup_{x\in\mathbb{R}^n}(1+|x|^{n+1})|\varphi(x)|.$$

Proof. We have

$$\begin{aligned}|F[\varphi]| &\leq \int_{\mathbb{R}^n}|F(x)\varphi(x)|\,dx=\int_0^\infty s^{n-1}\int_{S^{n-1}}|F(\theta)\varphi(s\theta)|\,d\sigma(\theta)\,ds\\ &= \int_{S^{n-1}}|F(\theta)|\int_0^\infty s^{n-1}|\varphi(s\theta)|\,ds\,d\sigma(\theta)=\heartsuit\end{aligned}$$

$$\begin{aligned}\int_0^\infty s^{n-1}|\varphi(s\theta)|\,ds &\leq \int_0^\infty\frac{s^{n-1}}{1+s^{n+1}}\sup_{|x|=s}(1+|x|^{n+1})|\varphi(x)|\,ds\\ &\leq C(n)\sup_{x\in\mathbb{R}^n}(1+|x|^{n+1})|\varphi(x)|.\end{aligned}$$

Hence

$$\heartsuit\leq C(n)\|F\|_{L^1(S^{n-1})}\sup_{\xi\in\mathbb{R}^n}(1+|\xi|^{n+1})|\varphi(\xi)|.$$

In particular taking[78] $\varphi(x) = (1+|x|^{n+1})^{-1}$ yields

$$\int_{\mathbb{R}^n} |F(x)|(1+|x|^{n+1})^{-1} \le C\|F\|_{L^1(S^{n-1})} < \infty$$

so F is a tempered L^1 function. □

The main result of this section generalizes Theorem 6.2.

Theorem 6.32. *Let $n \ge 2$ and $\Omega \in L^1(S^{n-1})$ be such that*

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0\,.$$

Then the Fourier transform of the distribution W_Ω is a tempered L^1 function, homogeneous of degree zero such that

$$\|\widehat{W_\Omega}\|_{L^1(S^{n-1})} \le C\|\Omega\|_{L^1(S^{n-1})} \quad \textit{and} \quad \int_{S^{n-1}} \widehat{W_\Omega}(\xi)\, d\sigma(\xi) = 0. \tag{6.25}$$

Therefore $\widehat{W_\Omega}$ is a tempered distribution satisfying

$$|\widehat{W_\Omega}[\varphi]| \le C\|\Omega\|_{L^1(S^{n-1})} \left(\sup_{\xi\in\mathbb{R}^n} (1+|\xi|^{n+1})|\varphi(\xi)|\right). \tag{6.26}$$

Moreover

$$\begin{aligned} \widehat{W_\Omega}(\xi) &= \int_{S^{n-1}} \Omega(\theta) \left(\log\frac{1}{|\xi\cdot\theta|} - \frac{i\pi}{2}\operatorname{sgn}(\xi\cdot\theta)\right) d\sigma(\theta) \\ &= \int_{S^{n-1}} \Omega(\theta) \left(\log\frac{1}{|\xi'\cdot\theta|} - \frac{i\pi}{2}\operatorname{sgn}(\xi'\cdot\theta)\right) d\sigma(\theta)\,, \end{aligned} \tag{6.27}$$

where $\xi' = \xi/|\xi|$.

Finally, if Ω is odd, then

$$\widehat{W_\Omega}(\xi) = -\frac{i\pi}{2}\int_{S^{n-1}} \Omega(\theta)\operatorname{sgn}(\xi'\cdot\theta)\, d\sigma(\theta) \quad \textit{is odd} \tag{6.28}$$

and if Ω is even, then

$$\widehat{W_\Omega}(\xi) = \int_{S^{n-1}} \Omega(\theta) \left(\log\frac{1}{|\xi'\cdot\theta|}\right) d\sigma(\theta) \quad \textit{is even.} \tag{6.29}$$

We will precede the proof with some technical lemmas.

Lemma 6.33. *For $\theta \in S^{n-1}$ define a function on S^{n-1} by $f_\theta(\xi') = \log\frac{1}{|\xi'\cdot\theta|}$. Then $f_\theta \in L^p(S^{n-1})$ for all $1 \le p < \infty$ and*

$$\sup_{\theta\in S^{n-1}} \|f_\theta\|_{L^p(S^{n-1})} = C(n,p) < \infty.$$

[78] $\varphi(x) = (1+|x|^{n+1})^{-1}$ does not belong to $\mathscr{S}_n$, but the above estimates work for this function too.

Proof. According to Lemma 6.30[79]

$$\int_{S^{n-1}} \log^p\left(\frac{1}{|\xi'\cdot\theta|}\right)\, d\sigma(\xi') = (n-1)\omega_{n-1}\int_{-1}^1 \log^p\left(\frac{1}{|s|}\right)(1-s^2)^{\frac{n-3}{2}}\, dx < \infty \tag{6.30}$$

The last integral has three singularities[80]

$$\log^p\frac{1}{|s|} \text{ at } s=0, \quad (1+s)^{\frac{n-3}{2}} \text{ at } s=-1, \quad (1-s)^{\frac{n-3}{2}} \text{ at } s=1$$

and all these singularities are integrable. □

Lemma 6.34. *If $h:[0,\infty)\to\mathbb{R}$ is continuous, bounded and the improper integral*

$$\int_1^\infty \frac{h(s)}{s}\, ds$$

converges, then for $\mu>\lambda>0$ and $N>\varepsilon>0$ we have

$$\left|\int_\varepsilon^N \frac{h(\lambda s)-h(\mu s)}{s}\, ds\right| \leq 2\|h\|_\infty \log\left(\frac{\mu}{\lambda}\right). \tag{6.31}$$

Moreover

$$\lim_{\substack{N\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^N \frac{h(\lambda s)-h(\mu s)}{s}\, ds = h(0)\log\left(\frac{\mu}{\lambda}\right). \tag{6.32}$$

Proof. We have

$$\begin{aligned}\int_\varepsilon^N \frac{h(\lambda s)-h(\mu s)}{s}\, ds &= \int_{\lambda\varepsilon}^{\lambda N}\frac{h(s)}{s}\, ds - \int_{\mu\varepsilon}^{\mu N}\frac{h(s)}{s}\, ds\\ &= \int_{\lambda\varepsilon}^{\mu\varepsilon}\frac{h(s)}{s}\, ds - \int_{\lambda N}^{\mu N}\frac{h(s)}{s}\, ds\,.\end{aligned}$$

Estimating the absolute value of the last two integrals gives (6.31). Since

$$\int_{\lambda\varepsilon}^{\mu\varepsilon}\frac{h(s)}{s}\, ds \to h(0)\log\left(\frac{\mu}{\lambda}\right) \quad \text{as } \varepsilon\to 0$$

and

$$\int_{\lambda N}^{\mu N}\frac{h(s)}{s}\, ds \to 0 \quad \text{as } N\to\infty$$

(6.32) follows. □

Corollary 6.35. *For $a\neq 0$ we have*

$$\lim_{\substack{N\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^N \frac{e^{-isa}-\cos s}{s}\, ds = \log\frac{1}{|a|} - i\frac{\pi}{2}\operatorname{sgn} a$$

and

$$\left|\int_\varepsilon^N \frac{e^{-isa}-\cos s}{s}\, ds\right| \leq 2\left|\log\frac{1}{|a|}\right| + C,$$

where C is some positive constant independent of a, ε and N.

[79] Since $|\xi'\cdot\theta|\leq 1$, $\log\frac{1}{|\xi'\cdot\theta|}\geq 0$ and we do not have to take the absolute value of the log term.

[80] The last two are singularities only when $n=2$.

Proof. We have

$$\begin{aligned}\int_\varepsilon^N \frac{e^{-isa}-\cos s}{s}\,ds &= \int_\varepsilon^N \frac{\cos(sa)-\cos s}{s}\,ds - i\int_\varepsilon^N \frac{\sin(sa)}{s}\,ds\\ &= \int_\varepsilon^N \frac{\cos(s|a|)-\cos s}{s} - i\,\mathrm{sgn}\,(a)\int_\varepsilon^N \frac{\sin(s|a|)}{s}\,ds\\ &= \int_\varepsilon^N \frac{\cos(s|a|)-\cos s}{s} - i\,\mathrm{sgn}\,(a)\int_{\varepsilon|a|}^{N|a|} \frac{\sin s}{s}\,ds\end{aligned}$$

and the result follows from Lemma 6.34. □

Proof of Theorem 6.32. The structure of the proof is as follows

(1) We will prove that the two integrals at (6.27) are equal.
(2) Define

$$F(\xi) = \int_{S^{n-1}} \Omega(\theta)\left(\log\frac{1}{|\xi'\cdot\theta|} - \frac{i\pi}{2}\,\mathrm{sgn}\,(\xi'\cdot\theta)\right)d\sigma(\theta)\,,$$

Clearly F is homogeneous of degree zero. We will prove that

$$\|F\|_{L^1(S^{n-1})} \le C\|\Omega\|_{L^1(S^{n-1})}, \qquad \int_{S^{n-1}} F(\xi)\,d\sigma(\xi) = 0.$$

Hence according to Lemma 6.31, F is a tempered L^1 function so $F\in\mathscr{S}_n'$ and

$$|F[\varphi]| \le C\|\Omega\|_{L^1(S^{n-1})}\left(\sup_{\xi\in\mathbb{R}^n}(1+|\xi|^{n+1})|\varphi(\xi)|\right).$$

(3) Since $\xi\mapsto\log\frac{1}{|\xi'\cdot\theta|}$ is even and $\xi\mapsto\mathrm{sgn}\,(\xi'\cdot\theta)$ is odd, it follows that if Ω is odd or even, then

$$F(\xi) = -\frac{i\pi}{2}\int_{S^{n-1}}\Omega(\theta)\,\mathrm{sgn}\,(\xi'\cdot\theta)\,d\sigma(\theta) \quad\text{or}\quad F(\xi) = \int_{S^{n-1}}\Omega(\theta)\left(\log\frac{1}{|\xi'\cdot\theta|}\right)d\sigma(\theta)$$

are odd and even respectively.
(4) Finally we will prove that $\widehat{W_\Omega}(\xi) = F(\xi)$.

First observe that the equality of the two integrals in (6.27) follows from

$$\log\frac{1}{|\xi\cdot\theta|} = \log\frac{1}{|\xi|} + \log\frac{1}{|\xi'\cdot\theta|} \quad\text{and}\quad \int_{S^{n-1}}\Omega(\theta)\log\frac{1}{|\xi|}\,d\sigma(\theta) = 0\,.$$

This proves (1).

Let

$$F(\xi) = \int_{S^{n-1}}\Omega(\theta)\left(\log\frac{1}{|\xi'\cdot\theta|} - \frac{i\pi}{2}\,\mathrm{sgn}\,(\xi'\cdot\theta)\right)d\sigma(\theta)$$

be the function defined by the integral in the second line of (6.27).

It is clear that $F(\xi)$ is homogeneous of degree zero. We will show that it is integrable on the unit sphere and

$$\|F\|_{L^1(S^{n-1})} \le C\|\Omega\|_{L^1(S^{n-1})}.$$

Note that

$$\xi \mapsto \int_{S^{n-1}} \Omega(\theta)\, \frac{i\pi}{2}\, \mathrm{sgn}\, (\xi' \cdot \theta)\, d\sigma(\theta)$$

is a bounded by $\frac{\pi}{2}\|\Omega\|_{L^1(S^{n-1})}$, so this component of $F(\xi)$ does not cause any troubles and hence we only need to estimate

$$G(\xi) = \int_{S^{n-1}} \Omega(\theta) \log \frac{1}{|\xi' \cdot \theta|}\, d\sigma(\theta)\,.$$

Lemma 6.33 with $p = 1$ yields

$$\begin{aligned} \|G\|_{L^1(S^{n-1})} &\leq \int_{S^{n-1}} \left| \int_{S^{n-1}} \Omega(\theta) \log \frac{1}{|\xi' \cdot \theta|}\, d\sigma(\theta) \right| d\sigma(\xi') \\ &\leq \int_{S^{n-1}} |\Omega(\theta)| \left(\int_{S^{n-1}} \log \frac{1}{|\xi' \cdot \theta|}\, d\sigma(\xi') \right) d\sigma(\theta) = C(n)\|\Omega\|_{L^1(S^{n-1})}. \end{aligned}$$

It easily follow now from the Fubini theorem that $\int_{S^{n-1}} F(\xi)\, d\sigma(\xi) = 0$. That completes the proof of (2).

Since $F \in L^1(S^{n-1})$, the step (3) is obvious.

To prove (4) we need to show that $\widehat{W_\Omega} = F$ in the sense of distributions i.e.,

$$\widehat{W_\Omega}[\varphi] = \int_{\mathbb{R}^n} \varphi(\xi) \int_{S^{n-1}} \Omega(\theta) \left(\log \frac{1}{|\xi' \cdot \theta|} - \frac{i\pi}{2}\, \mathrm{sgn}\, (\xi' \cdot \theta) \right) d\sigma(\theta)\, d\xi.$$

We have

$$\begin{aligned} &\widehat{W_\Omega}[\varphi] = W_\Omega[\hat{\varphi}] \\ &= \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon \leq |x| \leq N} \frac{\Omega(x/|x|)}{|x|^n}\, \hat{\varphi}(x)\, dx \\ &= \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\mathbb{R}^n} \varphi(\xi) \int_{\varepsilon \leq |x| \leq N} \frac{\Omega(x/|x|)}{|x|^n}\, e^{-2\pi i x \cdot \xi}\, dx\, d\xi \\ &= \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\mathbb{R}^n} \varphi(\xi) \int_{S^{n-1}} \Omega(\theta) \int_\varepsilon^N e^{-2\pi i s \theta \cdot \xi}\, \frac{ds}{s}\, d\sigma(\theta)\, d\xi \\ &= \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\mathbb{R}^n} \varphi(\xi) \int_{S^{n-1}} \Omega(\theta) \int_\varepsilon^N \left(e^{-2\pi i s \theta \cdot \xi} - \cos(2\pi s|\xi|) \right) \frac{ds}{s}\, d\sigma(\theta)\, d\xi \\ &= \diamondsuit\,. \end{aligned}$$

The last equality follows from the fact that the integral of Ω over the sphere vanishes. We have

$$\begin{aligned} \int_\varepsilon^N \left(e^{-2\pi i s \theta \cdot \xi} - \cos(2\pi s|\xi|) \right) \frac{ds}{s} &= \int_{2\pi|\xi|\varepsilon}^{2\pi|\xi|N} \frac{e^{-is\theta \cdot \xi'} - \cos s}{s}\, ds \\ &\longrightarrow \log \frac{1}{|\theta \cdot \xi'|} - i\frac{\pi}{2}\, \mathrm{sgn}\, (\theta \cdot \xi') \end{aligned}$$

by Corollary 6.35. Also Corollary 6.35 and integrability of G easily imply that

$$\left| \varphi(\xi)\Omega(\theta) \int_\varepsilon^N \left(e^{-2\pi i s \theta \cdot \xi} - \cos(2\pi s|\xi|) \right) \frac{ds}{s} \right|$$

$$\leq \quad C|\varphi(\xi)|\,|\Omega(\theta)|\left(1+\log\frac{1}{|\xi'\cdot\theta|}\right)\in L^1(\mathbb{R}^n\times S^{n-1})$$

so the Dominated Convergence Theorem gives

$$\diamondsuit \quad = \quad \int_{\mathbb{R}^n}\varphi(\xi)\int_{S^{n-1}}\Omega(\theta)\left(\log\frac{1}{|\theta\cdot\xi'|}-i\frac{\pi}{2}\operatorname{sgn}(\theta\cdot\xi')\right)d\sigma(\theta)\,d\xi\,.$$

The proof is complete. □

If Ω is an odd function, then

$$\widehat{W_\Omega}(\xi)=-\frac{i\pi}{2}\int_{S^{n-1}}\Omega(\theta)\operatorname{sgn}(\xi'\cdot\theta)\,d\sigma(\theta)\,.$$

In particular the Fourier transform $\widehat{W_\Omega}$ is bounded. More generally any function Ω on S^{n-1} can be decomposed into its even and odd parts

$$\Omega_e(\theta)=\frac{1}{2}(\Omega(\theta)+\Omega(-\theta)),\qquad \Omega_o(\theta)=\frac{1}{2}(\Omega(\theta)-\Omega(-\theta))\,.$$

Corollary 6.36. *Let $\Omega\in L^1(S^{n-1})$ be such that*

$$\int_{S^{n-1}}\Omega(\theta)\,d\sigma(\theta)=0\,.$$

If $\Omega_o\in L^1(S^{n-1})$ and $\Omega_e\in L^q(S^{n-1})$ for some $q>1$, then the Fourier transform of W_Ω is a bounded function. In particular the operator $T_\Omega\varphi=W_\Omega*\varphi$ for $\varphi\in\mathscr{S}_n$ extends to a bounded operator in L^2.

Proof. Lemma 6.33 implies that $\log(1/|\xi'\cdot\theta|)$ belongs to $L^{q'}(S^{n-1})$ and hence

$$\left|\int_{S^{n-1}}\Omega_e(\theta)\left(\log\frac{1}{|\xi'\cdot\theta|}\right)d\sigma(\theta)\right|\leq C\|\Omega_e\|_{L^q(S^{n-1})}\,,$$

with a constant C independent of ξ. Now the formulas (6.28) and (6.29) yield

$$\widehat{W_\Omega}(\xi)=-\frac{i\pi}{2}\int_{S^{n-1}}\Omega_o(\theta)\operatorname{sgn}(\xi'\cdot\theta)\,d\sigma(\theta)+\int_{S^{n-1}}\Omega_e(\theta)\left(\log\frac{1}{|\xi'\cdot\theta|}\right)d\sigma(\theta)$$

from which we have

$$\|\widehat{W_\Omega}\|_\infty\leq C(\|\Omega_o\|_{L^1(S^{n-1})}+\|\Omega_e\|_{L^q(S^{n-1})})\,.$$

Since $T_\Omega\varphi=(\hat{\varphi}\widehat{W_\Omega})^\vee$ and multiplication by a bounded function constitutes a bounded operator in L^2 it follows that

$$\|T_\Omega\varphi\|_2\leq\|\widehat{W_\Omega}\|_\infty\|\varphi\|_2\quad\text{for }\varphi\in\mathscr{S}_n.$$

□

It is natural to expect that if Ω in Theorem 6.32 is smooth, then $\widehat{W_\Omega}$ is smooth away from the origin. As we will see this it true. The next result is quite surprising since it gives a complete characterization of such Fourier transforms $\widehat{W_\Omega}$.

Theorem 6.37. *If $\Omega \in C^\infty(S^{n-1})$ satisfies*

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0, \tag{6.33}$$

then

$$m = \widehat{W_\Omega} \in C^\infty(\mathbb{R}^n \setminus \{0\}) \tag{6.34}$$

is homogeneous of degree zero and

$$\int_{S^{n-1}} m(\xi)\, d\sigma(\xi) = 0. \tag{6.35}$$

Conversely if $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree zero and it satisfies (6.35)*, then there is $\Omega \in C^\infty(S^{n-1})$ satisfying* (6.33) *such that $m = \widehat{W_\Omega}$.*

Lemma 6.38. *If $\Omega \in C^\infty(S^{n-1})$ and*

$$\int_{-1}^{1} |K(s)|(1-s^2)^{\frac{n-3}{2}}\, ds < \infty \tag{6.36}$$

then $m \in C^\infty(S^{n-1})$ where

$$m(\xi) = \int_{S^{n-1}} \Omega(\theta) K(\xi \cdot \theta)\, d\sigma(\theta).$$

Remark 6.39. We *do not* assume here that the integral of Ω vanishes.

Remark 6.40. According to Lemma 6.30 inequality (6.36) implies that for every $\xi \in S^{n-1}$

$$\int_{S^{n-1}} |K(\xi\cdot\theta)|\, d\sigma(\theta) = (n-1)\omega_{n-1} \int_{-1}^{1} |K(s)|(1-s^2)^{\frac{n-3}{2}}\, ds < \infty.$$

Proof. Observe that it suffices to prove that m is smooth in a neighborhood of e_1. Indeed, if $\xi_0 \in S^{n-1}$ and $\rho(e_1) = \xi_0$ for some $\rho \in O(n)$, then

$$\begin{aligned} \tilde{m}(\xi) &:= (m\circ\rho)(\xi) = \int_{S^{n-1}} \Omega(\theta) K(\rho(\xi)\cdot\theta)\, d\sigma(\theta) \\ &= \int_{S^{n-1}} \Omega(\theta) K(\xi \cdot \rho^{-1}(\theta))\, d\sigma(\theta) = \int_{S^{n-1}} \Omega(\rho(\theta)) K(\xi\cdot\theta)\, d\sigma(\theta) \end{aligned}$$

and $\Omega\circ\rho \in C^\infty(S^{n-1})$. Thus smoothness of $\tilde{m}$ is a neighborhood of e_1 will imply smoothness of m in a neighborhood of ξ_0, because $m = \tilde{m}\circ\rho^{-1}$ and $\rho^{-1} : S^{n-1} \to S^{n-1}$ is a diffeomorphism that maps a neighborhood of ξ_0 onto a neighborhood of e_1, where $\tilde{m}$ is smooth.

Thus we will prove that m is smooth in a neighborhood of e_1.

If ξ is in the right hemisphere generated by e_1, we can represent it in a local coordinate system

$$\xi = q(\xi_2, \ldots, \xi_n) = \left(\sqrt{1 - \sum_{j=2}^{n} \xi_j^2}, \xi_2, \ldots, \xi_n \right), \quad (\xi_2, \ldots, \xi_n) \in B^{n-1}(0,1).$$

It suffices to prove that $(m\circ q)(\xi_2, \ldots, \xi_n)$ is smooth in $B^{n-1}(0,1)$.

Given $\xi = q(\xi_2, \dots, \xi_n)$ let
$$\rho(\xi_2, \dots, \xi_n) \in SO(n), \quad \rho(\xi_2, \dots, \xi_n)(\xi) = e_1$$
be a rotation that rotates ξ to e_1 in the plane span $\{e_1, \xi\}$ and fixes the orthogonal complement of that plane. Such a rotation is unique and it is easy to see that components of the matrix $\rho(\xi_2, \dots, \xi_n)$ smoothly depend in $(\xi_2, \dots, \xi_n)$. By an argument similar to the one used at the beginning of the proof
$$(m \circ q)(\xi_2, \dots, \xi_n) = \int_{S^{n-1}} \Omega(\theta) K(\xi \cdot \theta)\, d\sigma(\theta) = \int_{S^{n-1}} (\Omega \circ \rho^{-1}(\xi_2, \dots, \xi_n))(\theta) K(\theta_1)\, d\sigma(\theta).$$
Now it is easy to see that we can differentiate with respect to $\xi_2, \dots, \xi_n$ under the sign of the integral infinitely many times.[81] □

Proof of Theorem 6.37. Let $\Omega \in C^\infty(S^{n-1})$ satisfy (6.33). For $\xi \neq 0$,
$$m(\xi) = \widehat{W_\Omega}(\xi) = \int_{S^{n-1}} \Omega(\theta) K(\xi \cdot \theta)\, d\sigma(\theta)$$
where
$$K(s) = \log \frac{1}{|s|} - \frac{i\pi}{2} \mathrm{sgn}\,(s)$$
and we already proved in (6.30) that
$$\int_{-1}^{1} |K(s)| (1 - s^2)^{\frac{n-3}{2}}\, ds < \infty.$$
Thus $m \in C^\infty(S^{n-1})$ by Lemma 6.38 and since m is homogeneous of degree zero, $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$. The fact that the integral of m over the sphere vanishes was already proved in Theorem 6.32, see (6.25).

Suppose now that $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree zero and it satisfies (6.35). Since m is a tempered distribution, $\hat{m}$ is a tempered distribution too and
$$\left(\frac{\partial^n m}{\partial x_j^n} \right)^{\wedge} (\xi) = (2\pi i \xi_j)^n \hat{m}(\xi).$$
The function $\partial^n m / \partial x_j^n$ is homogeneous of degree $-n$. Since
$$\frac{\partial^n m}{\partial x_j^n} = \frac{\partial}{\partial x_j} \left(\frac{\partial^{n-1} m}{\partial x_j^{n-1}} \right)$$
it is the derivative of a function that is homogeneous of degree $1-n$, Theorem 5.57 yield
$$\int_{S^{n-1}} \frac{\partial^n m(\theta)}{\partial x_j^n}\, d\sigma(\theta) = 0 \quad \text{and} \quad \frac{\partial^n m}{\partial x_j^n} = c\delta_0 + \text{p.v.}\, \frac{\partial^n m}{\partial x_j^n}\,.$$

[81]Perhaps it will be easier to see it if we rewrite the integral in a way that it does not involve integration of functions on manifolds (sphere). Let $F \in C^\infty(\mathbb{R}^n)$ be a smooth extension of Ω from S^{n-1} to $\mathbb{R}^n$. Then
$$(m \circ q)(\xi_2, \dots, \xi_n) = \int_{\mathbb{R}^n} F(\rho^{-1}(\xi_2, \dots, \xi_n)(x)) d\mu(x)$$
where μ is a measure concentrated on the sphere S^{n-1}. Namely μ is the extension of $K(\theta_1) d\sigma(\theta)$ from S^{n-1} to $\mathbb{R}^n$ by zero, i.e., $\mu(E) = \int_{S^{n-1} \cap E} K(\theta_1)\, d\sigma(\theta)$.

Taking the Fourier transform yields

$$(2\pi i)^n \xi_j^n \hat{m}(\xi) = c + \left(\text{p.v.}\,\frac{\partial^n m}{\partial x_j^n}\right)^\wedge(\xi).$$

Note that the function on the right hand side is of class $C^\infty(\mathbb{R}^n \setminus \{0\})$, homogeneous of degree zero (by the first part of the proof).

The above equality yields

$$(2\pi i)^n \sum_{j=1}^n (\xi_j^{2n})\hat{m}(\xi) = \sum_{j=1}^n \xi_j^n \left(c + \left(\text{p.v.}\,\frac{\partial^n m}{\partial x_j^n}\right)^\wedge(\xi)\right). \tag{6.37}$$

We claim that the distribution $\hat{m}$ coincides in $\mathbb{R}^n \setminus \{0\}$ with the function that is smooth in $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree $-n$

$$\hat{m}(\xi) = \frac{\sum_{j=1}^n \xi_j^n \left(c + \left(\text{p.v.}\,\frac{\partial^n m}{\partial x_j^n}\right)^\wedge(\xi)\right)}{(2\pi i)^n \sum_{j=1}^n (\xi_j^{2n})} \quad \text{in } \mathbb{R}^n \setminus \{0\}. \tag{6.38}$$

That means, if $\varphi \in \mathscr{S}_n$ with $\operatorname{supp}\varphi \subset \mathbb{R}^n \setminus \{0\}$, then

$$\hat{m}[\varphi] = \int_{\mathbb{R}^n} \frac{\sum_{j=1}^n \xi_j^n \left(c + \left(\text{p.v.}\,\frac{\partial^n m}{\partial x_j^n}\right)^\wedge(\xi)\right)}{(2\pi i)^n \sum_{j=1}^n (\xi_j^{2n})}\,\varphi(x)\,dx. \tag{6.39}$$

To see this observe that since φ vanishes in a neighborhood of 0, the function $\sum_{j=1}^n \xi_j^{2n}$ is positive on the support of φ so

$$\psi = \frac{\varphi}{(2\pi i)^n \sum_{j=1}^n (\xi_j^{2n})} \in \mathscr{S}_n$$

is in the Schwarz class.[82] Now it is easy to see that if we evaluate both sides of the distributional equality (6.37) on ψ we obtain equality (6.39).

We proved that in $\mathbb{R}^n \setminus \{0\}$, $\hat{m}$ coincides with a smooth function, homogeneous of degree $-n$ which is given by the formula on the right hand side of (6.38). We still do not know what happens at 0. There is a possibility that $\hat{m}$ has a part supported at 0.

Let $\tilde{\Omega}(\xi) = \hat{m}(\xi)$ for $\xi \in S^{n-1}$. Thus

$$\hat{m}(\xi) = \frac{\tilde{\Omega}(\xi/|\xi|)}{|\xi|^n} \quad \text{for } \xi \neq 0.$$

We claim that

$$\int_{S^{n-1}} \tilde{\Omega}(\theta)\,d\sigma(\theta) = 0, \tag{6.40}$$

Indeed, let $\varphi \in \mathscr{S}_n$ be a radial function supported in the annulus $1 \leq |x| \leq 2$ and positive in the interior of the annulus. Then the integration in the spherial coordinates gives

$$\hat{m}[\varphi] = \int_{\mathbb{R}^n} \frac{\tilde{\Omega}(x/|x|)}{|x|^n}\,\varphi(x)\,dx = C\int_{S^{n-1}} \tilde{\Omega}(\theta)\,d\sigma(\theta), \quad \text{where } C > 0. \tag{6.41}$$

[82]Why?

On the other hand the Fourier transform commutes with orthogonal transformations and hence $\hat{\varphi}(\xi)$ is also a radial function $\hat{\varphi}(\xi) = f(|\xi|)$ so

$$\hat{m}[\varphi] = m[\hat{\varphi}] = \int_{\mathbb{R}^n} m(\xi)\hat{\varphi}(\xi)\, d\xi = \int_{S^{n-1}} m(\theta)d\sigma(\theta) \int_0^\infty s^{n-1} f(s)\, ds = 0. \tag{6.42}$$

Now (6.40) follows from the equality between (6.41) and (6.42).

We proved that $\hat{m}(x)$ coincides with $\tilde{\Omega}(x/|x|)/|x|^n$ in $\mathbb{R}^n \setminus \{0\}$. Hence the distribution

$$\hat{m} - \text{ p.v. } \frac{\tilde{\Omega}(x/|x|)}{|x|^n} \tag{6.43}$$

is supported at the origin. According to Corollary 5.34 its Fourier transform is a polynomial. However, the Fourier transform of (6.43) is a bounded function so the polynomial must be constant. The constant must be zero because m and $(\text{p.v.}\, \tilde{\Omega}(x/|x|)/|x|^n)^\wedge$ have the zero integrals over S^{n-1}. Hence

$$\hat{m} = \text{p.v.}\, \frac{\tilde{\Omega}(x/|x|)}{|x|^n} \quad \text{so} \quad m(\xi) = \left(\text{p.v.}\, \frac{\Omega(x/|x|)}{|x|^n}\right)^\wedge(\xi)$$

where $\Omega(\theta) = \tilde{\Omega}(-\theta)$. □

6.6. **Algebra of singular integrals.** In this section we will investigate compositions of singular integrals

$$T_\Omega f(x) = W_\Omega * f(x) = \lim_{\varepsilon\to 0} \int_{|y|\geq\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy \quad \text{where} \quad \int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0.$$

We will assume that $\Omega \in C^\infty(S^{n-1})$.

In general we cannot expect that the composition of singular integrals is a singular integral. This can be seen in the following example

Example 6.41. There is a function $\Omega \in C^\infty(S^{n-1})$ with the vanishing integral such that that the square of a Riesz transform satisfies[83]

$$R_j^2 = -\frac{1}{n} I + T_\Omega \tag{6.44}$$

Indeed,

$$\left(\left(R_j^2 + \frac{1}{n} I\right)[\varphi]\right)^\wedge = \left(\frac{1}{n} - \frac{\xi_j^2}{|\xi|^2}\right)\hat{\varphi}(\xi) = m(\xi)\hat{\varphi}(\xi)$$

and

$$\int_{S^{n-1}} m(\xi)\, d\sigma(\xi) = \int_{S^{n-1}} \left(\frac{1}{n} - \xi_j^2\right) d\sigma(\xi) = \frac{1}{n}\int_{S^{n-1}} \left(1 - \sum_{k=1}^n \xi_k^2\right) d\sigma(\xi) = 0$$

so the existence of Ω satisfying (6.44) follows from Theorem 6.37.

However, the class of operators of the form $aI + T_\Omega$ is closed under compositions.

[83] I stands for the identity.

Theorem 6.42. *If $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree zero, then there is $a \in \mathbb{C}$ and $\Omega \in C^\infty(S^{n-1})$ with vanishing integral such that*

$$T\varphi := (m\hat{\varphi})^\vee = a\varphi + T_\Omega\varphi \quad \textit{for } \varphi \in \mathscr{S}_n.$$

Proof. Let $a \in \mathbb{C}$ be such that $\int_{S^{n-1}}(m(\theta) - a)\, d\sigma(\theta) = 0$. Then by Theorem 6.37 there is Ω such that $Tf - af = T_\Omega$. $\square$

Corollary 6.43. *The class $\mathcal{A}$ of operators of the form*

$$aI + T_\Omega$$

where $a \in \mathbb{C}$ and $\Omega \in C^\infty(S^{n-1})$ has vanishing integral coincides with the class of operators of the form

$$T\varphi = (m\hat{\varphi})^\vee,$$

where $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree zero. Hence $\mathcal{A}$ is a commutative algebra. An operator in the class $\mathcal{A}$ is invertible if and only if $m = a + \widehat{W_\Omega}$ does not vanish at any point of S^{n-1}.

7. Riesz potentials and fractional powers of the Laplacian

Recall that for $0 < \alpha < n$ the *Riesz potential* I_α is defined by[84]

$$(I_\alpha f)(x) = (U_\alpha * f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy\,,$$

where

$$\gamma(\alpha) = \frac{\pi^{\frac{n}{2}}\, 2^\alpha\, \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \quad \text{so} \quad U_\alpha = (2\pi)^{-\alpha} \frac{\pi^{\alpha-\frac{n}{2}}\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} |x|^{\alpha-n}\,.$$

Clearly, $I_\alpha\varphi$ is smooth, slowly increasing, and all its derivatives are slowly increasing when $\varphi \in \mathscr{S}_n$.[85]

A relationship between Riesz potentials and Riesz transforms is explained in the next result

Proposition 7.1. *If $n \geq 2$ and $\varphi \in \mathscr{S}_n$, then for $1 \leq j \leq n$*

$$\frac{\partial}{\partial x_j}(I_1\varphi)(x) = -R_j\varphi(x)\,.$$

Proof. This result easily follows from Proposition 5.53

$$\begin{aligned}
\frac{\partial}{\partial x_j}(I_1\varphi) &= \left(\frac{\partial U_2}{\partial x_j}\right) * \varphi = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \left(\frac{\partial}{\partial x_j}|x|^{1-n}\right) * \varphi \\
&= (1-n)\frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \left(\text{p.v. } \frac{x_j}{|x|^{n+1}}\right) * \varphi = -R_j\varphi.
\end{aligned}$$

□

According to Theorem 5.42, $\widehat{U_\alpha}(\xi) = (2\pi)^{-\alpha}|\xi|^{-\alpha} = (4\pi^2|\xi|^2)^{-\alpha/2}$. Since $(I_\alpha\varphi)^\wedge = \hat{\varphi}\widehat{U_\alpha}$ we get

Theorem 7.2. *For $0 < \alpha < n$ and $\varphi \in \mathscr{S}_n$,*[86]

$$(I_\alpha\varphi)^\wedge(\xi) = (4\pi^2|\xi|^2)^{-\alpha/2}\hat{\varphi}(\xi), \tag{7.1}$$

[84] c.f. Definition 2.14.

[85] However, in general $I_\alpha\varphi \notin \mathscr{S}_n$, see Remark 7.5.

[86] The Fourier transform in (7.1) is understood as a Fourier transform of a tempered distribution $U_\alpha * \varphi$. On the other hand $(4\pi^2|\xi|^2)^{-\alpha/2}\hat{\varphi} \in L^1$ so the inverse Fourier transform in (7.2) is just a regular inverse Fourier transform of an integrable function.

i.e.,

$$I_\alpha\varphi = \big((4\pi^2|\xi|^2)^{-\alpha/2}\hat{\varphi}(\xi)\big)^\vee. \tag{7.2}$$

Recall that when $n\geq 3$ the fundamental solution of the Laplace operator is[87]

$$\Phi(x) = -\frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}} = -U_2(x).$$

That means

$$-\Delta(I_2\varphi) = -\Delta(U_2*\varphi) = U_2*((-\Delta)\varphi) = I_2((-\Delta)\varphi) = \Delta(\Phi*\varphi) = \delta_0*\varphi = \varphi. \tag{7.3}$$

Hence in some sense the Riesz potential I_2 is the inverse of $-\Delta$ so it is reasonable to use notation

$$(-\Delta)^{-1}\varphi = I_2\varphi = \big((4\pi^2|\xi|^2)^{-1}\hat{\varphi}(\xi)\big)^\vee.$$

With this notation (7.3) reads as

$$(-\Delta)(-\Delta)^{-1}\varphi = (-\Delta)^{-1}(-\Delta)\varphi = \varphi.$$

Also for $\varphi\in\mathscr{S}_n$ and a positive integer k we have

$$-\Delta\varphi = \big(4\pi^2|\xi|^2\hat{\varphi}(\xi)\big)^\vee \quad\text{so}\quad (-\Delta)^k\varphi = \big((4\pi^2|\xi|^2)^k\hat{\varphi}(\xi)\big)^\vee.$$

This suggests how to define fractional powers of the Laplace operator, including powers with negative exponents

Definition 7.3. For $\alpha>-n$ and $\varphi\in\mathscr{S}_n$ we define

$$(-\Delta)^{\alpha/2}\varphi = \big((4\pi^2|\xi|^2)^{\alpha/2}\hat{\varphi}(\xi)\big)^\vee.$$

Thus

$$I_\alpha\varphi = \big((4\pi^2|\xi|^2)^{-\alpha/2}\hat{\varphi}(\xi)\big)^\vee = (-\Delta)^{-\alpha/2}\varphi \quad \text{for } 0<\alpha<n.$$

Proposition 7.4. *For any $\alpha>-n$ there is $u\in\mathscr{S}_n'$ such that $(-\Delta)^{\alpha/2}\varphi = u*\varphi$ for $\varphi\in\mathscr{S}_n$. In particular $(-\Delta)^{\alpha/2}\varphi$ is smooth, slowly increasing, and all its derivatives are slowly increasing.*

Proof. If $0>\alpha>-n$, then $(-\Delta)^{\alpha/2}\varphi = I_{-\alpha}\varphi = U_{-\alpha}*\varphi$ so it remains to consider the case $\alpha\geq 0$. However, our argument will work for all $\alpha>-n$. Let[88]

$$v(\xi)=\tilde{v}(\xi) = (4\pi^2|\xi|^2)^{\alpha/2}\in\mathscr{S}_n' \quad\text{and}\quad u=\hat{v} \quad\text{so}\quad \hat{u}=\tilde{v}=v.$$

Then

$$(u*\varphi)^\wedge = \hat{\varphi}\hat{u} = \hat{\varphi}v = (4\pi^2|\xi|^2)^{\alpha/2}\hat{\varphi}(\xi) = \big((-\Delta)^{\alpha/2}\varphi\big)^\wedge(\xi).$$

□

[87] Theorem 5.60.

[88] Since $\alpha>-n$, the singularity of v at $\xi=0$ is integrable and hence the function v defines a tempered distribution.

Remark 7.5. Although $(-\Delta)^{\alpha/2}\varphi$ is always smooth, it is not always in the Schwarz class $\mathscr{S}_n$. Namely for every $\alpha \notin \{0,2,4,6,\ldots\}$, $\alpha > -n$, there is $\varphi \in \mathscr{S}_n$ such that $(-\Delta)^{\alpha/2}\varphi \notin \mathscr{S}_n$. Suppose to the contrary that for some $\alpha \notin \{0,2,4,6,\ldots\}$, $\alpha > -n$, and all $\varphi \in \mathscr{S}_n$ we have $(-\Delta)^{\alpha/2}\varphi \in \mathscr{S}_n$. Then the Fourier transform of this function also belongs to $\mathscr{S}_n$ i.e.,

$$(4\pi^2|\xi|^2)^{\alpha/2}\hat{\varphi}(\xi) \in \mathscr{S}_n. \tag{7.4}$$

However $|\xi|^\alpha$ is not C^∞ smooth at $\xi = 0$ so if $\hat{\varphi}(0) \neq 0$, the function (7.4) is not C^∞ smooth at $\xi = 0$ and hence it cannot belong to $\mathscr{S}_n$.

On the other hand if $\alpha = 2k \in \{0,2,4,6,\ldots\}$, then

$$(-\Delta)^{\alpha/2} = (-\Delta)^k : \mathscr{S}_n \to \mathscr{S}_n$$

since the Laplace operator maps $\mathscr{S}_n$ to $\mathscr{S}_n$ and so does the k-fold composition of $(-\Delta)$.

Proposition 7.6. *If $\alpha > -n$, then for $\varphi \in \mathscr{S}_n$ we have*

$$(-\Delta)(-\Delta)^{\alpha/2}\varphi = (-\Delta)^{\alpha/2}(-\Delta)\varphi = (-\Delta)^{\frac{\alpha+2}{2}}\varphi.$$

Remark 7.7. We need to understand this result as follows. The function $(-\Delta)^{\alpha/2}\varphi$ is smooth so we can apply the classical Laplace operator $-\Delta$ to it. Also since $(-\Delta)^{\alpha/2}\varphi$ is slowly increasing and all its derivatives are slowly increasing, the classical Laplace operator coincides with the distributional one (since there is no problem with the integration by parts). This is how we understand the left hand side. In the middle term $(-\Delta)\varphi \in \mathscr{S}_n$ so we can apply $(-\Delta)^{\alpha/2}$ to it. And finally the the term on the right hand side is well defined because $\varphi \in \mathscr{S}_n$.

Proof. According to Proposition 7.4, $(-\Delta)^{\alpha/2}\varphi = u * \varphi$ for some $u \in \mathscr{S}'_n$. Hence

$$\begin{aligned}(-\Delta)((-\Delta)^{\alpha/2}\varphi) &= -\Delta(u*\varphi) = u * (\underbrace{-\Delta\varphi}_{\text{in } \mathscr{S}_n}) = (-\Delta)^{\alpha/2}(-\Delta\varphi)\\ &= \Big((4\pi^2|\xi|^2)^{\alpha/2}(\widehat{-\Delta\varphi})\Big)^\vee = \Big((4\pi^2|\xi|^2)^{\alpha/2}(4\pi^2|\xi|^2)\hat{\varphi}\Big)^\vee = (-\Delta)^{\frac{\alpha+2}{2}}\varphi.\end{aligned}$$

□

Remark 7.8. *Formally* we can extend the above result to composition of operators $(-\Delta)^{\alpha/2}(-\Delta)^{\beta/2}$ as follows

$$\begin{aligned}\big((-\Delta)^{\alpha/2}(-\Delta)^{\beta/2}\varphi)\big)^\wedge &= (4\pi^2|\xi|^2)^{\alpha/2}(4\pi^2|\xi|^2)^{\beta/2}\hat{\varphi}(\xi)\\ &= (4\pi^2|\xi|^2)^{\frac{\alpha+\beta}{2}}\hat{\varphi}(\xi) = \Big((-\Delta)^{\frac{\alpha+\beta}{2}}\varphi\Big)^\wedge\end{aligned}$$

so *formally*

$$(-\Delta)^{\alpha/2}(-\Delta)^{\beta/2} = (-\Delta)^{\frac{\alpha+\beta}{2}} \quad \text{on } \mathscr{S}_n \tag{7.5}$$

provided $\alpha, \beta > -n$ and $\alpha + \beta > -n$.

The problem is that in general if $\varphi \in \mathscr{S}_n$, then $(-\Delta)^{\beta/2}\varphi \notin \mathscr{S}_n$ so in the composition

$$(-\Delta)^{\alpha/2}\big((-\Delta)^{\beta/2}\varphi\big)$$

we apply the operator $(-\Delta)^{\alpha/2}$ to a function which is not in $\mathscr{S}_n$ and Definition 7.3 requires the function to which we apply $(-\Delta)^{\alpha/2}$ to be in the space $\mathscr{S}_n$. Well, the formula that defines $(-\Delta)^{\alpha/2}$ actually applies to a much larger class of function than just $\mathscr{S}_n$ and with this extension in mind we can justify the above composition formula. There is however, a price we have to pay for it. If $\varphi \in \mathscr{S}_n$, then[89] $(-\Delta)^{\alpha/2}\varphi = u_\alpha * \varphi$ and $(-\Delta)^{\beta/2}\varphi = u_\beta * \varphi$ for some $u_\alpha, u_\beta \in \mathscr{S}_n'$ and we would like to write

$$(-\Delta)^{\alpha/2}(-\Delta)^{\beta/2}\varphi = u_\alpha * (u_\beta * \varphi) \tag{7.6}$$

and this does not make any sense if $u_\beta * \varphi = (-\Delta)^{\beta/2}\varphi \notin \mathscr{S}_n$. An example where this creates a problem will be seen in the next result. A formal proof, similar to the arguments used above will be very short and elegant, but the actual rigorous proof will be quite long and boring.

Theorem 7.9. *If $\alpha, \beta > 0$, $\alpha + \beta < n$, then*

$$I_\alpha(I_\beta\varphi) = I_{\alpha+\beta}\varphi \quad \textit{for } \varphi \in \mathscr{S}_n. \tag{7.7}$$

Remark 7.10. Formally we can write this equality as

$$(-\Delta)^{-\alpha/2}(-\Delta)^{-\beta/2}\varphi = (-\Delta)^{-\frac{\alpha+\beta}{2}}\varphi \tag{7.8}$$

when $\alpha, \beta > 0$, $\alpha + \beta < n$ and $\varphi \in \mathscr{S}_n$. This looks like a special case of the situation considered in Remark 7.8 so, does it complete the proof of Theorem 7.9? Not really, because we run into the problem described in (7.6). The equality can be written as

$$U_\alpha * (U_\beta * \varphi) = U_{\alpha+\beta} * \varphi$$

If we would know that $I_\beta\varphi = U_\beta * \varphi \in \mathscr{S}_n$ we could apply the Fourier transform and easily prove the identity, but in general, without this information, we have no reason to claim that $(U_\alpha * (U_\beta * \varphi))^\wedge = \widehat{U_\beta * \varphi}\, \widehat{U_\alpha}$.

Proof.

Lemma 7.11. *If $\alpha, \beta > 0$, $\alpha + \beta < n$, then there is a constant*[90] *$C_0 = C_0(\alpha, \beta, n)$ such that*

$$\int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n-\alpha}|y|^{n-\beta}} = \frac{C_0}{|x|^{n-(\alpha+\beta)}}\, .$$

Proof. First observe that the integral on the left hand side is finite whenever $x \neq 0$. To see this let $R = |x|$ and divide the integral over $\mathbb{R}^n$ into four integrals over mutually disjoint regions

$$\int_{B(0,\frac{R}{2})} \dots dy, \quad \int_{B(x,\frac{R}{2})} \dots dy, \quad \int_{B(0,2R)\setminus(B(0,\frac{R}{2})\cup B(x,\frac{R}{2}))} \dots dy, \quad \int_{\mathbb{R}^n\setminus B(0,2R)} \dots dy.$$

It is very easy to see that the first three integrals are finite. Indeed, in the first integral we have a singularity $|y|^{n-\beta}$ at $y = 0$ only and this singularity is integrable. In the second integral we have a singularity $|x-y|^{n-\alpha}$ at $y = x$ only which is integrable again and in the third integral we do not have singularities at all. It remains to prove finiteness of the last integral.

[89] Proposition 7.4.

[90] In Corollary 7.12 we will find the exact value of C_0.

If $|y| \geq 2R$, then $|x-y| \geq |y| - |x| \geq |y| - |y|/2 = |y|/2$ so

$$\int_{\mathbb{R}^n \setminus B(0,2R)} \frac{dy}{|x-y|^{n-\alpha}|y|^{n-\beta}} \leq 2^{n-\alpha} \int_{\mathbb{R}^n \setminus B(0,2R)} \frac{dy}{|y|^{2n-(\alpha+\beta)}} < \infty,$$

because $2n - (\alpha+\beta) > n$.

Note that the integral from the lemma is invariant under rotations and hence

$$\int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n-\alpha}} |y|^{n-\beta} = f(|x|).$$

For $x \neq 0$ let $x_0 = x/|x|$ and $t = |x|$ so $x = tx_0$. A simple change of variables gives

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n-\alpha}|y|^{n-\beta}} &= \int_{\mathbb{R}^n} \frac{dy}{|tx_0-y|^{n-\alpha}|y|^{n-\beta}} \\ &= t^{\alpha+\beta-n} \int_{\mathbb{R}^n} \frac{dy}{|x_0-y|^{n-\alpha}|y|^{n-\beta}} = |x|^{\alpha+\beta-n} f(1). \end{aligned}$$

□

We have

$$\begin{aligned} I_\alpha(I_\beta\varphi) &= \frac{1}{\gamma(\alpha)\gamma(\beta)} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} \left(\int_{\mathbb{R}^n} \frac{\varphi(z)}{|y-z|^{n-\beta}}\, dz \right) dy \\ &= \frac{1}{\gamma(\alpha)\gamma(\beta)} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n-\alpha}|y-z|^{n-\beta}} \right) \varphi(z)\, dz \\ &= \frac{1}{\gamma(\alpha)\gamma(\beta)} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{dy}{|(x-z)-y|^{n-\alpha}|y|^{n-\beta}} \right) \varphi(z)\, dz \\ &= \frac{C_0}{\gamma(\alpha)\gamma(\beta)} \int_{\mathbb{R}^n} \frac{\varphi(z)}{|x-z|^{n-(\alpha+\beta)}}\, dz \\ &= \frac{C_0\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} I_{\alpha+\beta}\varphi(x). \end{aligned}$$

We could use the Fubini theorem here because the function that we integrated over $\mathbb{R}^n \times \mathbb{R}^n$ was integrable. To see the integrability, repeat the above computations with φ replaced by $|\varphi|$.

It remains to show that

$$C_0 = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha+\beta)}.$$

To prove this it suffices to verify that $I_\alpha(I_\beta\varphi) = I_{\alpha+\beta}\varphi$ for just one non-zero function φ. To this end let $\varphi \in \mathscr{S}_n$ be such that $\hat{\varphi} = 0$ in a neighborhood of 0. Then

$$I_\alpha(I_\beta\varphi) = I_\alpha\Big(\big(\underbrace{(4\pi^2|\xi|^2)^{-\beta/2}\hat{\varphi}}_{\in \mathscr{S}_n} \big)^\vee \Big) = \big((4\pi^2|\xi|^2)^{-\alpha/2}(4\pi^2|\xi|^2)^{-\beta/2}\hat{\varphi} \big)^\vee = I_{\alpha+\beta}\varphi.$$

The proof is complete. □

As a corollary we obtain

Corollary 7.12. *If $\alpha, \beta > 0$, $\alpha+\beta<n$, then*

$$\int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n-\alpha}|y|^{n-\beta}} = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha+\beta)}\, \frac{1}{|x|^{n-(\alpha+\beta)}}\, .$$

In the next result we will prove an interesting formula for the fractional Laplacian $(-\Delta)^{\alpha/2}$ when $0<\alpha<2$.

Theorem 7.13. *For $0<\alpha<2$ and $\varphi\in\mathscr{S}_n$ we have*

$$\begin{aligned}(-\Delta)^{\alpha/2}\varphi &= \lim_{\varepsilon\to 0}\frac{1}{\gamma(-\alpha)}\int_{|x-y|\geq\varepsilon}\frac{\varphi(y)-\varphi(x)}{|y-x|^{n+\alpha}}\, dy\\ &= \frac{1}{2\gamma(-\alpha)}\int_{\mathbb{R}^n}\frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+\alpha}}\, dy,\end{aligned}$$

where[91]

$$\gamma(-\alpha) = \frac{\pi^{\frac{n}{2}}2^{-\alpha}\Gamma\left(-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\, .$$

Proof. First we will prove equality of the integrals

$$\frac{1}{2}\int_{\mathbb{R}^n}\frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+\alpha}}\, dy = \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{\varphi(y)-\varphi(x)}{|y-x|^{n+\alpha}}\, dy. \tag{7.9}$$

It easily follows from Taylor's formula that

$$\frac{|\varphi(x+y)+\varphi(x-y)-2\varphi(x)|}{|y|^{n+\alpha}} \leq C\frac{\Vert D^2\varphi\Vert_\infty}{|y|^{n+\alpha-2}}\, ,$$

where

$$\Vert D^2\varphi\Vert_\infty = \sup_{x\in\mathbb{R}^n}\left(\sum_{j,k=1}^n\left|\frac{\partial^2\varphi}{\partial x_j\partial x_k}(x)\right|^2\right)^{1/2}\, .$$

Since $n+\alpha-2<n$, both sides of this inequality are integrable over the ball $\{|y|\leq 1\}$. They are also integrable in $\{|y|>1\}$ because of rapid decay of φ. Thus the first integral at (7.9) is well defined and finite. For the second integral we have

$$\int_{|x-y|\geq\varepsilon}\frac{\varphi(y)-\varphi(x)}{|y-x|^{n+\alpha}}\, dy = \int_{|y|\geq\varepsilon}\frac{\varphi(x+y)-\varphi(x)}{|y|^{n+\alpha}}\, dy = \int_{|y|\geq\varepsilon}\frac{\varphi(x-y)-\varphi(x)}{|y|^{n+\alpha}}\, dy$$

so

$$\int_{|x-y|\geq\varepsilon}\frac{\varphi(y)-\varphi(x)}{|y-x|^{n+\alpha}}\, dy = \frac{1}{2}\int_{|y|\geq\varepsilon}\frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+\alpha}}\, dy$$

and hence passing to the limit as $\varepsilon\to 0^+$ yields (7.9).

Observe that

$$(x,y)\mapsto\frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+\alpha}}\in L^1(\mathbb{R}^n\times\mathbb{R}^n). \tag{7.10}$$

[91]We use here Γ evaluated at a negative number. Actually, $\Gamma(-\frac{\alpha}{2}) = -\frac{2}{\alpha}\Gamma(1-\frac{\alpha}{2})$, where $1-\frac{\alpha}{2}>0$, see Definition 3.35. Note also that $\gamma(-\alpha)$ is given by the same formula as the constant in the definition of the Riesz potential.

Indeed, for $(x,y)\in\mathbb{R}^n\times B(0,1)$ we have

$$\left|\frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+\alpha}}\right|\le C\,\frac{\|D^2\varphi\|_{L^\infty(B(x,1))}}{|y|^{n+\alpha-2}}\le C\,\frac{(1+|x|)^{-2n}}{|y|^{n+\alpha-2}}\in L^1(\mathbb{R}^n\times B(0,1))$$

and

$$\int_{|y|\ge 1}\int_{\mathbb{R}^n}\frac{|\varphi(x+y)+\varphi(x-y)-2\varphi(x)|}{|y|^{n+\alpha}}\,dx\,dy\le 3\|\varphi\|_1\int_{|y|\ge 1}\frac{dy}{|y|^{n+\alpha}}<\infty.$$

This allows us to compute the Fourier transform of the integrable function

$$f(x)=\int_{\mathbb{R}^n}\frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+\alpha}}\,dy \tag{7.11}$$

by changing the order of integration. We have

$$\begin{aligned}\hat f(\xi) &= \int_{\mathbb{R}^n}e^{-2\pi i x\cdot\xi}\int_{\mathbb{R}^n}\frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+\alpha}}\,dy\,dx\\ &= \int_{\mathbb{R}^n}\frac{1}{|y|^{n+\alpha}}\left(\int_{\mathbb{R}^n}e^{-2\pi i x\cdot\xi}(\varphi(x+y)+\varphi(x-y)-2\varphi(x))\,dx\right)dy\\ &= \hat\varphi(\xi)\int_{\mathbb{R}^n}\frac{e^{2\pi i y\cdot\xi}+e^{-2\pi i y\cdot\xi}-2}{|y|^{n+\alpha}}\,dy\\ &= -2\hat\varphi(\xi)\int_{\mathbb{R}^n}\frac{1-\cos(2\pi y\cdot\xi)}{|y|^{n+\alpha}}\,dy.\end{aligned}$$

Let $\rho\in O(n)$ be such that $\rho(\xi)=|\xi|e_1$. Then the change of variables $\tilde y=\rho(y)$ gives

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{1-\cos(2\pi y\cdot\xi)}{|y|^{n+\alpha}}\,dy &= \int_{\mathbb{R}^n}\frac{1-\cos(2\pi\rho^{-1}(y)\cdot\xi)}{|\rho^{-1}(y)|^{n+\alpha}}\,dy=\int_{\mathbb{R}^n}\frac{1-\cos(2\pi y\cdot\rho(\xi))}{|y|^{n+\alpha}}\,dy\\ &= \int_{\mathbb{R}^n}\frac{1-\cos(2\pi|\xi|y_1)}{|y|^{n+\alpha}}\,dy\\ &= (2\pi|\xi|)^\alpha\int_{\mathbb{R}^n}\frac{1-\cos y_1}{|y|^{n+\alpha}}\,dy\end{aligned}$$

where in the last equality we used another change of variables $\tilde y=2\pi|\xi|y$. Actually it is a consequence of the above calculation that the function $(1-\cos y_1)/|y|^{n+\alpha}$ is integrable on $\mathbb{R}^n$, but one can see it more directly by using the estimate $|1-\cos y_1|\le C|y|^2$ in a neighborhood of the origin. Let

$$C(\alpha)=\int_{\mathbb{R}^n}\frac{1-\cos y_1}{|y|^{n+\alpha}}\,dy.$$

We proved that

$$\hat f(\xi)=-2\hat\varphi(\xi)(2\pi|\xi|)^\alpha C(\alpha)=-2C(\alpha)\big((-\Delta)^{\alpha/2}\varphi\big)^\wedge(\xi).$$

Hence

$$\begin{aligned}(-\Delta)^{\alpha/2}\varphi(x)=-\frac{1}{2C(\alpha)}f(x) &= -\frac{1}{2C(\alpha)}\int_{\mathbb{R}^n}\frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+\alpha}}\,dy\\ &= -\frac{1}{C(\alpha)}\lim_{\varepsilon\to 0}\int_{|y|\ge\varepsilon}\frac{\varphi(x+y)-\varphi(x)}{|y|^{n+\alpha}}\,dy.\end{aligned}$$

It remains to prove that

$$C(\alpha) = -\frac{\pi^{\frac{n}{2}} 2^{-\alpha}\Gamma\left(-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\,. \tag{7.12}$$

Let $\varphi(x) = e^{-\pi|x|^2}$. Since $\hat{\varphi} = \varphi$ we have

$$(-\Delta)^{\alpha/2}\varphi(0) = -\frac{1}{C(\alpha)} \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} \frac{\varphi(0+y)-\varphi(0)}{|y|^{n+\alpha}}\, dy = -\frac{1}{C(\alpha)}\int_{\mathbb{R}^n} \frac{e^{-\pi|y|^2}-1}{|y|^{n+\alpha}}\, dy. \tag{7.13}$$

Both sides of this equality are easy to compute. Recall that according to Lemma 5.44 for $\gamma > -n$ we have

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^\gamma\, dx = n\omega_n \int_0^\infty e^{-\pi s^2} s^{n+\gamma-1}\, ds = \frac{\Gamma\left(\frac{n+\gamma}{2}\right)}{\pi^{\frac{\gamma}{2}}\Gamma\left(\frac{n}{2}\right)}\,.$$

Since for $g\in L^1$, $\check{g}(0) = \int_{\mathbb{R}^n} g(x)\, dx$, we obtain

$$\begin{aligned} (-\Delta)^{\alpha/2}\varphi(0) &= \left((4\pi^2|\xi|^2)^{\alpha/2}\hat{\varphi}(\xi)\right)^\vee(0) = \int_{\mathbb{R}^n} (2\pi|\xi|)^\alpha e^{-\pi|\xi|^2}\, d\xi \\ &= (2\pi)^\alpha \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{\frac{\alpha}{2}}\Gamma\left(\frac{n}{2}\right)} = \frac{2^\alpha \pi^{\frac{\alpha}{2}}\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \end{aligned} \tag{7.14}$$

On the other hand

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{e^{-\pi|y|^2}-1}{|y|^{n+\alpha}}\, dy &= n\omega_n \int_0^\infty s^{n-1}\frac{e^{-\pi s^2}-1}{s^{n+\alpha}}\, ds = n\omega_n \int_0^\infty (e^{-\pi s^2}-1)\left(\frac{s^{-\alpha}}{-\alpha}\right)'\, ds \\ &= \frac{n\omega_n}{\alpha}\int_0^\infty (e^{-\pi s^2}-1)' s^{-\alpha}\, ds = -2\pi\,\frac{n\omega_n}{\alpha}\int_0^\infty e^{-\pi s^2}s^{1-\alpha}\, ds \\ &\overset{\gamma=2-\alpha-n}{=} -\frac{2\pi}{\alpha}\, n\omega_n \int_0^\infty e^{-\pi s^2} s^{n+\gamma-1}\, ds = -\frac{2\pi}{\alpha}\frac{\Gamma\left(\frac{n+\gamma}{2}\right)}{\pi^{\frac{\gamma}{2}}\Gamma\left(\frac{n}{2}\right)} \\ &= \pi^{\frac{\alpha+n}{2}}\frac{\Gamma\left(-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \end{aligned}$$

This identity, (7.13) and (7.14) yield

$$\frac{2^\alpha \pi^{\frac{\alpha}{2}}\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = -\frac{1}{C(\alpha)}\,\pi^{\frac{\alpha+n}{2}}\,\frac{\Gamma\left(-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

which easily implies (7.12). □

Corollary 7.14. *For* $0<\alpha<2$

$$\int_{\mathbb{R}^n} \frac{1-\cos y_1}{|y|^{n+\alpha}}\, dy = -\frac{\pi^{\frac{n}{2}}2^{-\alpha}\Gamma\left(-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\,.$$

Corollary 7.15. *If* $0<\alpha<2$ *and* $\varphi\in\mathscr{S}_n$, *then* $(-\Delta)^{\alpha/2}\varphi \in L^1(\mathbb{R}^n)$.

Proof.

$$(-\Delta)^{\alpha/2}\varphi = \frac{1}{2\gamma(-\alpha)}\int_{\mathbb{R}^n} \frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+\alpha}}\, dy = \frac{1}{2\gamma(-\alpha)} f(x) \in L^1$$

since we proved in (7.10) that the function f defined in (7.11) is integrable. □

It follows from Proposition 7.6 that is $0 < \alpha < 2$ and $k \geq 0$ is an integer, then

$$(-\Delta)^{\frac{\alpha}{2}+k}\varphi = (-\Delta)^{\frac{\alpha}{2}}(-\Delta)^k\varphi = (-1)^k(-\Delta)^{\frac{\alpha}{2}}(\Delta^k\varphi)$$

and hence Theorem 7.13 can be generalized to the case of higher powers as follows

Corollary 7.16. *If* $0 < \alpha < 2$*,* $k \geq 0$ *is an integer and* $\varphi \in \mathscr{S}_n$*, then*

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}+k}\varphi &= \lim_{\varepsilon\to 0}\frac{(-1)^k}{\gamma(-\alpha)}\int_{|x-y|\geq\varepsilon}\frac{\Delta^k\varphi(y)-\Delta^k\varphi(x)}{|y-x|^{n+\alpha}}\,dy \\ &= \frac{(-1)^k}{2\gamma(-\alpha)}\int_{\mathbb{R}^n}\frac{\Delta^k\varphi(x+y)+\Delta^k\varphi(x-y)-2\Delta^k\varphi(x)}{|y|^{n+\alpha}}\,dy,\end{aligned}$$

where

$$\gamma(-\alpha) = \frac{\pi^{\frac{n}{2}}2^{-\alpha}\Gamma\left(-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}.$$