NON-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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1. The classical Dirichlet problem and the origin of Sobolev spaces

The classical *Dirichlet problem* reads as follows. Given an open domain $\Omega \subset \mathbb{R}^n$ and $g \in C^0(\partial\Omega)$. Find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega\,, \\ u|_{\partial\Omega} = g\,. \end{cases} \tag{1.1}$$

This problem originates from the XIX'th century physics. To begin with, we shall provide a brief explanation of the physical context of the equation. This will also provide us with an intuition for the principles of the calculus of variations that will be developed later.

Let $\Omega \subset \mathbb{R}^3$ be a vacuum region and let $E : \Omega \to \mathbb{R}^3$ be an electric field (electrostatic case). Given two points $x, y \in \Omega$, the integral

$$\int_x^y E \cdot ds \tag{1.2}$$

does not depend on the choice of the curve that joins x with y inside Ω (provided Ω is simply connected). By the definition, (1.2) equals to

$$\int_a^b E(\gamma(t)) \cdot \dot{\gamma}(t)\, dt,$$

where γ is a parametrization of the given curve that joins x and y. Here and in the sequel $A \cdot B$ denotes the scalar product. We will also denote the scalar product by $\langle A, B \rangle$.

Fix $x_0 \in \Omega$ and define the potential u as follows

$$u(x) = -\int_{x_0}^x E \cdot ds.$$

The potential u is a scalar function defined up to a constant (since we can change the base point x_0). We have

$$E = -\nabla u\,.$$

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It is well known that the electric field is divergence free, $\operatorname{div} E = 0$, and hence

$$\Delta u = \operatorname{div} \nabla u = -\operatorname{div} E = 0.$$

Thus the potential u is a harmonic function inside Ω. Assume that the boundary $\partial\Omega$ of the vacuum region contains electrical charge that induces potential g on $\partial\Omega$. This electrical charge induces an electric field in Ω, so the induced potential u in Ω satisfies $\Delta u = 0$ in Ω and $u|_{\partial\Omega} = g$. Thus u is a solution to the Dirichlet problem stated at the beginning.

The energy of the electric field (up to a constant factor) is given by the formula

$$\text{Energy} = \int_\Omega |E|^2 = \int_\Omega |\nabla u|^2 .$$

It is a general principle in physics that all the systems approach a configuration with the minimal energy. Thus given potential g on the boundary $\partial\Omega$ one may expect that induced potential u in Ω has the property that it minimizes the *Dirichlet integral*

$$I(u) = \int_\Omega |\nabla u|^2$$

among all the functions[1] $u \in C^2(\overline{\Omega})$ such that $u|_{\partial\Omega} = g$. Indeed, we have

Theorem 1.1 (Dirichlet principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open and bounded set and let $u \in C^2(\Omega)$. Then the following conditions are equivalent:*

(a) $\Delta u = 0$ *in* Ω,
(b) *u is a critical point of the functional I in the sense that*

$$\left.\frac{d}{dt}\right|_{t=0} I(u + t\varphi) = 0 \quad \textit{for all } \varphi \in C_0^\infty(\Omega).$$

If in addition $u \in C^2(\overline{\Omega})$, and $u|_{\partial\Omega} = g$, then we have one more equivalent condition:

(c) *u minimizes I in the sense that $I(u) \le I(w)$ for all $w \in C^2(\overline{\Omega})$ with $w|_{\partial\Omega} = g$.*

Remark 1.2. The assumption $u \in C^2(\overline{\Omega})$ is certainly too strong, but it is not our purpose to prove the result under minimal assumptions.

[1] $C^2(\overline{\Omega})$ is the class of functions on Ω that can be extended as C^2 functions to $\mathbb{R}^n$. We do not impose any regularity conditions on $\partial\Omega$. By $C_0^\infty(\Omega)$ we shall denote the class of smooth functions compactly supported in Ω.

Proof of Theorem 1.1. To prove the equivalence between (a) and (b) observe that for any $u \in C^2(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\int_\Omega \nabla(u+t\varphi)|^2 &= \frac{d}{dt}\Big|_{t=0}\int_\Omega \langle \nabla u + t\nabla\varphi, \nabla u + t\nabla\varphi\rangle \\ = \ & \frac{d}{dt}\Big|_{t=0}\left(\int_\Omega |\nabla u|^2 + 2t\int_\Omega \langle \nabla u, \nabla \varphi\rangle + t^2\int_\Omega |\nabla\varphi|^2\right) \\ = \ & 2\int_\Omega \langle \nabla u, \nabla\varphi\rangle = -\int_\Omega (\Delta u)\varphi.\end{aligned}$$

The last equality follows from integration by parts. Now the implication (a)⇒(b) is obvious. The implication (b)⇒(a) follows from the following important lemma.

Lemma 1.3. *If $f \in L^1_{\mathrm{loc}}(\Omega)$ satisfies $\int_\Omega f\varphi = 0$ for any $\varphi \in C_0^\infty(\Omega)$, then $f = 0$ a.e.*

Proof. Suppose that $f \not\equiv 0$. We can assume f is positive on a set of positive measure (otherwise we replace f by $-f$). Then there is a compact set $K \subset \Omega$, $|K| > 0$ and $\varepsilon > 0$ such that $f \geq \varepsilon$ on K.

Let G_i be a sequence of open sets such that $K \subset G_i \subset\subset \Omega$, $|G_i \setminus K| \to 0$ as $i \to \infty$. Take $\varphi_i \in C_0^\infty(G_i)$ with $0 \leq \varphi \leq 1$, $\varphi_i|_K \equiv 1$. Then

$$0 = \int_\Omega f\varphi_i \geq \varepsilon|K| - \int_{G_i\setminus K} |f| \to \varepsilon|K|\,,$$

as $i \to \infty$, which is a contradiction. The proof is complete. □

We are left with the proof of the equivalence with (c) under the given additional regularity assumptions. For $u = w$ on $\partial\Omega$ we have

$$\begin{aligned}\int_\Omega |\nabla w|^2 &= \int_\Omega |\nabla(w-u) + \nabla u|^2 \\ &= \int_\Omega |\nabla(w-u)|^2 + \int_\Omega |\nabla u|^2 + 2\int_\Omega \langle \nabla(w-u), \nabla u\rangle \\ &= \int_\Omega |\nabla(w-u)|^2 + \int_\Omega |\nabla u|^2 - 2\int_\Omega (w-u)\Delta u. \qquad (1.3)\end{aligned}$$

The last equality follows from the integration by parts and the fact that $w - u = 0$ on $\partial\Omega$.

(a)⇒(c) $\Delta u = 0$ and hence the last term in (1.3) equals zero, so $\int_\Omega |\nabla w|^2 > \int_\Omega |\nabla u|^2$ unless $w = u$.

(c)⇒(b) Take $w = u + t\varphi$. Then $w \in C^2(\overline{\Omega})$, $w|_{\partial\Omega} = g$, so $I(t + t\varphi) \geq I(u)$. Thus $t \mapsto I(u + t\varphi)$ attains minimum at $t = 0$ and (b) follows.

1.1. **Direct method in the calculus of variations.** Riemann concluded that the Dirichlet problem was solvable, reasoning that I is nonnegative thus attains a minimum value. Choosing a function u such $I(u) = \min I$, solves the problem.

Of course this "proof" of the existence of the solution is not correct. The function I is defined on an infinite dimensional object: the space of functions and there is no reason why the minimum of I should be attained.

The first rigorous proof of the existence of the solution of the Dirichlet problem was obtained by a different method. Later, however, Hilbert showed that it was possible to solve the Dirichlet problem using Riemann's strategy. This was the beginning of the so called *direct methods in the calculus of variations.* We will explain the principles of this method in a general setting of Banach spaces.

Till the end of the section we assume that $I : X \to \mathbb{R}$ is a function (called functional) defined on a Banach space $(X, \|\cdot\|)$. We want to emphasize that despite the name "functional" we do not assume linearity of I. Actually, in all the interesting instances I will not be linear. We want to find reasonable conditions that will guarantee existence of $\bar{u}$ such that

$$I(\bar{u}) = \inf_{u \in X} I(u). \tag{1.4}$$

Function $\bar{u}$ as in (1.4) is called a *minimizer* of I and the problem of finding a minimizer is called a *variational problem.*

Definition 1.4. We say that I is *sequentially weakly lower semicontinuous* (SWLSC) if for every weakly convergent sequence[2] $u_n \rightharpoonup u$ in X, $I(u) \leq \liminf_{n\to\infty} I(u_n)$. We say that the functional I is *coercive* if $\|u_n\| \to \infty$ implies $I(u_n) \to \infty$.

We shall use the following well known result from functional analysis.

Theorem 1.5. *Every bounded sequence in a reflexive Banach space contains a weakly convergent subsequence.* □

The following result explains the main idea behind the direct method in the calculus of variations.

Theorem 1.6. *If X is a reflexive Banach space and $I : X \to \mathbb{R}$ is SWLSC and coercive, then there exists $\bar{u} \in X$ such that $I(\bar{u}) = \inf_{u\in X} I(u)$.*

Proof. Let u_n be a sequence such that $I(u_n) \to \inf_X I$. Such a sequence shall be called a *minimizing sequence.*

[2]Recall that the weak convergence $u_n \rightharpoonup u$ in X means that for every linear continuous functional $e \in X^*$ we have $\langle e, u_n\rangle \to \langle e, u\rangle$.

Coercivity yields boundedness of the sequence u_n. Since the space is reflexive, we can find a subsequence $u_{n_k} \rightharpoonup \bar{u}$ weakly convergent to some element $\bar{u} \in X$. Then

$$I(\bar{u}) \leq \liminf_{k\to\infty} I(u_{n_k}) = \inf_{u\in X} I(u),$$

and the theorem follows. □

In general, the most difficult condition to deal with is the SWLSC condition. Note that it does not follow from the continuity of I which would be much easier to check.

An important class of functionals for which it is relatively easy to verify the SWLSC condition is the class of convex functionals.

Definition 1.7. A functional $I : X \to \mathbb{R}$ is called *convex* if $I(tu+(1-t)v) \leq tI(u)+(1-t)I(v)$ whenever $t \in [0,1]$ and $u, v \in X$. We say that I is *strictly convex* if $I(tu+(1-t)v) < tI(u)+(1-t)I(v)$ whenever $t \in (0,1)$ and $u \neq v$.

We say that I is *lower semicontinuous* if the convergence in norm $u_n \to u$ implies $I(u) \leq \liminf_{n\to\infty} I(u_n)$.

Theorem 1.8. *If X is a normed space and $I : X \to \mathbb{R}$ is convex and lower semicontinuous, then I is SWLSC.*

Proof. In the proof we will need Mazur's lemma which states that for a weakly convergent sequence $u_n \rightharpoonup u$ in X, a sequence of convex combinations of u_n converges to u in norm.

Lemma 1.9 (Mazur's lemma)**.** *Let X be a normed space and let $u_n \rightharpoonup u$ be a weakly convergent sequence in X. Then $v_n \to u$ in norm for some sequence v_n of the form*

$$v_n = \sum_{k=n}^{N(n)} a_k^n u_k$$

where $a_k^n \geq 0$, $\sum_{k=n}^{N(n)} a_k^n = 1$. □

To prove the theorem we have to prove that $u_n \rightharpoonup u$ implies $I(u) \leq \liminf_{n\to\infty} I(u_n)$. Denote $\liminf_{n\to\infty} I(u_n) = g$. We can find a subsequence of u_n (still denoted by u_n) such that $I(u_n) \to g$. Let v_n be a sequence as in Mazur's lemma, constructed to a selected subsequence u_n. Then by lower semicontinuity and convexity we have

$$I(u) \leq \liminf_{n\to\infty} I(v_n) \leq \liminf_{n\to\infty} \sum_{k=n}^{N(n)} a_k^n I(u_k) = g = \liminf_{n\to\infty} I(u_n).$$

This completes the proof of the theorem. □

Corollary 1.10. *If $I : X \to \mathbb{R}$ is a convex, lower semicontinuous, and coercive functional defined an a reflexive Banach space, then I attains minimum on X, i.e. there exists $\bar{u} \in X$ such that $I(\bar{u}) = \inf_X I(u)$. If, in addition, the functional is strictly convex, the minimum is unique.* □

As we will see, in many cases it is very easy to verify assumptions of the above corollary. Such an abstract approach to the existence of minimizers of variational problems was proposed by Mazur and Schauder in 1936 in the International Congress of Mathematics in Oslo.

1.2. **Origin of Sobolev spaces.** According to the Dirichelt principle, in order to solve the Dirichlet problem (1.1) it suffices to show that the functional $I(u) = \int_\Omega |\nabla u|^2$ attains a minimum in the space

$$C^2_g(\Omega) = \left\{ u \in C^2(\overline{\Omega}) :\, u|_{\partial\Omega} = g \right\} .$$

Let's try to apply Corollary 1.10. The first problem is that the space $C^2_g(\Omega)$ is not linear, but it is easy to handle: Let $w \in C^2(\overline{\Omega})$ be any function such that $w|_{\partial\Omega} = g$. Then

$$C^2_g(\Omega) = w + C^2_b(\overline{\Omega}) = \left\{ w + u :\, u \in C^2_b(\overline{\Omega}) \right\} ,$$

where $C^2_b(\overline{\Omega})$ is a subspace of $C^2(\overline{\Omega})$ consisting of functions vanishing at the boundary. Let $J : C^2_b(\overline{\Omega}) \to \mathbb{R}$ be defined by $J(u) = I(w + u)$. Clearly u is a minimizer of $J : C^2_b(\overline{\Omega}) \to \mathbb{R}$ if and only if $w + u$ is a minimizer of $I : C^2_g(\Omega) \to \mathbb{R}$. Thus an equivalent problem is to find a minimizer of J in the space $C^2_b(\overline{\Omega})$.

If Ω is bounded, $C^2_b(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{C^2} = \sup_{x \in \Omega} \left(|u(x)| + |\nabla u(x)| + |\nabla^2 u(x)| \right) ,$$

where $\nabla^2 u$ denotes the matrix of all second order partial derivatives.

The functional J is convex and continuous. Unfortunately neither the space $C^2_b(\overline{\Omega})$ is reflexive nor the functional J is coercive. If $n \geq 2$, then one can construct a sequence of $C^2_b(\overline{\Omega})$ functions with the supremum norm (and hence the C^2 norm) divergent to infinity, but with the L^2 norm of the gradient approaching to zero. This proves that the functional is not coercive. It is also easy to construct a relevant example when $n = 1$. We leave details to the reader.

To obtain both coercivity and reflexivity we need introduce a different norm

$$\|u\|_{1,2} = \|u\|_2 + \|\nabla u\|_2 .$$

This causes, however, another problem: the space $C^2_b(\overline{\Omega})$ is not complete with respect to this norm, so we need to take a completion of the space.

More precisely, let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \le p < \infty$. The *Sobolev space* $W^{1,p}(\Omega)$ is defined as a completion of the set of $C^\infty(\Omega)$ functions with respect to the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p \, .$$

Of course, we take into account only those C^∞ functions for which the norm is finite.

Since I defined on $C^\infty(\Omega) \cap W^{1,2}(\Omega)$ is locally uniformly continuous[3] with respect to $\| \cdot \|_{1,2}$, it is easy to see that it uniquely extends to a continuous function on $W^{1,2}(\Omega)$.

$W^{1,2}(\Omega)$ is a Hilbert space with the inner product[4]

$$\langle u, v \rangle = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v.$$

In particular it is reflexive. It is still not a correct setting for the Dirichlet problem since we are looking for a minimizer in the class of functions with fixed restriction (called *trace*) on the boundary.

The subspace $W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ is defined as the closure of the subset $C_0^\infty(\Omega)$ in the Sobolev norm. Roughly speaking $W_0^{1,p}(\Omega)$ is a subspace of $W^{1,p}(\Omega)$ consisting of functions which vanish on the boundary.

Fix $w \in W^{1,p}(\Omega)$ and define $W_w^{1,p}(\Omega) = w + W_0^{1,p}(\Omega)$. Thus $W_w^{1,p}(\Omega)$ consists of all those functions in the Sobolev space that, in some sense, have the same trace at the boundary as w. Note that $W_w^{1,p}(\Omega)$ is not linear, but an affine subspace of $W^{1,p}(\Omega)$. The elements of the Sobolev space need not be continuous, so it does not make sense to take restriction to the boundary and we should understand that elements of $W_w^{1,p}(\Omega)$ have the same trace on the boundary as w only in a generalized sense explained above.

Now we are ready to formulate the variational problem of finding minimizer of $I(u) = \int_\Omega |\nabla u|^2$ with given trace on the boundary in the setting of Sobolev spaces.

Variational problem. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $w \in W^{1,2}(\Omega)$. Find $\bar{u} \in W_w^{1,2}(\Omega)$ such that*

$$\int_\Omega |\nabla \bar{u}|^2 = \inf_{u \in W_w^{1,2}(\Omega)} \int_\Omega |\nabla u|^2. \tag{1.5}$$

Theorem 1.11. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $w \in W^{1,2}(\Omega)$. Then there exists unique $\bar{u} \in W_w^{1,2}(\Omega)$ which minimizes the Dirichlet integral in the sense of (1.5).*

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[3] $|I(u) - I(v)| = | \, \|\nabla u\|_2^2 - \|\nabla v\|_2^2 \, | \le \|\nabla u - \nabla v\|_2 (\|\nabla u\|_2 + \|\nabla v\|_2)$.

[4] We consider only real valued functions, so we do not need to take the complex conjugate.

Proof. Let $J(u) = I(u + w)$. Now $J : W_0^{1,2}(\Omega) \to \mathbb{R}$ and the problem of finding a minimizer of I over $W_w^{1,2}(\Omega)$ is equivalent to the problem of finding a minimizer of J over $W_0^{1,2}(\Omega)$.

The functional J is strictly convex and continuous on a reflexive (Hilbert) space $W_0^{1,2}(\Omega)$. To prove existence of a minimizer it remains to prove that J is coercive. Then uniqueness will follow from strict convexity of the functional J.

Lemma 1.12 (Poincaré)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $u \in W_0^{1,p}(\Omega)$, $1 \le p < \infty$, we have*

$$\left(\int_\Omega |u|^p\, dx\right)^{1/p} \le C(p,\Omega) \left(\int_\Omega |\nabla u|^p\, dx\right)^{1/p}.$$

Proof. Assume that $u \in C_0^\infty(\Omega)$. The general case follows by the approximation argument[5]. Let $M > 0$ be such that $\Omega \subset [-M, M]^n$. Then for every $x \in [-M, M]^n$

$$u(x) = \int_{-M}^{x_1} D_1 u(t, x_2, \ldots, x_n)\, dt \le \int_{-M}^{M} |D_1 u|\, dt.$$

Here $D_1 u$ denotes the partial derivative with respect to the first variable. Hölder's inequality yields

$$|u(x)|^p \le 2^{p-1} M^{p-1} \int_{-M}^{M} |D_1 u|^p\, dt,$$

and the assertion follows upon integration of this inequality with respect to x. □

The space $W_0^{1,2}(\Omega)$ is equipped with a norm $\|u\|_{1,2} = \|u\|_2 + \|\nabla u\|_2$. The Poincaré inequality for $p = 2$ states that $\|\nabla u\|_2$ is an equivalent norm on the space $W_0^{1,2}(\Omega)$. Hence for a sequence $u_k \in W_0^{1,2}(\Omega)$, $\|u_k\|_{1,2} \to \infty$ if and only if $\|\nabla u_k\|_2 \to \infty$.

Now the inequality $J(u)^{1/2} = \|\nabla(u+w)\|_2 \ge \|\nabla u\|_2 - \|\nabla w\|_2$ implies that if $\|u_k\|_{1,2} \to \infty$ in $W_0^{1,2}(\Omega)$, then $J(u_k) \to \infty$ which means the functional J is coercive. The proof of Theorem 1.11 is complete. □

Remark 1.13. Observe that $I(u) = \int_\Omega |\nabla u|^2$ is strictly convex and continuous on any of the spaces $W^{1,p}(\Omega)$ for $p \ge 2$. As we will see all the spaces $W^{1,p}$, $1 < p < \infty$ are reflexive. However I is not coercive when $p > 2$.

In the classical setting of C^2 functions, the Dirichlet principle asserts that finding minimizer of the Dirichlet integral is equivalent to solving the Dirichlet problem. We proved that the Dirichlet integral has unique minimizer in

[5]By definition $C_0^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$.

the setting of Sobolev spaces. Does it solve the Dirichlet problem? This is far from being obvious. For the Dirichlet problem the function needs to be twice differentiable, but we only know that the minimizer exists in the Sobolev space. Is there any interpretation of the Dirichlet principle in the setting of Sobolev spaces? How is it related to the classical Dirichlet problem. We will answer all the questions raised here, but to do this we have to develop machinery of Sobolev spaces.

The method of solving variational problems (more general than the one described above) consists very often of two main steps. First we prove the existence of the solution in a Sobolev space. This space is very large. Too large. Then using the theory of Sobolev spaces one can prove that this solution is in fact more regular. For example later we will prove that the Sobolev minimizer of (1.5) is C^∞ smooth, which will imply that $\bar{u}$ is actually the classical harmonic function. Moreover we will prove that under some regularity conditions for the boundary, the Sobolev minimizer of the Dirichlet integral solves the classical Dirichlet problem.

2. Sobolev spaces

2.1. **Basic definitions and results.** We will develop now a rigorous theory of Sobolev spaces. We will use a different definition than above, but as we shall see both definitions are equivalent.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $u, v \in L^1_{\rm loc}(\Omega)$ and let α be a multiindex. We say that $D^\alpha u = v$ in the *weak sense* if

$$\int_\Omega v\varphi = (-1)^{|\alpha|}\int_\Omega uD^\alpha\varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

The weak derivative is unique. Indeed, if $D^\alpha u = v_1$ and $D^\alpha u = v_2$ in the weak sense, then

$$\int_\Omega v_1\varphi = \int_\Omega v_2\varphi = (-1)^{|\alpha|}\int_\Omega uD^\alpha\varphi$$

for all $\varphi \in C_0^\infty(\Omega)$, so

$$\int_\Omega (v_1 - v_2)\varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega)$$

and hence $v_1 = v_2$ a.e. by Lemma 1.3.

If $u \in C^1(\Omega)$, then integration by parts formula yields that weak partial derivatives of u coincide with the classical partial derivatives of u.

Definition 2.2. Let $1 \le p \le \infty$ and let m be an integer. The *Sobolev space* $W^{m,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ such that the partial derivatives of order less than or equal to m exist in the weak sense and belong to $L^p(\Omega)$. The space is equipped with the norm

$$\|u\|_{m,p} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_p.$$

Theorem 2.3. *$W^{m,p}(\Omega)$ is a Banach space.*

Proof. If $\{u_k\}$ is a Cauchy sequence in $W^{m,p}(\Omega)$, then for every $|\alpha| \leq m$, $D^\alpha u_k$ converges in $L^p(\Omega)$ to some $u_\alpha \in L^p(\Omega)$ (we will write u instead of u_0). Since

$$\int_\Omega u D^\alpha \varphi \leftarrow \int_\Omega u_k D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega D^\alpha u_k \varphi \to (-1)^{|\alpha|} \int_\Omega u_\alpha \varphi,$$

we conclude that $u_\alpha = D^\alpha u$ and that u_k converges to u in the norm of $W^{m,p}$. □

Exercise 2.4 (Leibnitz formula)**.** *Prove that if $u \in W^{m,p}(\Omega)$ and $\varphi \in C^\infty(\Omega)$ has bounded derivatives for all $|\alpha| \leq m$, then $\varphi u \in W^{m,p}(\Omega)$ and*

$$D^\alpha(u\varphi) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta}\varphi\,.$$

Theorem 2.5 (Meyers-Serrin)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, $m \geq 1$ and $1 \leq p < \infty$. Then $C^\infty(\Omega)$ functions are dense in $W^{m,p}(\Omega)$.* □

The idea of the proof is a follows: using partition of unity we represent the function as a series of functions with compact support. Next we approximate each compactly supported function by convolution which gives us a compactly supported smooth function. Finally we add the resulting smooth functions to obtain a smooth approximation of the original function.

According to the Meyers-Serrin theorem, equivalently, we could define the Sobolev space as a completion of the space of smooth functions with respect to the Sobolev norm which is to say that the definition involving weak derivative is equivalent to the one discussed in Section 1.

Definition 2.6. We define $W^{m,p}_0(\Omega) \subset W^{m,p}(\Omega)$ as a closure of the $C^\infty_0(\Omega)$ functions in the Sobolev norm. Clearly $W^{m,p}_0(\Omega)$ is a closed linear subspace of $W^{m,p}(\Omega)$. We also define the local Sobolev space $W^{m,p}_{\mathrm{loc}}(\Omega)$ by assuming L^p integrability of a function and its derivatives on compact subsets of Ω.

Theorem 2.7. *C^∞_0 is dense in $W^{m,p}(\mathbb{R}^n)$, provided $1 \leq p < \infty$.*

In other words $W^{m,p}(\mathbb{R}^n) = W^{m,p}_0(\mathbb{R}^n)$. The proof is based on the following idea: using approximation by convolution we can approximate $u \in W^{m,p}(\mathbb{R}^n)$ by smooth functions. Then multiplication with a cut-off functions yields approximation by compactly supported smooth functions.

If Ω is a bounded domain, then $C_0^\infty(\Omega)$ functions are not dense in $W^{m,p}(\Omega)$, so $W^{m,p}(\Omega) \neq W_0^{m,p}(\Omega)$. Indeed, Poincaré inequality, Lemma 1.12 yields that nonzero constant functions belong to $W^{m,p}(\Omega) \setminus W_0^{m,p}(\Omega)$. That is consistent with our intuition: functions in $W_0^{m,p}(\Omega)$ have zero trace on the boundary (in a weak sense), but certainly constant nonzero functions should not have zero trace.

Exercise 2.8. *Let Ω be a two dimensional disc with one radius removed. Prove that $C^\infty(\Omega)$ functions are not dense in $W^{1,p}(\Omega)$.*

However, we have

Theorem 2.9. *If Ω is a bounded domain whose boundary if locally a graph of a continuous function, then $C^\infty(\overline{\Omega})$ is a dense subset of $W^{m,p}(\Omega)$ for all $1 \leq p < \infty$, $m \geq 1$.* □

2.2. **Dirichlet principle again.**

Definition 2.10. We say that $u \in W^{1,2}(\Omega)$ is *weakly harmonic* or *weak solution* to the Laplace equation $\Delta u = 0$, if

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{2.1}$$

For any function $u \in C^2(\Omega)$, integration by parts gives

$$\int_\Omega \nabla u \cdot \nabla \varphi = -\int_\Omega (\Delta u)\varphi,$$

so Lemma 1.3 yields that u is weakly harmonic if and only if u is harmonic in the classical sense.

The notion of weakly harmonic function provides the missing link between Sobolev minimizers of the Dirichlet integral and solutions to the Dirichlet problem. We proved that if $\Omega \subset \mathbb{R}^n$ is open and bounded and $w \in W^{1,2}(\Omega)$, then there is unique $\bar{u} \in W_w^{1,2}(\Omega)$ such that

$$I(\bar{u}) = \inf_{u \in W_w^{1,2}(\Omega)} I(u), \quad \text{where} \quad I(u) = \int_\Omega |\nabla u|^2. \tag{2.2}$$

The following result is a version of the Dirichlet principle in the setting of Sobolev spaces.

Theorem 2.11. *u is a minimizer of the Dirichlet integral* (2.2) *in a bounded domain Ω if and only if u is a weak solution to the Dirichelt problem*

$$\begin{cases} -\Delta u = 0 & \textit{in } \Omega, \textit{ in the weak sense,} \\ u \in W_w^{1,2}(\Omega). \end{cases} \tag{2.3}$$

Proof. $\Rightarrow$ Let u be a minimizer of (2.2). Then $u+t\varphi\in W^{1,2}_w(\Omega)$ for all $\varphi\in C_0^\infty(\Omega)$ and hence the function $g(t)=I(u+t\varphi)$ attains minimum at $t=0$. Hence

$$\begin{aligned} 0 &= g'(0)=\frac{d}{dt}\Big|_{t=0}\int_\Omega |\nabla(u+t\varphi)|^2\,dx \\ &= \frac{d}{dt}\Big|_{t=0}\left(\int_\Omega|\nabla u|^2+2t\int_\Omega\nabla u\cdot\nabla\varphi+t^2\int_\Omega|\nabla\varphi|^2\right) \\ &= 2\int_\Omega\nabla u\cdot\nabla\varphi\,. \end{aligned}$$

This, however, means that u is a weakly harmonic function in Ω.

$\Leftarrow$ Let u be a weak solution to the Dirichlet problem (2.3). We need to show that u is a minimizer of (2.2). The variational problem (2.2) has a unique minimizer $\bar{u}$, which, as proved above, is a solution to the Dirichlet problem and thus it suffices to show that the Dirichelt problem has a unique solution. Thus we need to show that if u_1 and u_2 are solutions to (2.3), then $u_1=u_2$ a.e. We have

$$\begin{cases} -\Delta(u_1-u_2)=0 & \text{in } \Omega \\ u_1-u_2\in W_0^{1,2}(\Omega). \end{cases}$$

That means

$$\int_\Omega\nabla(u_1-u_2)\cdot\nabla\varphi=0\quad\text{for all }\varphi\in C_0^\infty(\Omega).$$

Applying this identity to $\varphi_k\in C_0^\infty(\Omega)$, $\varphi_k\to u_1-u_2$ in $W^{1,2}(\Omega)$ and passing to the limit we obtain

$$\int_\Omega|\nabla(u_1-u_2)|^2=0.$$

Now the Poincaré inequality, Lemma 1.12 yields

$$\int_\Omega|u_1-u_2|^2\le C\int_\Omega|\nabla(u_1-u_2)|^2=0$$

and hence $u_1=u_2$ a.e. □

We proved a version of the Dirichelt principle in the Sobolev setting. However, it is still unclear how the weak solutions to the Dirichlet problem (whose existence we proved) are related to the classical Dirichlet problem (1.1). Later we will prove (Weyl's lemma, Theorem 3.1) that weakly harmonic functions are C^∞ classical harmonic functions and that under some regularity conditions for the boundary, weak solutions of (2.3) are continuous up to the boundary (Theorems 3.23 and 3.24), so in such a case, weak solutions to (2.3) are actually classical solutions to the Dirichlet problem. This illustrates a modern approach to partial differential equations and variational problems: instead of finding a direct proof of the existence of a classical solution, first we prove existence of a solution in the Sobolev space and then we prove its regularity. In the example discussed above we considered the most simple elliptic equation $\Delta u = 0$, but soon we will see that variational approach applies also to complicated nonlinear equations.

2.3. **Reflexivity.** We have already seen that reflexivity plays an important role in the direct methods of the calculus of variations.

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p < \infty$ and let $m \geq 1$ be an integer. Then the Sobolev space $W^{m,p}(\Omega)$ is reflexive.*

Proof. For the sake of simplicity of notation we will prove the theorem in the case $m = 1$ only, but a very similar argument can be used to cover the case of higher order derivatives as well. Closed subspaces of reflexive spaces are reflexive. Thus it suffices to find an isomorphism between $W^{1,p}(\Omega)$ and a closed subspace of $L^p(\Omega, \mathbb{R}^{n+1})$. Such an isomorphism is given by a mapping

$$W^{1,p}(\Omega) \ni u \mapsto (u, \nabla u) \in L^p(\Omega, \mathbb{R}^{n+1}).$$

This mapping is actually an isometry between $W^{1,p}$ and a closed subspace of L^p. □

The space $W^{m,1}$ is not reflexive and this is one of the major reasons for difficulty of variational problems which are formulated in the $W^{1,1}$ setting.

Reflexivity of L^p, $1 < p < \infty$ implies also the following useful characterization of $W^{1,p}$.

Theorem 2.13. *Let $1 < p < \infty$. Then $u \in W^{1,p}(\mathbb{R}^n)$ if and only if there is a constant $C > 0$ such that*

$$\|\tau_h u - u\|_p \leq C|h|$$

for all $h \in \mathbb{R}^n$, where $\tau_u(x) = u(x+h)$ is the translation of u by a vector h.

Proof. ⇒ It suffices to assume that $u \in C^\infty(\mathbb{R}^n)$ — the general case follows upon taking the smooth approximation of u. We have

$$u(x+h) - u(x) = \int_0^1 \frac{d}{dt} u(x+th)\, dt = \int_0^1 \nabla u(x+th) \cdot h\, dt.$$

Hölder's inequality and Fubini's theorem yield

$$\begin{aligned}
\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p\, dx &\leq \int_{\mathbb{R}^n} \left| \int_0^1 \nabla u(x+th) \cdot h\, dt \right|^p dx \\
&\leq \int_{\mathbb{R}^n} \int_0^1 |\nabla u(x+th)|^p |h|^p\, dt dx \\
&= |h|^p \int_0^1 \underbrace{\int_{\mathbb{R}^n} |\nabla u(x+th)|^p\, dx}_{\int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx}\, dt \\
&= |h|^p\, \|\nabla u\|_p^p
\end{aligned}$$

and hence

$$\|\tau_h u - u\|_p = \int_{\mathbb{R}^n} |u(x+h) - u(x)|^p \, dx \leq |h| \, \|\nabla u\|_p.$$

$\Leftarrow$ Denote by e_k one of the coordinate directions. Let $\mathbb{R} \ni h_i \to 0$. It follows from the assumption that

$$\left(\int_{\mathbb{R}^n} \left| \frac{u(x + h_i e_k) - u(x)}{h_i} \right|^p \right)^{1/p} \leq C.$$

Thus the sequence $(u(x + h_i e_k) - u(x))/h_i$ is bounded in $L^p(\mathbb{R}^n)$. By the reflexivity of L^p we can extract a weakly convergent subsequence to some $u_k \in L^p(\mathbb{R}^n)$. It remains to show that weak partial derivative $\partial u/\partial x_k$ equals u_k. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. We have

$$\begin{aligned} \int_{\mathbb{R}^n} u_k \varphi &= \lim_{j\to\infty} \int_{\mathbb{R}^n} \left(\frac{u(x + h_{i_j} e_k) - u(x)}{h_{i_j}} \right) \varphi(x) \, dx \\ &= \lim_{j\to\infty} \int_{\mathbb{R}^n} u(x) \left(\frac{\varphi(x - h_{i_j} e_k) - \varphi(x)}{h_{i_j}} \right) dx \\ &= -\int_{\mathbb{R}^n} u(x) \, \frac{\partial \varphi}{\partial x_k}(x) \, dx. \end{aligned}$$

The proof is complete. □

In the second part of the above proof we used boundedness in L^p of the difference quotients only for small h_i, so the same proof gives the following variant of the above result.

Corollary 2.14. *Let $1 < p < \infty$. Then $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and $\limsup_{h\to 0} \|\tau_h u - u\|_p/|h| < \infty$.*

Also the argument used in the proof can be localized. That leads to the following result

Theorem 2.15. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open. Then $u \in W^{1,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and there is a constant $C > 0$ such that*

$$\|u(\cdot + h) - u(\cdot)\|_{L^p(\Omega')} \leq C_{\Omega'}|h|,$$

provided $|h| < \min\{\frac{1}{2}\mathrm{dist}\,(\Omega', \partial\Omega), 1)\}$.

Exercise 2.16. *Prove that if $u \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$, then*

$$\frac{u(x + he_k) - u(x)}{h} \to \frac{\partial u}{\partial x_k} \quad \textit{as } h \to 0$$

in the norm of $L^p(\mathbb{R}^n)$.

2.4. **ACL characterization.**

Definition 2.17. We say that a function $f : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if

$$(x_1, x_1 + h_1), \ldots, (x_k, x_k + h_k)$$

are pairwise disjoint intervals in $[a, b]$ of total length less than δ, i.e. $\sum_{i=1}^k h_i < \delta$, then

$$\sum_{i=1}^{k} |u(x_i + h_i) - u(x_i)| < \varepsilon.$$

In particular Lipschitz functions are absolutely continuous.

Proposition 2.18. *If $u, v : [a, b] \to \mathbb{R}$ are absolutely continuous, then $u \pm v$, uv are absolutely continuous. If in addition $v \geq c > 0$ on $[a, b]$, then u/v is absolutely continuous.* □

The following result provides an amazing characterization of absolutely continuous functions.

Theorem 2.19. *Let $f \in L^1([a, b])$. Then the function $F(x) = \int_a^x f(t)\, dt$ is absolutely continuous. On the other hand if F is absolutely continuous, then F is differentiable a.e., $F' \in L^1([a, b])$ and*

$$F(x) = F(a) + \int_a^x F'(t)\, dx \quad \textit{for all } x \in [a, b].$$

Thus the absolutely continuous functions are exactly the functions for which the fundamental theorem of calculus is true. Hence one can easily prove

Theorem 2.20 (Integration by parts)**.** *If the functions $f, g : [a, b] \to \mathbb{R}$ are absolutely continuous, then*

$$\int_a^b fg' = fg|_a^b - \int_a^b f'g.$$

The weak derivative of a function is defined through the validity of integration by parts. Thus for absolutely continuous functions the pointwise derivative which exists a.e. equals to the weak derivative. That means if $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then $f \in W^{1,1}(a, b)$. It turns out that the converse implication is also true.

Theorem 2.21. *$u \in W^{1,p}(a, b)$, $1 \leq p \leq \infty$ if and only if u can be modified on a set of measure zero in such a way that*

(a) *$u \in L^p(a, b)$;*
(b) *u is absolutely continuous on every compact interval in (a, b) and the pointwise derivative u' (which exists a.e.) belongs to $L^p(a, b)$.*

Moreover the pointwise derivative u' equals the weak one.

This characterization has a higher dimensional analog.

Definition 2.22. If $U \subset \mathbb{R}$ is open, we say that $u \in AC(U)$ if u is absolutely continuous on every compact interval in U. Let $\Omega \subset \mathbb{R}^n$. We say that u is *absolutely continuous on lines*, $u \in ACL(\Omega)$, if the function u is Borel measurable and for almost every line ℓ parallel to one of the coordinate axes, $u|_\ell \in AC(\Omega \cap \ell)$. Since absolutely continuous functions in dimension one are differentiable a.e., $u \in ACL(\Omega)$ has partial derivatives a.e. and hence ∇u is defined a.e. Now we say that $u \in ACL^p(\Omega)$ if $u \in L^p(\Omega) \cap ACL(\Omega)$ and $|\nabla u| \in L^p(\Omega)$.

Theorem 2.23 (Nikodym; ACL characterization)**.** *For $1 \leq p \leq \infty$ and any open set $\Omega \subset \mathbb{R}^n$*

$$W^{1,p}(\Omega) = ACL^p(\Omega).$$

Moreover the pointwise partial derivatives of an $ACL^p(\Omega)$ function equal to the weak partial derivatives.

The theorem asserts that each $ACL^p(\Omega)$ function belongs to $W^{1,p}(\Omega)$ and that the classical partial derivatives (which exist a.e. for elements of $ACL^p(\Omega)$) are equal to weak partial derivatives. On the other hand every element $u \in W^{1,p}(\Omega)$ can be alternated on a set of measure zero in a way that the resulting function belongs to $ACL^p(\Omega)$.

The proof of the inclusion $ACL^p(\Omega) \subset W^{1,p}(\Omega)$ is easy. It follows from the fact that integration by parts holds for the absolutely continuous functions, from the Fubini theorem, and from the definition of the weak derivative. The opposite implication is more involved and we will not prove it.

Exercise 2.24. *Given $n > 1$ and $1 \leq p < \infty$, find all $\alpha > 0$ such that $u(x) = |x|^{-\alpha} \in W^{1,p}(B^n(0,1))$.*

Solution. Denote $\Omega = B^n(0,1)$. The function u is smooth away from 0, but it is discontinuous at the 0. For $x \neq 0$ we have

$$\frac{\partial u}{\partial x_i} = -\frac{\alpha x_i}{|x|^{\alpha+2}}\,,$$

so

$$|\nabla u(x)| = \frac{\alpha}{|x|^{\alpha+1}}, \quad x \neq 0.$$

This implies that $|\nabla u| \in L^p(\Omega)$ if and only if $(\alpha+1)p < n$, i.e. if and only if $\alpha < (n-p)/p$. In this range of α we also have $u \in L^p(\Omega)$. We would like to conclude that $u \in W^{1,p}(\Omega)$ if and only if $\alpha < (n-p)/p$, but there is a problem. Since u is smooth in $\Omega \setminus \{0\}$, then clearly the partial derivatives of u are weak derivatives in $\Omega \setminus \{0\}$. However, that we do not know yet if the partial derivatives are weak derivatives in Ω with 0 included: they are discontinuous! The problem disappears if we apply the ACL characterization. The function u is continuous on almost all lines. It is actually smooth on all lines except the lines passing through the origin. Thus according to the ACL characterization of the Sobolev space it suffices to find p such that $u \in L^p$ and the pointwise gradient, defined everywhere but at 0, is in L^p. But as was observed above, it is the case if and only if $\alpha < (n-p)/p$.

Note that if $p > n$, then $(n-p)/p < 0$, so there is no such α.

One can also solve the problem more directly without referring to the ACL characterization of the Sobolev space.

We will show that the pointwise derivatives of u, defined on $\Omega \setminus \{0\}$, and hence defined a.e. in Ω are weak partial derivatives on Ω (with 0 included) if $0 \leq \alpha < n-1$. Let $\varphi \in C_0^\infty(\Omega)$. Fix $\varepsilon > 0$. We have

$$\int_{\Omega \setminus B^n(0,\varepsilon)} u \frac{\partial \varphi}{\partial x_i} = -\int_{\Omega \setminus B^n(0,\varepsilon)} \frac{\partial u}{\partial x_i}\, \varphi + \int_{\partial B^n(0,\varepsilon)} u \varphi \nu^i \, d\sigma \tag{2.4}$$

where $\nu = (\nu^1, \ldots, \nu^n)$ is the inward pointing normal on $\partial B^n(0,\varepsilon)$. If $0 \leq \alpha < n-1$, $|\nabla u| \in L^1(\Omega)$ and in this case

$$\left| \int_{\partial B^n(0,\varepsilon)} u \varphi \nu^i \, d\sigma \right| \leq \|\varphi\|_\infty \int_{\partial B^n(0,\varepsilon)} \varepsilon^{-\alpha} \, d\sigma \leq C \varepsilon^{n-1-\alpha} \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence passing to the limit in (2.4) yields

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i}\, \varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Recall that $u, |\nabla u| \in L^p(\Omega)$ if and only if $\alpha < (n-p)/p$. Since this implies $\alpha < n-1$ we see that the pointwise derivatives are equal to the weak ones on all of Ω and hence $u \in W^{1,p}(\Omega)$ if and only if $\alpha < (n-p)/p$. □

The above example shows that Sobolev functions need not be continuous when $1 \leq p < n$. However, if $p > n$, then $(n-p)/p < 0$ and we cannot find α satisfying $0 < \alpha < (n-p)/p$. Actually, we will prove later that if $p > n$, then Sobolev functions are Hölder continuous (after being redefined on a set of measure zero). This still does not answer what happens when $p = n$. It turns out that in this case Sobolev functions need not be continuous as well, but the example has to be different.

Exercise 2.25. *Prove that* $u(x) = \log|\log|x||$ *belongs to* $W^{1,n}(B^n(0, e^{-1}))$.

The above two examples show that Sobolev functions in $W^{1,p}$ for $1 \leq p \leq n$ may have one point singularity. Adding a finite number of such functions we may easily construct Sobolev functions with a finite number of singularities. However, it turns out that it is also possible to construct a Sobolev function that is essentially discontinuous everywhere.

Example 2.26. Fix $1 \le p < n$ and $0 < \alpha < (n-p)/p$. Let $D = \{x_i\}_{i=1}^{\infty}$ be a countable and dense subset of of $\Omega = B^n(0,1/2)$. Define

$$u(x) = \sum_{i=1}^{\infty} 2^{-i}|x - x_i|^{-\alpha}, \quad x \in B^n(0,1) \setminus D \tag{2.5}$$

Since

$$\begin{aligned} \sum_{i=1}^{\infty} \|2^{-i}|x-x_i|^{-\alpha}\|_{1,p} &= \sum_{i=1}^{\infty} 2^{-i}\||x-x_i|^{-\alpha}\|_{1,p} \\ &\le \||x|^{-\alpha}\|_{W^{1,p}(B^n(0,1))} \sum_{i=1}^{\infty} 2^{-i} < \infty \end{aligned}$$

we conclude that the series (2.5) converges[6] in the space $W^{1,p}(\Omega)$ and thus defines an element of $W^{1,p}(\Omega)$. This shows that Sobolev functions in $W^{1,p}(\Omega)$, $1 \le p < n$, $n \ge 2$ may be essentially unbounded on every open subset of Ω. Similar example can be constructed in the space $W^{1,n}(\Omega)$. This can be obtained by constructing a similar series based on functions $\log|\log x||$.

Here is another important example.

Example 2.27. The radial projection mapping

$$u_0(x) = \frac{x}{|x|} : B^n(0,1) \to S^{n-1}(0,1) \subset \mathbb{R}^n,$$

is discontinuous at $x = 0$. The coordinate functions $x_i/|x|$ of u_0 are absolutely continuous on almost all lines. Moreover

$$\frac{\partial}{\partial x_j}\left(\frac{x_i}{|x|}\right) = \frac{\delta_{ij}|x| - x_i x_j/|x|}{|x|^2} \in L^p(B^n(0,1)),$$

for all $1 \le p < n$. Hence by the ACL characterization $u_0 \in W^{1,p}(B^n, \mathbb{R}^n)$ for all $1 \le p < n$. Here δ_{ij} is the Kronecker symbol i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \ne j$.

The ACL characterization also easily implies the following results.

Corollary 2.28. *Functions in the space $W^{1,\infty}(\Omega)$ are locally Lipschitz continuous. If in addition Ω is a bounded Lipschitz domain (i.e. $\partial\Omega$ is locally a graph of a Lipschitz function), then $W^{1,\infty}(\Omega) = \mathrm{Lip}\,(\Omega)$.* □

Corollary 2.29. *If $u \in W^{1,p}(\Omega)$, where Ω is connected and $\nabla u = 0$ a.e., then u is constant.* □

Corollary 2.30. *If $u \in W^{1,p}(\Omega)$ is constant in a measurable set $E \subset \Omega$, then $\nabla u = 0$ a.e. in E.* □

[6]We use the fact that Sobolev space is a Banach space.

Corollary 2.31. *Let $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$. Then $u_\pm \in W^{1,p}(\Omega)$, where $u_+ = \max\{u, 0\}$, $u_- = \min\{u, 0\}$ and*

$$\nabla u_+(x) = \begin{cases} \nabla u(x) & \text{if} \quad u(x) > 0\,, \\ 0 & \text{if} \quad u(x) \le 0\,, \end{cases}$$

almost everywhere. Similar formula holds for ∇u_-. □

Corollary 2.32. *Let*

$$\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n :\, x_n > 0\}\,.$$

For any $1 \le p \le \infty$ there is a bounded linear operator

$$E : W^{1,p}(\mathbb{R}^n_+) \to W^{1,p}(\mathbb{R}^n)$$

such that

$$(Eu)|_{\mathbb{R}^n_+} = u, \quad \text{for } u \in W^{1,p}(\mathbb{R}^n_+).$$

Indeed, we can define $Eu(x)$ to be equal $u(x_1, \ldots, x_n)$ if $x_n > 0$ and $u(x_1, \ldots, -x_n)$ if $x_n < 0$ and then the result follows from the ACL characterization. Operator E is called an *extension operator*.

2.5. **Riesz potentials and Poincaré inequality.** The lemma below provides a very powerful integral estimate for Sobolev functions.

Lemma 2.33. *Let $B \subset \mathbb{R}^n$ be a ball. Then for every $u \in W^{1,p}(B)$, $1 \le p \le \infty$,*

$$|u(x) - u_B| \le C(n) \int_B \frac{|\nabla u(z)|}{|x - z|^{n-1}}\, dz \ \text{ a.e.}, \tag{2.6}$$

where

$$u_B = |B|^{-1} \int_B u\, dx.$$

Proof. First we prove the inequality for $u \in C^\infty(B)$. Fix $x \in B$. For $y \in B$, $y \ne x$ set

$$y = x + t\frac{y - x}{|y - x|} = x + tz, \qquad z \in S^{n-1}$$

and let $\delta(z) = \max\{t > 0 :\, x + tz \in B\}$. We have

$$|u(x) - u(y)| \le \int_0^{|y-x|} \left|\nabla u(x + s\frac{y - x}{|y - x|})\right| ds \le \int_0^{\delta(z)} |\nabla u(x + sz)|\, ds.$$

Denoting by $d\sigma(z)$ the surface measure on S^{n-1} we get

$$
\begin{aligned}
|u(x)-u_B| &\leq |B|^{-1}\int_B |u(x)-u(y)|\,dy \\
&\quad \text{(polar coordinates)} \\
&= |B|^{-1}\int_{S^{n-1}}\int_0^{\delta(z)} t^{n-1}|u(x)-u(x+tz)|\,dt\,d\sigma(z) \\
&\leq |B|^{-1}\int_{S^{n-1}}\int_0^{\delta(z)} t^{n-1}\int_0^{\delta(z)} |\nabla u(x+sz)|\,ds\,dt\,d\sigma(z) \\
&\leq |B|^{-1}\int_{S^{n-1}}\int_0^{2r} t^{n-1}\,dt\int_0^{\delta(z)} |\nabla u(x+sz)|\,ds\,d\sigma(z) \\
&= C(n)\int_{S^{n-1}}\int_0^{\delta(z)} \frac{|\nabla u(x+sz)|}{s^{n-1}}s^{n-1}\,ds\,d\sigma(z) \\
&\quad \text{(polar coordinates)} \\
&= C(n)\int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.
\end{aligned}
$$

The case of general $u\in W^{1,p}(B)$ follows by approximating u by C^∞ smooth functions. To this end we have to know that if $u_k\to u$ in $W^{1,p}$, then after subtracting a subsequence, $I_1^B|\nabla u_k|\to I_1^B|\nabla u|$ a.e. where $I_1^B g(x)=\int_B g(y)|x-y|^{1-n}\,dy$. This will follow from Lemma 2.35 below.

To get further estimates we introduce Riesz potentials. The Riesz potential is an integral operator I_α, $0<\alpha<n$, defined by the formula

$$I_\alpha g(x)=\int_{\mathbb{R}^n}\frac{g(z)}{|x-z|^{n-\alpha}}\,dz.$$

If $\Omega\subset\mathbb{R}^n$, then we set

$$I_\alpha^\Omega g(x)=\int_\Omega\frac{g(z)}{|x-z|^{n-\alpha}}\,dz.$$

We start with an elementary, but very useful observation which will be employed in the sequel.

Lemma 2.34. *If $E\subset\mathbb{R}^n$ is a measurable set of finite measure, then*

$$\int_E\frac{dz}{|x-z|^{n-1}}\leq C(n)|E|^{1/n},$$

for all $x\in\mathbb{R}^n$.

Proof. Let $B=B(x,r)$ be a ball with $|B|=|E|$. Then it easily follows that

$$\int_E\frac{dz}{|x-z|^{n-1}}\leq\int_B\frac{dz}{|x-z|^{n-1}}=C(n)r=C'(n)|E|^{1/n}.$$

The proof is complete. □

Lemma 2.35. *If $|\Omega| < \infty$, then for $1 \le p < \infty$ we have*

$$\|I_1^\Omega g\|_{L^p(\Omega)} \le C(n,p)|\Omega|^{1/n}\|g\|_{L^p(\Omega)}.$$

Proof. It follows from the previous lemma that

$$\int_\Omega \frac{dz}{|x-z|^{n-1}} \le C(n)|\Omega|^{1/n}\,.$$

Now if $p > 1$, then Hölder's inequality with respect to the measure $|x - z|^{1-n}dz$ implies

$$\begin{aligned}\int_\Omega \frac{|g(z)|}{|x-z|^{n-1}}\,dz &\le \left(\int_\Omega \frac{|g(z)|^p}{|x-z|^{n-1}}\,dz\right)^{1/p}\left(\int_\Omega \frac{dz}{|x-z|^{n-1}}\,dz\right)^{1-1/p}\\ &\le C|\Omega|^{\frac{p-1}{np}}\left(\int_\Omega \frac{|g(z)|^p}{|x-z|^{n-1}}\,dz\right)^{1/p}.\end{aligned}$$

If $p = 1$, then the above inequality is obvious. Now we can conclude the proof using Fubini's theorem.

$$\begin{aligned}\int_\Omega |I_1^\Omega g(x)|^p\,dx &\le C|\Omega|^{\frac{p-1}{n}}\int_\Omega\int_\Omega \frac{|g(z)|^p}{|x-z|^{n-1}}\,dz\,dx\\ &\le C|\Omega|^{\frac{p-1}{n}}|\Omega|^{\frac{1}{n}}\int_\Omega |g(z)|^p\,dz.\end{aligned}$$

This completes the proof of Lemma 2.35 and hence that for Lemma 2.33. □

Remark. The proof of Lemma 2.33 easily extends to the case of an arbitrary bounded, convex domain.

As a direct consequence of Lemma 2.33 and Lemma 2.35 we obtain

Corollary 2.36. *If $u \in W^{1,p}(B)$, where B is a ball of radius r, and $1 \le p < \infty$, then*

$$\left(⨍_B |u-u_B|^p\,dx\right)^{1/p} \le C(n,p)r\left(⨍_B |\nabla u|^p\,dx\right)^{1/p}.$$

Here and in what follows we will use notation

$$f_E = ⨍_E f d\mu := \mu^{-1}(E)\int_E f\,d\mu.$$

A variant of the above calculation leads to the following result.

Lemma 2.37. *Let $u \in W_0^{1,p}(\Omega)$, $1 \le p \le \infty$. Then*

$$u(x) = \frac{1}{n\omega_n}\int_\Omega \frac{(x-y)\cdot\nabla u(y)}{|x-y|^n}\,dy \quad a.e.$$

Here and in what follows ω_n stands for the volume of the unit ball.

2.6. **Hölder's continuity and Rademacher's theorem.** We will prove a result, already mentioned before, that if $n < p < \infty$, then Sobolev functions are Hölder continsuous. Then we will apply it to prove Rademacher's theorem which states that Lipschitz functions are differentiable a.e.

Lemma 2.38. *If $u \in W^{1,p}(B)$, $n < p < \infty$, then*

$$|u(x) - u_B| \leq C(n,p) r^{1-\frac{n}{p}} \left(\int_B |\nabla u|^p \right)^{1/p} \qquad a.e.$$

Here B is a ball of radius r.

Proof. We have

$$\begin{aligned} |u(x) - u_B| &\leq C(n) \int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz \\ &\leq C(n) \left(\int_B |\nabla u(z)|^p\, dz \right)^{1/p} \left(\int_B \frac{dz}{|x-z|^{(n-1)p/(p-1)}} \right)^{1-1/p} \\ &\leq C(n,p) r^{1-\frac{n}{p}} \left(\int_B |\nabla u(z)|^p\, dz \right)^{1/p} . \end{aligned}$$

□

In what follows $2b$ will denote a ball concentric with B and with twice the radius.

Corollary 2.39. *If $u \in W^{1,p}(2B)$, $n < p < \infty$, then u (after being redefined on a set of measure zero) if Hölder continuous on B, $u \in C^{0,1-\frac{n}{p}}(B)$ and*

$$|u(x) - u(y)| \leq C(n,p) |x-y|^{1-\frac{n}{p}} \left(\int_{2B} |\nabla u|^p \right)^{1/p} . \tag{2.7}$$

for all $x, y \in B$.

Proof. Given $x, y \in B$, let $\tilde{B} \subset 2B$ be a ball such that $x, y \in \tilde{B}$ and $\operatorname{diam} \tilde{B} < 2|x-y|$. Then

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u_{\tilde{B}}| + |u(y) - u_{\tilde{B}}| \\ &\leq C(\operatorname{diam} \tilde{B})^{1-\frac{n}{p}} \left(\int_{\tilde{B}} |\nabla u|^p \right)^{1/p} \\ &\leq C' |x-y|^{1-\frac{n}{p}} \left(\int_{2B} |\nabla u|^p \right)^{1/p} . \end{aligned}$$

More precisely, the argument shows that (2.7) is true for a.e. $x, y \in B$. However, a function which is Hölder continuous on a set of full measure in B uniquely extends to a Hölder continuous function on all of B. □

Corollary 2.40. *If $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and $n < p < \infty$, then u is locally Hölder continuous with exponent $1 - n/p$.*

Later we will provide more detailed results regarding Hölder continuity in the case $n < p < \infty$. However, the above results are enough to prove Calderón's and Rademacher's differentiability theorems.

Theorem 2.41 (Calderón)**.** *If $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and $n < p \leq \infty$, then u is differentiable a.e.*

Proof. It is enough to consider the case $n < p < \infty$ only, because locally $W^{1,\infty} \subset W^{1,p}$ for any $p < \infty$.

Fix $x_0 \in \Omega$ and set $v(x) = u(x) - \nabla u(x_0)(x - x_0)$. Obviously $v \in W^{1,p}_{\rm loc}(\Omega)$. If x is sufficiently close to x_0, then

$$B = B(x_0, 2|x - x_0|) \subset 2B = B(x_0, 4|x - x_0|) \subset \Omega$$

and hence Lemma 2.39 yields

$$\begin{aligned} &|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)| = |v(x) - v(x_0)| \\ &\leq \ C|x - x_0|^{1-\frac{n}{p}} \left(\int_{2B} |\nabla u(z) - \nabla u(x_0)|^p \, dz \right)^{1/p} \\ &= \ C'|x - x_0| \left(\fint_{2B} |\nabla u(z) - \nabla u(x_0)|^p \, dz \right)^{1/p}, \end{aligned}$$

so

$$\frac{|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|}{|x - x_0|} \leq C' \left(\fint_{2B} |\nabla u(z) - \nabla u(x_0)|^p \, dx \right)^{1/p}.$$

This implies that u is differentiable at x_0 whenever x_0 is a p-Lebesgue point[7] of u. □

As an immediate corollary we obtain

Theorem 2.42 (Rademacher)**.** *Lipschitz functions defined on open sets in an Euclidean space are differentiable a.e.*

One can use Rademacher's theorem do find a necessary and sufficient condition for a function to be differentiable a.e.

Theorem 2.43 (Stepanov)**.** *let u be a function defined in an open set $\Omega \subset \mathbb{R}^n$. then u is differentiable a.e. if and only if*

$$\limsup_{y \to x} \frac{|u(y) - u(x)|}{|y - x|} < \infty \quad a.e.$$

[7]Recall that according to the Lebesgue differentiability theorem, if $g \in L^p(\Omega)$, $1 \leq p < \infty$, then for a.e. $x_0 \in \Omega$, $\fint_{B(x_0,r)} |g(z) - g(x_0)|^p \, dz \to 0$ as $r \to 0$. Every such point x_0 is called a *p-Lebesfue point* of g.

2.7. **Sobolev embedding theorem.** By the definition a function $u \in W^{1,p}(\Omega)$ is a priori only L^p integrable. We have already seen that if $n < p < \infty$, then u is actually Hölder continuous. If $1 \le p \le n$, then we do not get continuity – we have seen examples of Sobolev functions which are essentially unbounded on every open subset of Ω, however, we can still prove that u is integrable with a higher exponent than p.

Theorem 2.44 (Sobolev embedding theorem)**.** *Let $1 \le p < n$ and $p^* = np/(n-p)$. Then for $u \in W^{1,p}(\mathbb{R}^n)$ we have*

$$\left(\int_{\mathbb{R}^n} |u(x)|^{p^*}\, dx\right)^{1/p^*} \le C(n,p)\left(\int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx\right)^{1/p}. \tag{2.8}$$

Proof. Step 1. $p = 1$. This is the crucial step in the proof. As we will see later, the general case $1 \le p < n$ easily follows from the case $p = 1$.

By the density argument we can assume that $u \in C_0^\infty(\mathbb{R}^n)$. We have

$$|u(x)| \le \int_{-\infty}^{x_1} |D_1 u(t_1, x_2, \ldots, x_n)|\, dt_1 \le \int_{-\infty}^{\infty} |D_1 u(t_1, x_2, \ldots, x_n)|\, dt_1.$$

Here by D_i we denote the partial derivative with respect to i-th coordinate. Analogous inequalities hold with x_1 replaced by $x_2, \ldots, x_n$. Hence

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |D_i u|\, dt_i\right)^{\frac{1}{n-1}}.$$

Now we integrate both sides with respect to $x_1 \in \mathbb{R}$. Note that exactly one integral in the product on the right hand side does not depend on x_1. Applying Hölder's inequality to the remaining $n-1$ integrals yields

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1 \le \left(\int_{-\infty}^{\infty} |D_1 u|\, dt_1\right)^{\frac{1}{n-1}} \prod_{i=2}^n \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |D_i u|\, dt_i\, dx_1\right)^{\frac{1}{n-1}}.$$

Next, we integrate both sides with respect to $x_2 \in \mathbb{R}$ and apply Hölder's inequality in a similar way as above. This leads to an inequality which we then integrate with respect to $x_3 \in \mathbb{R}$ etc. In the end, we obtain the inequality

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}}\, dx \le \prod_{i=1}^n \left(\int_{\mathbb{R}^n} |D_i u|\, dx\right)^{\frac{1}{n-1}},$$

which readily implies (2.8) with $p = 1$.

Step 2. General case. Let $u \in C_0^\infty(\mathbb{R}^n)$. Define a nonnegative function f of class C^1 by

$$f^{\frac{n}{n-1}} = |u|^{\frac{np}{n-p}}.$$

Applying (2.8) with $p = 1$ to f yields

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\right)^{\frac{n-1}{n}} = \left(\int_{\mathbb{R}^n} f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le C\int_{\mathbb{R}^n} |\nabla f|.$$

Since

$$|\nabla f| = \frac{p(n-1)}{n-p}|u|^{\frac{n(p-1)}{n-p}}|\nabla u|,$$

the theorem easily follows by Hölder's inequality (with suitable exponents) applied to $\int_{\mathbb{R}^n}|\nabla f|$. The proof is complete. □

2.8. **Change of variables.** We say that the mapping $T:\Omega\to\mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$ is *bi-Lipschitz* if there is a constant $C\geq 1$ such that

$$C^{-1}|x-y|\leq |T(x)-T(y)|\leq C|x-y|\,,$$

for all $x,y\in\Omega$. Obviously a bi-Lipschitz mapping is a homeomorphism. It follows from the Rademacher theorem that bi-Lipschitz mappings are differentiable a.e. In particular the Jacobian

$$J_T(x)=\det DT(x)$$

is defined a.e. It turns out that the classical change of variables formula, usually stated for C^1 diffeomorphisms, is also true for bi-Lipschitz mappings.

Theorem 2.45. *Let $T:\Omega\to\mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$, be a bi-Lipschitz homeomorphism and let $f:T(\Omega)\to\mathbb{R}$ be measurable. Then*

$$\int_\Omega (f\circ T)|J_T| = \int_{T(\Omega)} f\,,$$

in the sense that if one of the integrals exists then the second one exists and the integrals are equal one to another. □

As a consequence we obtain that the Sobolev space $W^{1,p}$ is invariant under the bi-Lipschitz change of variables. Namely we have

Theorem 2.46. *Let $T:\Omega_1\to\Omega_2$, be a bi-Lipschitz homeomorphism between domains $\Omega_2,\Omega_2\subset\mathbb{R}^n$. Then $u\in W^{1,p}(\Omega_2)$, $1\leq p\leq\infty$ if and only if $v=u\circ T\in W^{1,p}(\Omega_1)$, and*

$$Dv(x)=Du(T(x))\cdot DT(x) \tag{2.9}$$

for almost all $x\in\Omega_1$. Moreover the transformation $T^*:W^{1,p}(\Omega_2)\to W^{1,p}(\Omega_1)$ given by $T^*u=u\circ T$ is an isomorphism of Sobolev spaces.

Proof. Assume first that u is locally Lipschitz. Then (2.9) is obvious. Since T is Lipschitz, we have $|DT|\leq C$, so the chain rule (2.9) implies

$$|Dv(x)|^p\leq C|Du(T(x))|^p.$$

The fact that T is bi-Lipschitz implies $|J_T|>C$ and hence

$$|Dv(x)|^p\leq C|Du(T(x))|^p|J_T(x)|$$

Now applying the change of variables formula we conclude

$$\int_{\Omega_1}|Dv|^p\leq C\int_{\Omega_1}|Du(T(x))|^p|J_T(x)| = C\int_{\Omega_2}|Du|^p.$$

By a similar argument $\int_{\Omega_1} |v|^p \leq C \int_{\Omega_2} |u|^p$, and hence $\|T^*u\|_{W^{1,p}(\Omega_1)} \leq C\|u\|_{W^{1,p}(\Omega_2)}$. We proved the inequality when u is locally Lipschitz. By the density argument Theorem 2.5 and by Corollary 2.28 it is true for any $u \in W^{1,p}(\Omega_2)$.

Applying the above argument to T^{-1} we conclude that T^* is an isomorphism of Sobolev spaces. Finally the density argument proves (2.9) for any $u \in W^{1,p}(\Omega_2)$. □

Similarly one can prove that diffeomorphisms with higher order regularity preserve higher order Sobolev spaces.

Theorem 2.47. *Let $T : \Omega_1 \to \Omega_2$ be a C^m-diffeomorphism of domains $\Omega_1, \Omega_2 \subset \mathbb{R}^n$. Assume that derivatives of order up to m of T and T^{-1} are bounded. Then $u \in W^{m,p}(\Omega_2)$ if and only if $u \circ T \in W^{m,p}(\Omega_1)$. Moreover the transformation $T^* : W^{m,p}(\Omega_2) \to W^{m,p}(\Omega_1)$ given by $T^*u = u \circ T$ is an isomorphism of Sobolev spaces.* □

The change of variables transformation shows that one can define Sobolev spaces on manifolds: If M is a compact smooth manifold without boundary, we say that $u \in W^{m,p}(M)$ if for every coordinate map $\varphi : \mathbb{R}^n \supset U \to M$ end every $U' \Subset U$, the function $u \circ \varphi|_{U'} \in W^{m,p}(U')$. We need to restrict to compact sub-domains of U, because otherwise the derivatives of the mapping φ would not be necessary bounded which could affect integrability of derivatives of $u \circ \varphi$. With a minor modifications we can also define Sobolev functions on compact manifolds with boundary.

2.9. **Extension operator.**

Definition 2.48. Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p \leq \infty$. A bounded linear operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that $Eu|_\Omega = u$ for all $u \in W^{1,p}(\Omega)$ is called an *extension operator*.

We have already seen that there is an extension operator

$$E : W^{1,p}(\mathbb{R}^n_+) \to W^{1,p}(\mathbb{R}^n)$$

defined by reflecting the function u across the boundary.

Definition 2.49. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ is a *Lipschitz domain* if the boundary $\partial\Omega$ is locally a graph of a Lipschitz function.

Theorem 2.50. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \leq p \leq \infty$. Then there is an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$.*

Proof. We will only sketch the proof and leave details to the reader. If Ω is a bounded Lipschitz domain we use partition of unity to localize u near the boundary, next we flat small parts of the boundary using bi-Lipschitz homeomorphisms and we extend the localized pieces of the function across the boundary using the reflection. Finally we come back using the inverse bi-Lipschitz homeomorphism and we are done. Note that in this argument we use the fact that Sobolev functions are invariant under a bi-Lipschitz change of variables. □

The extension operator can be used to prove a version of the Sobolev embedding theorem for Lispchitz domains. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be an extension operator. Then invoking Theorem 2.44 for $1 \leq p < n$ we have

$$\begin{aligned} \left(\int_\Omega |u|^{p^*} \, dx \right)^{1/p^*} &\leq \left(\int_{\mathbb{R}^n} |Eu|^{p^*} \, dx \right)^{1/p^*} \leq C \left(\int_{\mathbb{R}^n} |\nabla (Eu)|^p \, dx \right)^{1/p} \\ &\leq C \left(\left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p} + \left(\int_\Omega |u|^p \, dx \right)^{1/p} \right). \end{aligned}$$

Hence we proved the following result

Proposition 2.51. *If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $1 \leq p < n$, then $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$. Moreover*

$$\|u\|_{L^{p^*}(\Omega)} \leq C(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}) \, .$$

2.10. **Compactness.** One of the most important results in the theory of Sobolev spaces is the following theorem.

Theorem 2.52 (Rellich–Kondrachov)**.** *Let Ω be a bounded Lipschitz domain. The the embedding*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

provided $1 \leq p < n$ and $q < p^* = np/(n-p)$ or $p \geq n$ and $q < \infty$. □

Remark 2.53. In the limiting case, $1 \leq p < n$ and $q = p^*$, the embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ *not* compact.

As an application of the theorem we prove the following result.

Theorem 2.54 (Sobolev–Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \leq p < n$. Then for every $u \in W^{1,p}(\Omega)$*

$$\left(\int_\Omega |u - u_\Omega|^{p^*} \, dx \right)^{1/p^*} \leq C(\Omega, p) \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p},$$

where $p^* = np/(n-p)$.

Proof. Applying Proposition 2.51 to $u - u_\Omega$ we see that it remains to prove that

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)},$$

for all $u \in W^{1,p}(\Omega)$ with $\int_\Omega u = 0$.

Suppose this is not true. Then there is a sequence $u_k \in W^{1,p}(\Omega)$, such that $\int_\Omega u_k\, dx = 0$ and

$$\int_\Omega |u_k|^p\, dx \geq k \int_\Omega |\nabla u_k|^p\, dx. \tag{2.10}$$

Multiplying u_k by a suitable constant we may further assume that

$$\int_\Omega |u_k|^p = 1.$$

Since the embedding $W^{1,p}(\Omega) \subset L^p(\Omega)$ is compact we may subtract a subsequence u_{k_i} such that $u_{k_i} \to u$ in $L^p(\Omega)$. Hence $\int_\Omega |u|^p = 1$ and $\int_\Omega u = 0$. Inequality (2.10) implies that $\nabla u_{k_i} \to 0$ in $L^p(\Omega)$. Hence u_{k_i} is a Cauchy sequence in $W^{1,p}(\Omega)$ and thus $u \in W^{1,p}(\Omega)$, $\nabla u = 0$ a.e. which means u is constant. This is a contradiction because $\int_\Omega u = 0$ and $\|u\|_p = 1$. □

Almost the same argument implies the following result.

Proposition 2.55. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $1 \leq p < n$, and let $E \subset \Omega$, be a measurable set of positive measure, $|E| > 0$. Then*

$$\left(\int_\Omega |u|^{p^*}\, dx \right)^{1/p^*} \leq C(\Omega, E, p) \left(\int_\Omega |\nabla u|^p\, dx \right)^{1/p}$$

for all $u \in W^{1,p}(\Omega)$ with $u|_E \equiv 0$.

The argument employed in the proof of Theorem 2.54 establishes also the following result.

Theorem 2.56. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \leq p < \infty$. Then*

$$\left(\int_\Omega |u - u_\Omega|^p\, dx \right)^{1/p} \leq C(n,p) \left(\int_\Omega |\nabla u|^p\, dx \right)^{1/p}$$

for all $u \in W^{1,p}(\Omega)$.

Later we will need the following special case.

Corollary 2.57. *Let $1 \leq p < \infty$, and $n \geq 2$. Then for any $u \in W^{1,p}(B(2r) \setminus B(r))$ we have*

$$\left(\int_{B(2r)\setminus B(r)} |u - u_{B(2r)\setminus B(r)}|^p\, dx \right)^{1/p} \leq C(n,p) r \left(\int_{B(2r)\setminus B(r)} |\nabla u|^p\, dx \right)^{1/p}.$$

Proof. The assumption $n \geq 2$ is to guarantee that the annulus $B(2r) \setminus B(r)$ is connected. It remains to prove that the constant in the inequality is proportional to the radius of the ball. This easily follows from the scaling argument: If the radius of the ball is one, then the inequality holds with some constant $C = C(n,p)$. If the radius is arbitrary, then by a linear change of variables we can reduce it to the case in which $r = 1$. □

Corollary 2.58. *Let $1 \leq p < \infty$. Then for any $u \in W^{1,p}(B(r))$ we have*

$$\left(\fint_{B(r)} |u(x) - u_{B(r)}|\, dx \right)^{1/p} \leq C(n,p) r \left(\fint_{B(r)} |\nabla u(x)|^p\, dx \right)^{1/p} .$$

If $1 \leq p < n$, then the Sobolev function is integrable with the exponent $p^* = np/(n-p)$. If $p > n$, it is Hölder continuous. What about the case $p = n$? Since $p^* \to \infty$ as $p \to n^-$, we see that a function $u \in W^{1,n}$ is integrable with any finite exponent, at least locally. Thus one may expect that u is bounded, but that is not true in general: we have seen en example of unbounded function in $W^{1,n}$, $u(x) = \log|\log|x||$. The best possible result in the limiting case is

Theorem 2.59 (Trudinger)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exist constants C_1, C_2 depending on Ω only such that*

$$\fint_\Omega \exp\left(\frac{|u - u_\Omega|}{C_1 \|\nabla u\|_{L^n(\Omega)}} \right)^{\frac{n}{n-1}} \leq C_2 \, .$$

for any $u \in W^{1,n}(\Omega)$.

The idea of the proof is as follows. Observe that

$$\int_\Omega \exp|f|^{\frac{n}{n-1}} = \sum_{k=0}^\infty \frac{1}{k!} \int_\Omega |f|^{\frac{nk}{n-1}} \, . \tag{2.11}$$

If

$$f = \frac{|u - u_\Omega|}{C_1 \|\nabla u\|_{L^n(\Omega)}} \, ,$$

then the integrals on the right hand side of (2.11) are finite by the Sobolev-Poincaré inequality and in order to prove that the series converges it suffices to obtain precise estimates for the constants. □

2.11. **Hölder continuity again.** Now we can establish slightly better versions of the Hölder continuity results in the case $n < p < \infty$, but to do this we have to slightly modify an extension operator.

If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, then, as we know, there is an extension operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

and hence

$$\|Eu\|_{L^p(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)} \leq C \left(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \right) .$$

In particular

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq C \left(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \right) .$$

It turns out that we can modify an extension operator to get a linear mapping

$$\tilde{E}: W^{1,p}(\Omega) \to W^{1,p}_{\rm loc}(\mathbb{R}^n)$$

such that

$$\|\nabla(\tilde{E}u)\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\Omega)}\, . \tag{2.12}$$

That is easy, we simply define

$$\tilde{E}u = E(u - u_\Omega) + u_\Omega.$$

Note that in general $\tilde{E}u \notin W^{1,p}(\mathbb{R}^n)$, because if u is a constant function, then $\tilde{E}u$ is the same constant function and thus it is not integrable over $\mathbb{R}^n$. The estimate (2.12) is easy to prove. Namely we have

$$\begin{aligned} \|\nabla(\tilde{E}u)\|_{L^p(\mathbb{R}^n)} &= \|\nabla E(u-u_\Omega)\|_{L^p(\mathbb{R}^n)} \\ &\leq C\left(\|u-u_\Omega\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\right) \\ &\leq C'\|\nabla u\|_{L^p(\Omega)}\, , \end{aligned}$$

where the last inequality follows from Theorem 2.56.

In particular if $\Omega = B$ is a ball, then

$$\tilde{E}: W^{1,p}(B) \to W^{1,p}(2B),$$

$$\|\nabla(\tilde{E}u)\|_{L^p(2B)} \leq C(n,p)\|\nabla u\|_{L^p(B)}\, . \tag{2.13}$$

A scaling argument[8] shows that the constant in (2.13) is independent of the radius. We can use this result to prove the following improvement of Corollary 2.39.

Theorem 2.60. *If $u \in W^{1,p}(B)$, $n < p < \infty$, then $u \in C^{0,1-\frac{n}{p}}(B)$ and*

$$|u(x) - u(y)| \leq C(n,p)|x-y|^{1-\frac{n}{p}}\left(\int_B |\nabla u|^p\right)^{1/p}$$

for all $x, y \in B$.

Proof. Applying Corollary 2.39 to the function $\tilde{E}u$ and $x, y \in B$ we have

$$\begin{aligned} |u(x)-u(y)| &= |\tilde{E}u(x) - \tilde{E}u(y)| \\ &\leq C|x-y|^{1-\frac{n}{p}}\left(\int_{2B} |\nabla(\tilde{E}u)|^p\right)^{1/p} \\ &\leq C'|x-y|^{1-\frac{n}{p}}\left(\int_B |\nabla u|^p\right)^{1/p}\, . \end{aligned}$$

□

The extension result (2.12) easily implies

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[8]Linear change of variables which reduces the inequality to the one in the unit ball.

Corollary 2.61. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then for $n < p < \infty$ we have $W^{1,p}(\Omega) \subset C^{0,1-\frac{n}{p}}(\Omega)$ and*

$$|u(x) - u(y)| \leq C(\Omega, p)|x - y|^{1-\frac{n}{p}} \left(\int_\Omega |\nabla u|^p \right)^{1/p}$$

for all $x, y \in \Omega$.

In many situations we can obtain Sobolev estimates for solutions to partial differential equations and then we can deduce that the solution is actually a classical C^∞ function using the following result. This will be precisely the way we will prove that the weakly harmonic functions are C^∞ classical harmonic functions.

Corollary 2.62. *Let $1 \leq p < \infty$. If $u \in W^{k,p}(\Omega)$ for all $k = 1, 2, \ldots$, then $u \in C^\infty(\Omega)$.*

Proof. If $p > n$, then $W^{1,p} \subset C^{0,\alpha}$, so $W^{k,p} \subset C^{k-1,\alpha}$ and the claim follows. Let $p < n$. Take k such that $kp < n$, but $(k+1)p > n$. Then $W^{1,p} \subset L^{np/(n-p)}$, so by induction

$$W^{k+1,p} \subset W^{k,np/(n-p)} \subset W^{k-1,np/(n-2p)} \subset \ldots \subset W^{1,np/(n-kp)} \subset C^{0,\alpha},$$

because $np/(n - kp) > n$. Hence $W^{m,p} \subset C^{m-k-1,\alpha}$ and then the claim follows. □

2.12. **Traces.** Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We want to restrict the function $u \in W^{1,p}(\Omega)$ to the boundary of Ω. If $p > n$, then the function u is Hölder continuous and such a restriction makes sense. However if $p \leq n$, then the function can be essentially discontinuous everywhere and the restriction makes no sense when understood in the usual way.

Thus in the case $p \leq n$ we want to describe the trace in the following way. Find a function space defined on the boundary $X(\partial\Omega)$ with a norm $\|\cdot\|_X$ such that the operator of restriction $Tu = u|_{\partial\Omega}$ defined for $u \in C^\infty(\overline{\Omega})$ is continuous in the sense that it satisfies the estimate $\|Tu\|_X \leq C\|u\|_{W^{1,p}(\Omega)}$. We will assume that Ω is a bounded Lipschitz domain. Observe that in this case the space $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ (Theorem 2.9) and hence T extends in a unique way to an operator defined on $W^{1,p}(\Omega)$.

Theorem 2.63. *Let Ω be a bounded Lipschitz domain, and $1 \leq p < n$. Then there exists a unique bounded operator*

$$\mathrm{Tr} : W^{1,p}(\Omega) \to L^{p(n-1)/(n-p)}(\partial\Omega),$$

such that $\mathrm{Tr}(u) = u|_{\partial\Omega}$, for all $u \in C^\infty(\overline{\Omega})$.

Remark 2.64. The operator Tr is called a *trace operator.*

Proof. Using the partition of unity and flatting the boundary argument, it suffices to assume that $\Omega = Q^n = Q^{n-1} \times [0,1]$, $u \in C^\infty(\overline{\Omega})$, $\operatorname{supp} u \subset Q^{n-1} \times [0, 1/2)$, and prove the estimate

$$\|u\|_{L^q(Q^{n-1})} \leq C\left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p},$$

where $q = p(n-1)/(n-p)$, and $Q^{n-1} = Q^{n-1} \times \{0\}$.

In Q^n we use coordinates (x', t), where $x' \in Q^{n-1}$, $t \in [0,1]$. Let $w = |u|^q$. We have

$$w(x', 0) = -\int_0^1 \frac{\partial w}{\partial t}(x', t) \, dt,$$

and hence

$$|u(x', 0)|^q \leq q \int_0^1 |u(x', t)|^{q-1} \left|\frac{\partial u}{\partial t}(x', t)\right| dt.$$

Now we integrate both sides with respect to $x' \in Q^{n-1}$. If $p = 1$, then $q = 1$ and the theorem follows. If $p > 1$, we use Hölder's inequality which yields

$$\begin{aligned}\int_{Q^{n-1}} |u(x', 0)|^{\frac{p(n-1)}{n-p}} \, dx' \quad &\leq \quad \frac{p(n-1)}{n-p} \left(\int_{Q^{n-1}} \int_0^1 |u(x', t)|^{\frac{np}{n-p}}, dt \, dx'\right)^{1-1/p} \\ &\times \left(\int_{Q^{n-1}} \int_0^1 \left|\frac{\partial u}{\partial t}(x', t)\right|^p dt \, dx'\right)^{1/p}.\end{aligned}$$

Now we use Sobolev embedding theorem (Proposition 2.55) to estimate the first integral on the right-hand side and the theorem follows. □

The trace theorem proved here is far from being optimal. Actually it is possible to obtain a complete characterization of functions on the boundary that arise as traces of Sobolev functions.

Definition 2.65. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $1 < p < \infty$. We say that $u \in W^{1-\frac{1}{p},p}(\partial\Omega)$ if $u \in L^p(\partial\Omega)$ and

$$A(u) := \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p-2}} \, dx \, dy\right)^{1/p} \leq \infty.$$

The space $W^{1-\frac{1}{p},p}(\partial\Omega)$ is a Banach space with the norm

$$\|u\|_{1-\frac{1}{p},p} = \|u\|_{L^p(\partial\Omega)} + A(u).$$

A complete characterization of traces is provided by the following result.

Theorem 2.66 (Gagliardo). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $u \in C^\infty(\overline{\Omega})$ we define $\operatorname{Tr}(u) = u|_{\partial\Omega}$. If $1 < p < \infty$, then the operator* Tr *extends uniquely to the bounded operator*

$$\operatorname{Tr} : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega).$$

Moreover there exists a bounded extension operator

$$\mathrm{Ext} : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$$

such that $\mathrm{Tr}(\mathrm{Ext}\, u) = u$*.*

If $p = 1$, it turns out that the space of traces coincides with $L^1(\partial\Omega)$, but, surprisingly, there is no *linear* extension operator $\mathrm{Ext} : L^1(\partial\Omega) \to W^{1,1}(\Omega)$.

The following result relates the zero boundary values to the trace theorem.

Theorem 2.67. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz boundary and* $1 \le p < \infty$*. Let* $u \in W^{1,p}(\Omega)$*. Then* $u \in W_0^{1,p}(\Omega)$ *if and only if* $\mathrm{Tr}\, u = 0$ *a.e. on* $\partial\Omega$*.* □

3. Linear elliptic equations

3.1. **Weyl lemma.** Now we are ready to prove the first important regularity result: weakly harmonic functions are C^∞ classical harmonic functions.

Theorem 3.1 (Weyl lemma)**.** *If* $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *is a weak solution to the Laplace equation* $\Delta u = 0$*, then* $u \in C^\infty(\Omega)$ *and hence* u *is a classical harmonic function.*

Proof. By the definition u is a weak solution to $\Delta u = 0$ if and only if

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0 \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{3.1}$$

Note that (3.1) holds also for $\varphi \in W^{1,2}(\Omega)$ with compact support — simply by approximating such φ by compactly supported smooth functions.

In the first step of the proof we will derive the so called *Caccioppoli estimates.* The idea is very simple and it easily generalizes to more complicated elliptic equations or systems, where it is frequently employed.

Fix concentric balls $B(r) \subset\subset B(R) \subset\subset \Omega$ and let $\eta \in C_0^\infty(B(R))$, $0 \le \eta \le 1$, $\eta|_{B(r)} \equiv 1$, $|\nabla\eta| \le 2/(R-r)$ be a cutoff function.

Applying $\varphi = (u-c)\eta^2$ to (3.1) we obtain

$$\int_\Omega \nabla u \cdot \left(\nabla u \eta^2 + 2(u-c)\eta\nabla\eta\right) = 0,$$

so

$$\int_\Omega |\nabla u|^2\eta^2 \le 2\int_\Omega |u-c|\eta|\nabla u|\,|\nabla\eta| \le 2\left(\int_\Omega |u-c|^2|\nabla\eta|^2\right)^{1/2}\left(\int_\Omega |\nabla u|^2\eta^2\right)^{1/2}.$$

Hence

$$\int_\Omega |\nabla u|^2\eta^2 \le 4\int_\Omega |u-c|^2|\nabla\eta|^2,$$

which yields the following Caccioppoli inequality

$$\int_{B(r)} |\nabla u|^2 \leq \frac{16}{(R-r)^2} \int_{B(R)\setminus B(r)} |u-c|^2 \,, \tag{3.2}$$

for any $c \in \mathbb{R}$. In particular we get

$$\int_{B(r)} |\nabla u|^2 \leq C(R,r) \int_{B(R)} |u|^2. \tag{3.3}$$

Assume for a moment that we already know that $u \in C^\infty$. Then also the derivatives of u are harmonic and hence (3.3) applies to derivatives of u, so for $r < r' < R$ we get

$$\int_{B(r)} |\nabla^2 u|^2 \leq C_1 \int_{B(r')} |\nabla u|^2 \leq C_2 \int_{B(R)} |u|^2.$$

Repeating the argument with higher order derivatives we obtain

$$\int_{B(R/2)} |\nabla^k u|^2 \, dx \leq C(R,k) \int_{B(R)} |u|^2,$$

for $k = 1, 2, 3, \ldots$ In other words

$$\|u\|_{W^{k,2}(B(R/2))} \leq C(R,k) \|u\|_{L^2(B(R))}. \tag{3.4}$$

The inequality was proved under the assumption that $u \in C^\infty$. We shall prove now that (3.4) holds also for any weakly harmonic function $u \in W^{1,2}_{\rm loc}(\Omega)$.

Let $u_\varepsilon(x) = \int u(x-y)\phi_\varepsilon(y)\, dy$ be a standard mollifier approximation where $\phi_\varepsilon(y) = \varepsilon^{-n}\phi(y/\varepsilon)$, $\phi \in C_0^\infty(B(0,1))$, $\phi \geq 0$, $\int \phi = 1$.

Then $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$, where Ω_ε consists of points in Ω with the distance to the boundary bigger than ε. Note that u_ε is harmonic in Ω_ε. Indeed, for any $\varphi \in C_0^\infty(\Omega_\varepsilon)$ we have

$$\begin{aligned}
\int_\Omega \nabla u_\varepsilon(x) \nabla\varphi(x)\, dx &= \int \left(\int u(y) \nabla_x \phi_\varepsilon(x-y)\, dy \right) \nabla\varphi(x)\, dx \\
&= \int \left(\int \nabla_x u(x-y) \phi_\varepsilon(y)\, dy \right) \nabla\varphi(x)\, dx \\
&= \int \left(\int \nabla_x u(x-y) \nabla\varphi(x)\, dx \right) \phi_\varepsilon(y)\, dy = 0.
\end{aligned}$$

Hence u_ε is a weakly harmonic function and since u_ε is C^∞ smooth we conclude that u_ε is a classical harmonic function. Thus (3.4) holds for each u_ε. Since $u_\varepsilon \to u$ in $L^2(B(R))$, we obtain that u_ε is a Cauchy sequence in $W^{k,2}(B(R/2))$ and passing to the limit yields (3.4) for u and all $k = 1, 2, 3, \ldots$ Hence by the Sobolev embedding theorem (Corollary 2.62) $u \in C^\infty$. The proof is complete. □

We used in the above proof the following argument: if u is harmonic, then the approximation by convolution u_ε is harmonic too. This argument, however, does not generalize to more complicated equations, only to equations with constant coefficients. Later we will use a method of difference quotients that applies to more general situations, but now we show some basic applications of the Caccioppli estimates.

Taking $R = 2r$ in (3.2) and applying Poincaré inequality (Corollary 2.57) we get

$$\begin{aligned}\int_{B(r)} |\nabla u|^2\,dx &\leq \frac{16}{r^2}\int_{B(2r)\setminus B(r)} |u - u_{B(2r)\setminus B(r)}|^2\,dx\\ &\leq C(n)\int_{B(2r)\setminus B(r)} |\nabla u|^2\,dx.\end{aligned}$$

We have obtained the estimate of the integral over $B(r)$ by an integral over an annulus. Now we add $C(n)\int_{B(r)}|\nabla u|^2$ to both sides of the inequality to fill the hole in the annulus. We get

$$\int_{B(r)} |\nabla u|^2\,dx \leq \underbrace{\frac{C(n)}{C(n)+1}}_{<1}\int_{B(2r)} |\nabla u|^2\,dx. \tag{3.5}$$

For obvious reasons the argument is called *hole–filling.*

It is crucial that the coefficient in (3.5) is strictly less than 1. We will show some applications of this fact.

Theorem 3.2. *If u is a harmonic function on $\mathbb{R}^n$ with $|\nabla u| \in L^2(\mathbb{R}^n)$, then u is constant.*

Proof. Passing to the limit as $r \to \infty$ in inequality (3.5) yields

$$\int_{\mathbb{R}^n} |\nabla u|^2\,dx \leq \theta\int_{\mathbb{R}^n} |\nabla u|^2, \qquad \theta < 1\,.$$

Hence $|\nabla u| = 0$ a.e. and thus u is constant. □

Corollary 3.3. *Any bounded harmonic function on $\mathbb{R}^2$ is constant.*

Proof. By Caccioppoli inequality (3.2) we get

$$\int_{B(r)} |\nabla u|^2 \leq \frac{C}{r^2}\int_{B(2r)} |u|^2 \leq C'.$$

Since u is bounded, the constant C' does not depend on r. Passing to the limit as $r \to \infty$ we conclude that $\int_{\mathbb{R}^2}|\nabla u|^2\,dx < \infty$ and hence u is constant by the previous result. □

Remark 3.4. The corollary holds in $\mathbb{R}^n$ for any $n \geq 1$, but the proof is different.

Remark 3.5. We proved that given Sobolev boundary conditions there is a Sobolev solution to the Dirichlet problem and by the Weyl lemma this solution is a classical harmonic function. However, it is not clear that the solution is continuous up to the boundary, even if the boundary conditions are smooth. We will address this problem later in Section 3.4.

3.2. **Linear elliptic equations: existence of solutions.** The Dirichlet problem

$$\begin{cases} -\Delta u = 0 \\ u \in W^{1,2}_w(\Omega). \end{cases}$$

can be rewritten as a non-homogeneous problem with zero boundary data. For simplicity assume that the function w is so regular that there is $h \in W^{2,2}(\Omega) \cap W^{1,2}_w(\Omega)$. Denote $\Delta h = f \in L^2(\Omega)$. Then $v = u - h \in W^{1,2}_0(\Omega)$ solves the following non-homogeneous[9] Dirichlet problem

$$\begin{cases} -\Delta u = f \in L^2(\Omega), \\ u \in W^{1,2}_0(\Omega). \end{cases}$$

That means

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{3.6}$$

We will consider now more general elliptic boundary value problems in a bounded domain Ω:

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u \in W^{1,2}_0(\Omega), \end{cases} \tag{3.7}$$

where

$$Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a^{ij}(x) \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i} + c(x) u.$$

We assume that the coefficients are bounded measurable functions

$$a^{ij}(x), b^i(x), c(x) \in L^\infty(\Omega)$$

and $f \in L^2(\Omega)$. We also assume that the matrix $A(x) = [a^{ij}(x)]$ is symmetric, i.e.

$$a^{ij}(x) = a^{ji}(x) \quad \text{a.e.}$$

Definition 3.6. The operator L is called *(uniformly) elliptic* if there is $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

This is to say that the matrix $A(x)$ is positive definite with the smallest eigenvalue bounded from below by a positive constant.

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[9] i.e. with non-zero right hand side.

Remark 3.7. We say that the operator L is in the *divergence form*, because we can write

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right) = -\mathrm{div}\,\left(A(x)\nabla u\right).$$

One also considers elliptic operators in *non-divergence form*

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u\,,$$

but we will discuss operators in the divergence from only.

By analogy with (3.6) we define weak solutions to $Lu = f$ as follows.

Definition 3.8. We say that u is a *weak solution* to (3.7) if

$$\int_\Omega \left(\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i}\varphi + c(x)u\varphi\right)\, dx = \int_\Omega f\varphi\, dx \tag{3.8}$$

for all $\varphi \in C_0^\infty(\Omega)$.

By a density argument (3.8) is true for all $\varphi \in W_0^{1,2}(\Omega)$.

It is convenient to associate a certain bilinear form with the operator L.

Definition 3.9. The bilinear form

$$B[\cdot,\cdot] : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$$

associated with the elliptic operator L is defined by

$$B[u,v] = \int_\Omega \left(\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i}v + c(x)uv\right)\, dx.$$

Clearly u is a weak solution of (3.7), if

$$B[u,v] = \langle f, v\rangle \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

Here $\langle f, v\rangle$ stands for the inner product in $L^2(\Omega)$. The existence of weak solutions will follows from the Lax-Milgram theorem.

Theorem 3.10 (Lax-Milgram)**.** *Assume that*

$$B : H \times H \to \mathbb{R}$$

is a bilinear form on a Hilbert space such that for some constants $\alpha, \beta > 0$ *we have*

$$|B[u,v]| \le \alpha\|u\|\,\|v\| \quad \textit{for all } u,v \in H, \tag{3.9}$$

and

$$B[u,u] \ge \beta\|u\|^2 \quad \textit{for all } u \in H. \tag{3.10}$$

Then for any bounded linear functional $f \in H^*$ *there is a unique element* $u \in H$ *such that*

$$B[u,v] = \langle f, v\rangle \quad \textit{for all } v \in H.$$

Thus in order to prove existence of solutions of the problem (3.7) it suffices to show that a bilinear form associated with the operator L satisfies estimates (3.9) and (3.10).

Theorem 3.11 (Energy estimates)**.** *Let* B *be a bilinear form associated with the elliptic operator* L*. Then there exist constants* $\alpha, \beta > 0$ *and* $\gamma \geq 0$ *such that*

$$|B[u,v]| \leq \alpha \|u\|_{1,2}\|v\|_{1,2} \tag{3.11}$$

and

$$B[u,u] + \gamma\|u\|_2^2 \geq \beta\|u\|_{1,2}^2 \tag{3.12}$$

for all $u, v \in W_0^{1,2}(\Omega)$*.*

Remark 3.12. The existence of the term $\gamma\|u\|_2^2$ in the above estimate does not allow us to apply directly the Lax-Milgram theorem. We will take care of it in the next result.

Proof. The estimate (3.11) is very easy to prove.

$$\begin{aligned} |B[u,v]| &\leq \sum_{i,j=1}^n \|a^{ij}\|_\infty \int_\Omega |\nabla u|\,|\nabla v|\,dx \\ &+ \sum_{i=1}^n \|b^i\|_\infty \int_\Omega |\nabla u||v|\,dx + \|c\|_\infty \int_\Omega |u||v|\,dx \\ &\leq \alpha\|u\|_{1,2}\|v\|_{1,2} \end{aligned}$$

for some $\alpha > 0$. To prove (3.12) we will need to use ellipticity of L. We have

$$\begin{aligned} \theta\int_\Omega |\nabla u|^2\,dx &\leq \int_\Omega \sum_{i,j=1}^n a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx \\ &= B[u,u] - \int_\Omega \left(\sum_{i=1}^n b^i(x)\frac{\partial u}{\partial x_i}\,u + c(x)u^2\right)\,dx \\ &\leq B[u,u] + \sum_{i=1}^n \|b^i\|_\infty \int_\Omega |\nabla u||u|\,dx + \|c\|_\infty \int_\Omega |u|^2\,dx \end{aligned}$$

It is easy to prove that for $a, b > 0$ and $\varepsilon > 0$ we have

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}\,. \tag{3.13}$$

Hence

$$\int_\Omega |\nabla u|\,|u|\,dx \leq \varepsilon\int_\Omega |\nabla u|^2\,dx + \frac{1}{4\varepsilon}\int_\Omega |u|^2\,dx\,.$$

Taking $\varepsilon > 0$ so small that

$$\varepsilon \sum_{i=1}^{n} \|b^i\|_\infty < \frac{\theta}{2}$$

we obtain

$$\frac{\theta}{2} \int_\Omega |\nabla u|^2 \, dx \le B[u,u] + C \int_\Omega |u|^2 \, dx \tag{3.14}$$

and it suffices to observe that by the Poincarś inequality, Lemma 1.12, the left hand side of (3.14) is comparable to $\|u\|_{1,2}^2$ for $u \in W_0^{1,2}(\Omega)$. The proof is complete. □

Theorem 3.13. *There is a constant $\gamma \ge 0$ such that for each $\mu \ge \gamma$ and each $f \in L^2(\Omega)$ the following Dirichlet problem*

$$\begin{cases} Lu + \mu u = f, & \textit{in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

has a unique solution.

Remark 3.14. The term μ appears, because the estimates from Theorem 3.11 are not good enough to apply the Lax-Milgram theorem.

Proof. Let γ be as in Theorem 3.11 and let $\mu \ge \gamma$. The form

$$B_\mu[u,v] = B[u,v] + \mu\langle u, v\rangle$$

corresponds to the operator $L_\mu u = Lu + \mu u$. Since

$$B_\mu[u,u] = B[u,u] + \mu\|u\|_2^2 \ge B[u,u] + \gamma\|u\|_2^2 \ge \beta\|u\|_{1,2}^2,$$

the Lax-Milgram theorem applies and the theorem follows.[10] □

In some cases the term μu is not needed.

Theorem 3.15. *If the operator is of the form*

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)u,$$

where $c \ge 0$, then the Dirichlet problem

$$\begin{cases} Lu = f, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

has a unique solution.

Proof. We have

$$B[u,u] \ge \theta \int_\Omega |\nabla u|^2 \, dx + \int_\Omega c(x)|u|^2 \, dx \ge \theta \int_\Omega |\nabla u|^2 \ge C\|u\|_{1,2}^2$$

by the Poincaré inequality, Lemma 1.12. □

[10] The L^2 inner product $v \mapsto \langle f, v\rangle$ defines a bounded linear functional on $W_0^{1,2}(\Omega)$.

Remark 3.16. In the result above, the function $c(x)$ can actually attain negative values as long as the function c is bounded from below by a certain constant that is sufficiently close to zero. Indeed, let μ be the best constant in the Poincaré inequality

$$\int_\Omega |u|^2\, dx \leq \mu \int_\Omega |\nabla u|^2\, dx \quad \text{for } u \in W_0^{1,2}(\Omega).$$

If $-\theta/\mu < \lambda_0 < 0$ and $c(x) \geq \lambda_0$ a.e., then

$$\begin{aligned} B[u,u] &\geq \theta \int_\Omega |\nabla u|^2\, dx + \int_\Omega c(x)|u|^2\, dx \\ &\geq \theta \int_\Omega |\nabla u|^2\, dx + \lambda_0 \int_\Omega |u|^2\, dx \\ &\geq \underbrace{(\theta + \lambda_0\mu)}_{>0} \int_\Omega |\nabla u|^2 \\ &\geq C\|u\|_{1,2}^2. \end{aligned}$$

Again, the last inequality follows from the Poincaré inequality. The above argument shows that estimates for the best constant in the Poincaré inequality play an important role in the theory of elliptic equations.

3.3. **Linear elliptic equations: interior regularity.** Let us start with a simple estimate that mimics the proof of the Caccioppoli estimate for the Laplace operator. As before we assume that L is uniformly elliptic.

Proposition 3.17. *Assume that*

$$a^{ij}, b^i, c \in L^\infty(\Omega)$$

and

$$f \in L^2(\Omega).$$

Assume that $u \in W^{1,2}(\Omega)$ is a weak solution to the equation[11]

$$Lu = f \quad \textit{in } \Omega.$$

Then for $W \Subset \Omega$ we have

$$\|u\|_{W^{1,2}(W)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right),$$

where the constant C depends on W, Ω and $\|a^{ij}\|_\infty$, $\|b^i\|_\infty$, $\|c\|_\infty$. If in addition $u \in W_0^{1,2}(\Omega)$, then

$$\|u\|_{W^{1,2}(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

Proof. Fix a cutoff function ζ such that

$$\zeta \in C_0^\infty(\Omega), \quad 0 \leq \zeta \leq 1 \quad \zeta|_W \equiv 1,$$

[11]We do not assume any boundary conditions.

and take
$$\varphi = \zeta^2 u$$
as a test function. We have
$$\int_\Omega \Big(\sum_{i,j=1}^n a^{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,\zeta^2 + \sum_{i,j=1}^n a^{ij}\frac{\partial u}{\partial x_i} 2\zeta \frac{\partial \zeta}{\partial x_j}\,u + \sum_{i=1}^n b^i \frac{\partial u}{\partial x_i}\zeta^2 u + cu^2\zeta^2\Big)\,dx$$
$$= \int_\Omega f\zeta^2 u\,dx.$$
Ellipticity of L gives
$$\int_\Omega \sum_{i,j=1}^n a^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,\zeta^2\,dx \geq \theta \int_\Omega |\nabla u|^2\zeta^2\,dx.$$
Hence
$$\begin{aligned} \theta\int_\Omega |\nabla u|^2\zeta^2\,dx &\leq C\int_\Omega \big(|\nabla u||\zeta|\,|u| + |u|^2 + |f|\,|u|\big)\ dx \\ &\leq \frac{\theta}{2}\int_\Omega |\nabla u|^2\zeta^2\,dx + C'\int_\Omega (|u|^2+|f|^2)\,dx, \end{aligned}$$
where in the last inequality we employed (3.13). Since $\zeta^2 = 1$ on W we conclude
$$\int_W |\nabla u|^2\,dx \leq C\int_\Omega (|u|^2+|f|^2)\,dx$$
and the first part of the result follows.

If in addition $u \in W^{1,2}_0(\Omega)$, then we can use $\varphi = u$ as a test function and a suitable estimate follows from a similar, but even easier arguments than those explained above. □

In the case of the Laplace equation we could prove C^∞ regularity of solutions by using the fact that if u is a weakly harmonic function, then its convolution approximation is a classical harmonic function. This argument is no longer available for equations $Lu = f$ considered above. Instead, we will use a difference quotient method of Nirenberg. As before we assume that the operator L is uniformly elliptic, but in addition to that we will impose higher regularity conditions for the coefficients. The next result generalizes a priori estimates (3.4) that we proved for the Laplace operator.

Theorem 3.18 (Interior regularity)**.** *Assume that*
$$a^{ij} \in C^1(\Omega),\ b^i, c \in L^\infty(\Omega)$$
and
$$f \in L^2(\Omega).$$
Suppose that $u \in W^{1,2}(\Omega)$ is a weak solution to the equation
$$Lu = f \quad in\ \Omega.$$

Then $u \in W^{2,2}_{\rm loc}(\Omega)$ *and for each* $U \Subset \Omega$ *we have*

$$\|u\|_{W^{2,2}(U)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

where the constant C *depends on* U, Ω *and the coefficients of* L.

Sketch of the proof. The proof is quite involved and instead of a detailed proof we will explain its main idea. The reader may find details in Evans' book, Theorem 6.3.1, page 309. We want to obtain estimates for the second order derivatives of a solution to $Lu = f$. Assume for a moment that u is sufficiently regular. Let $U \Subset W \Subset \Omega$ and let ζ be a *cutoff* function, i.e. $\zeta \in C_0^\infty(W)$, $0 \leq \zeta \leq 1$ and $\zeta|_U \equiv 1$. We will test the equation with the following test function

$$\varphi = -\frac{\partial}{\partial x_k}\left(\zeta^2 \frac{\partial u}{\partial x_k}\right). \tag{3.15}$$

This is a moment where we need to assume high regularity of u — the given test function involves second order derivatives of u. The test function needs to belong to $W_0^{1,2}(\Omega)$, so we need to assume that $u \in W^{3,2}(\Omega)$. We have

$$\begin{aligned}
&-\sum_{i,j}\int_\Omega a^{ij}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}\left(\zeta^2\frac{\partial u}{\partial x_k}\right)\right)\\
&= -\int_\Omega\left(f - \sum_i b^i\frac{\partial u}{\partial x_i} - cu\right)\frac{\partial}{\partial x_k}\left(\zeta^2\frac{\partial u}{\partial x_k}\right).
\end{aligned}$$

We write this equality as $A = B$ and we estimate A and B separately. We have

$$\begin{aligned}
A &= \sum_{i,j}\int_\Omega \frac{\partial}{\partial x_k}\left(a^{ij}\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_j}\left(\zeta^2\frac{\partial u}{\partial x_k}\right)\\
&= \sum_{i,j}\int_\Omega\left(\frac{\partial a^{ij}}{\partial x_k}\frac{\partial u}{\partial x_i} + a^{ij}\frac{\partial^2 u}{\partial x_k\partial x_i}\right)\left(\zeta^2\frac{\partial^2 u}{\partial x_k \partial x_j} + 2\zeta\frac{\partial \zeta}{\partial x_j}\frac{\partial u}{\partial x_k}\right)\\
&= \sum_{i,j}\int_\Omega a^{ij}\frac{\partial^2 u}{\partial x_k\partial x_i}\frac{\partial^2 u}{\partial x_k\partial x_j}\zeta^2 + \text{other terms.}
\end{aligned}$$

Using ellipticity we get

$$\sum_{i,j}\int_\Omega a^{ij}\frac{\partial^2 u}{\partial x_k\partial x_i}\frac{\partial^2 u}{\partial x_k\partial x_j}\zeta^2 \geq \theta\int_\Omega\left|\nabla\frac{\partial u}{\partial x_k}\right|^2\zeta^2,$$

while for other terms we only need a rough estimate

$$\text{other terms} \leq C\int_\Omega\left(|\nabla u|\left|\nabla\frac{\partial u}{\partial x_k}\right| + |\nabla u|^2\right)\zeta.$$

We used here the fact that the functions a^{ij} and $\partial a^{ij}/\partial x_k$ are bounded on the support of ζ and we estimated ζ and $\partial\zeta/\partial x_i$ by a constant. Recall that

$ab \leq \varepsilon a^2 + C(\varepsilon) b^2$, so

$$\text{other terms} \leq \varepsilon \int_\Omega \left| \nabla \frac{\partial u}{\partial x_k} \right|^2 \zeta^2 + C \int_\Omega |\nabla u|^2 .$$

Taking $\varepsilon = \theta/2$ we obtain

$$A \geq \frac{\theta}{2} \int_\Omega \left| \nabla \frac{\partial u}{\partial x_k} \right|^2 \zeta^2 - C \int_\Omega |\nabla u|^2 .$$

Now we estimate B:

$$\begin{aligned} B &\leq C \int_\Omega \big(|f| + |u| + \nabla u| \big) \left| \frac{\partial}{\partial x_k} \left(\zeta^2 \frac{\partial u}{\partial x_k} \right) \right| \\ &\leq \varepsilon \int_\Omega \left| \frac{\partial}{\partial x_k} \left(\zeta^2 \frac{\partial u}{\partial x_k} \right) \right|^2 + C(\varepsilon) \int_\Omega \big(|f|^2 + |u|^2 + |\nabla u|^2 \big) . \end{aligned}$$

We have

$$\begin{aligned} \varepsilon \int_\Omega \left| \frac{\partial}{\partial x_k} \left(\zeta^2 \frac{\partial u}{\partial x_k} \right) \right|^2 &= \varepsilon \int_\Omega \left| 2\zeta \frac{\partial \zeta}{\partial x_k} \frac{\partial u}{\partial x_k} + \zeta^2 \frac{\partial^2 u}{\partial x_k^2} \right|^2 \\ &\leq C\varepsilon \int_\Omega |\nabla u|^2 + C\varepsilon \int_\Omega \zeta^2 \left| \nabla \frac{\partial u}{\partial x_k} \right|^2 . \end{aligned}$$

Taking ε sufficiently small we obtain

$$B \leq \frac{\theta}{4} \int_\Omega \left| \nabla \frac{\partial u}{\partial x_k} \right|^2 \zeta^2 + C \int_\Omega \big(|f|^2 + |u|^2 + |\nabla u|^2 \big).$$

Thus

$$\begin{aligned} &\frac{\theta}{2} \int_\Omega \left| \nabla \frac{\partial u}{\partial x_k} \right|^2 \zeta^2 - C \int_\Omega |\nabla u|^2 \\ &\quad \leq \frac{\theta}{4} \int_\Omega \left| \nabla \frac{\partial u}{\partial x_k} \right|^2 \zeta^2 + C \int_\Omega \big(|f|^2 + |u|^2 + |\nabla u|^2 \big). \end{aligned}$$

Since $\zeta \equiv 1$ on U and $k \in \{1, 2, \dots, n\}$ is any index, we get

$$\int_U |\nabla^2 u|^2 \, dx \leq C \int_\Omega (|f|^2 + |u|^2 + |\nabla u|^2) \, dx, \tag{3.16}$$

so

$$\|u\|_{W^{2,2}(U)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)} \right) . \tag{3.17}$$

This is not yet the estimate we wanted to obtain, because we wanted to have an L^2 norm of u instead of its $W^{1,2}$ norm on the right hand side. Note that the estimate (3.17) is true if we replace Ω by W, where $U \Subset W \Subset \Omega$, so

$$\|u\|_{W^{2,2}(U)} \leq C \left(\|f\|_{L^2(W)} + \|u\|_{W^{1,2}(W)} \right)$$

and hence Proposition 3.17 gives

$$\|u\|_{W^{2,2}(U)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right) . \tag{3.18}$$

The real problem with the above argument is that the test function (3.15) requires high regularity of u. After all, we obtained estimates for the second order derivatives of u in terms of L^2 of u and one may expect that this should also be true for any weak solution $u \in W^{1,2}(\Omega)$. However, we need to use a different test function. The idea is to replace derivatives $\partial/\partial x_k$ in the test function (3.15) by difference quotients. Let

$$D_k^h u(x) = \frac{u(x+he_k)-u(x)}{h}, \quad h \in \mathbb{R}\setminus\{0\}$$

and let

$$\varphi = -D_k^{-h}(\zeta^2 D_k^h u). \tag{3.19}$$

This test function can be applied to $u \in W^{1,2}(\Omega)$. Recall that a difference quotient method has already been used in the proof of Theorem 2.13.

The difference quotients satisfy both the product rule:

$$D_k^h(vu) = (\tau_h v)(D_k^h w) + w D_k^h v,$$

where $(\tau_h v)(x) = v(x+he_k)$ and the integration by parts formula:[12]

$$\int_\Omega v D_k^{-h} w\, dx = -\int_\Omega w D_k^h v\, dx,$$

provided one of the functions is compactly supported in Ω and h is sufficiently small.[13] This is all what we need: the arguments used in the above proof are based mainly on the product rule and the integration by parts. Using the test function (3.19) and following verbatim the above argument one can show that for $U \Subset \Omega$

$$\int_U |D_k^h \nabla u|^2\, dx \leq C \int_\Omega (|f|^2 + |u|^2 + |\nabla u|^2)\, dx$$

Compare this estimate with (3.16). Since the estimate on the right hand side is independent of h, Theorem 2.15 implies that

$$\|u\|_{W^{2,2}(U)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)}\right)$$

and finally the same argument as before leads to

$$\|u\|_{W^{2,2}(U)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

□

Theorem 3.19 (Higher order regularity)**.** *Let m be a nonnegative integer. Assume that*

$$a^{ij}, b^i, c \in C^{m+1}(\Omega)$$

and

$$f \in W^{m,2}(\Omega).$$

[12]The $-h$ that appears in this formula explains why we took $-h$ in (3.19).

[13]Otherwise $x + he_k$ could lie outside Ω and the functions would not be well defined.

If $u \in W^{1,2}(\Omega)$ is a weak solution of $Lu = f$, then

$$u \in W^{m+2,2}_{\mathrm{loc}}(\Omega)$$

and for each $U \Subset \Omega$ we have

$$\|u\|_{W^{m+2,2}(U)} \leq C\left(\|f\|_{W^{m,2}(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

with the coefficient C depending on U, Ω and the coefficients of L only.

Proof. The proof is by induction. For $m = 0$ the result is contained in Theorem 3.18. It remains to show that if the theorem is true for m, it is also true for $m + 1$. For simplicity we will prove this implication for $m = 0$ only. The proof of the implication for general m follows from the same argument, but notation is much more complicated, so we restrict to the case $m = 0$ for the sake of clarity of presentation.

Thus suppose that

$$a^{ij}, b^i, c \in C^2(\Omega) \quad \text{and} \quad f \in W^{1,2}(\Omega)$$

We need to prove that $u \in W^{3,2}_{\mathrm{loc}}(\Omega)$ with suitable estimates. From the induction hypotheses ($m = 0$) we already know that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and

$$\|u\|_{W^{2,2}(U)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right). \tag{3.20}$$

for all $U \Subset \Omega$. Fix $W \Subset U \Subset \Omega$ and take an arbitrary $\varphi \in C^\infty_0(U)$. We will test the equation $Lu = f$ against the function $-\partial\varphi/\partial x_k$. We have

$$\int_\Omega \left(-\sum_{i,j} a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} - \sum_i b^i \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_k} - cu \frac{\partial \varphi}{\partial x_k} \right) dx = -\int_\Omega f \frac{\partial \varphi}{\partial x_k}\, dx.$$

Now we integrate by parts with respect to x_k

$$\int_\Omega \left(\sum_{i,j} \frac{\partial}{\partial x_k}\left(a^{ij}\frac{\partial u}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_j} + \sum_i \frac{\partial}{\partial x_k}\left(b^i \frac{\partial u}{\partial x_i}\right)\varphi + \frac{\partial}{\partial x_k}(cu)\varphi \right) dx$$
$$= \int_\Omega \frac{\partial f}{\partial x_k}\, \varphi\, dx.$$

For simplicity we will write $\partial u/\partial x_k = u_k$. We have

$$\begin{aligned} \int_\Omega \sum_{ij} a^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \quad &+ \quad \sum_{ij} \frac{\partial a^{ij}}{\partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \sum_i b^i \frac{\partial u_k}{\partial x_i} \varphi \\ &+ \quad \sum_i \frac{\partial b^i}{\partial x_k} \frac{\partial u}{\partial x_i} \varphi + c u_k \varphi + \frac{\partial c}{\partial x_k} u\varphi \\ &= \quad \int_\Omega \frac{\partial f}{\partial x_k}\, \varphi. \end{aligned}$$

We integrate by parts with respect to x_j the integral corresponding to the second sum and reorganize terms as follows

$$\int_\Omega \left(\sum_{i,j} a^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \sum_i b^i \frac{\partial u_k}{\partial x_i} \varphi + c u_k \varphi \right) dx$$

$$= \int_\Omega \underbrace{\Big(\frac{\partial f}{\partial x_k} + \sum_{i,j} \frac{\partial}{\partial x_j} \left(\frac{\partial a^{ij}}{\partial x_k} \frac{\partial u}{\partial x_i} \right) - \sum_i \frac{\partial b^i}{\partial x_k} \frac{\partial u}{\partial x_i} - \frac{\partial c}{\partial x_k} u \Big)}_{\tilde{f}} \varphi \, dx.$$

Since the equality is true for any $\varphi \in C_0^\infty(U)$ we see that u_k is a solution of the equation

$$Lu_k = \tilde{f} \quad \text{in } U.$$

The assumed regularity implies that $\tilde{f} \in L^2(U)$ and

$$\|\tilde{f}\|_{L^2(U)} \leq C \left(\|f\|_{W^{1,2}(U)} + \|u\|_{W^{2,2}(U)} \right) \leq C' \left(\|f\|_{W^{1,2}(\Omega)} + \|u\|_{L^2(\Omega)} \right),$$

where the last inequality follows from (3.20). Another application of (3.20) yields

$$\|u_k\|_{W^{2,2}(W)} \leq C \left(\|\tilde{f}\|_{L^2(U)} + \|u_k\|_{L^2(U)} \right) \leq C' \left(\|f\|_{W^{1,2}(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

This simply means that $u \in W^{3,2}(W)$ and

$$\|u\|_{W^{3,2}(W)} \leq C \left(\|f\|_{W^{1,2}(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

The proof is complete. □

As an immediate corollary of Corollary 2.62 we obtain

Theorem 3.20 (C^∞ regularity). *Assume that*

$$a^{ij}, b^i, c \in C^\infty(\Omega)$$

and

$$f \in C^\infty(\Omega).$$

Suppose that $u \in W^{1,2}(\Omega)$ is a weak solution of the equation

$$Lu = f \quad \text{in } \Omega.$$

Then $u \in C^\infty(\Omega)$.

3.4. **Linear elliptic equations: boundary regularity.** To make the theory complete we need to show that under suitable assumptions solutions of the Dirichlet problem are regular up to the boundary.

Theorem 3.21 (Boundary $W^{2,2}$ regularity). *Assume that Ω is a bounded domain with C^2 boundary,*

$$a^{ij} \in C^1(\overline{\Omega}), \quad b^i, c \in L^\infty(\Omega)$$

and

$$f \in L^2(\Omega).$$

If $u \in W^{1,2}_0(\Omega)$ is a weak solution of the boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u \in W^{1,2}_0(\Omega), \end{cases} \tag{3.21}$$

then $u \in W^{2,2}(\Omega)$ and

$$\|u\|_{W^{2,2}(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right), \tag{3.22}$$

where the constant C depends on Ω and the coefficients of L only.

Remark 3.22. One can also prove that if, in addition, u is a *unique* solution to (3.21), then we have even a better estimate

$$\|u\|_{W^{2,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Proof. Large part of the proof is similar to that of Theorem 3.18, but now we need to take test functions that touch the boundary. Consider first a special case when[14]

$$U = B^n(0,1) \cap \mathbb{R}^n_+.$$

Assume that $u \in W^{1,2}(U)$ solves the equation $Lu = f$ in U and $u = 0$ along the flat part of the boundary $\{x_n = 0\}$. Take $V = B^n(0,1/2) \cap \mathbb{R}^n_+$ and consider a cutoff function

$$\zeta \in C^\infty_0(B^n(0,1)), \quad 0 \leq \zeta \leq 1, \quad \zeta|_{B^n(0,1/2)} \equiv 1.$$

We want to use the following test function

$$\varphi = -D_k^{-h}\big(\zeta^2 D_k^h u\big).$$

Observe that if $k \in \{1, 2, \dots, n-1\}$, then $\varphi \in W^{1,2}_0(U)$. Indeed, for $x \in U$ we have

$$\begin{aligned} \varphi(x) &= -\frac{1}{h} D_k^{-h}\big(\zeta^2(x)(u(x+he_k) - u(x))\big) \\ &= \frac{1}{h^2}\big(\zeta^2(x-he_k)(u(x) - u(x-he_k)) - \zeta^2(x)(u(x+he_k) - u(x))\big). \end{aligned}$$

If h is sufficiently small, then $\varphi \in W^{1,2}_0(U)$. Indeed, it vanishes near the spherical part of the boundary (because of ζ) and since u has zero trace on the flat part of the boundary, φ has zero trace too and it suffices to apply Theorem 2.67.

In the above argument it was important that part of the boundary was flat and we could prove that $\varphi \in W^{1,2}_0(U)$ only for $k \neq n$. Since $\varphi \in W^{1,2}_0(U)$ we can use it as a test function in our equation. Calculations as in the proof of Theorem 3.18 lead to the estimate

$$\int_V |D_k^h \nabla u|^2\, dx \leq C \int_U (|f|^2 + |u|^2 + |\nabla u|^2)\, dx$$

[14] Recall that $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$.

and hence we conclude as in the proof of Theorem 3.18 that

$$\frac{\partial^2 u}{\partial x_i \partial x_k} \in L^2(V) \quad \text{for } i = 1, 2, \ldots, n$$

along with the estimate

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_k} \right\|_{L^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{W^{1,2}(U)} \right). \tag{3.23}$$

The problem is that we still miss the L^2 estimate for $\partial^2 u/\partial x_n^2$. Such an estimate can, however, be easily obtained from (3.23) using ellipticity of L. Indeed, since $a^{ij} \in C^1$ and $u \in W^{2,2}_{\rm loc}$ we can rewrite the equation $Lu = f$ as an equation in a non-divergence form

$$-\sum_{i,j=1}^n a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n \tilde{b}^i \frac{\partial u}{\partial x_i} + cu = f,$$

where

$$\tilde{b}^i = b_i - \sum_{j=1}^n \frac{\partial a^{ij}}{\partial x_j}.$$

Thus

$$a^{nn} \frac{\partial^2 u}{\partial x_n^2} = -\sum_{(i,j) \neq (n,n)} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n \tilde{b}^i \frac{\partial u}{\partial x_i} + cu - f. \tag{3.24}$$

The ellipticity condition

$$\sum_{i,j} a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

applied to $\xi = e_n = (0, \ldots, 0, 1)$ gives

$$a^{nn}(x) \geq \theta.$$

Applying this to the left hand side of (3.24) and using a rough estimate for the right hand side we get

$$\left| \frac{\partial^2 u}{\partial x_n^2} \right| \leq C \left(\sum_{(i,j) \neq (n,n)} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| + |\nabla u| + |u| + |f| \right).$$

Thus (3.23) yields

$$\left\| \frac{\partial^2 u}{\partial x_n^2} \right\|_{L^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{W^{1,2}}(U) \right)$$

and hence

$$\|u\|_{W^{2,2}(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{W^{1,2}(U)} \right).$$

We are left with the general case when the boundary is not necessarily flat. The idea is to use a (local) diffeomorphic change of variables to make the boundary flat. To do this we have to investigate what happens to the solution of the equation $Lu = f$ when we apply such a change of variables. More precisely, let

$$\Phi : \Omega_1 \to \Omega_2, \quad \Psi = \Phi^{-1} : \Omega_2 \to \Omega_1$$

be a C^2 diffeomorphism of bounded domains $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ and its inverse. We will also write

$$\Phi(x) = y, \quad \Psi(y) = x.$$

Suppose for simplicity that the diffeomorphisms Φ and Ψ are C^2 up to the boundary.

Suppose L is an elliptic operator in Ω_1, and the function $u \in W^{1,2}(\Omega_1)$ satisfies

$$Lu = f \quad \text{in } \Omega_1. \tag{3.25}$$

We want to find out what equation is satisfied by the function

$$v = u \circ \Phi \quad \text{in } \Omega_2.$$

The idea is very simple: equation (3.25) in the weak form can be written as a family of integral identities in Ω_1 and we can rewrite it as an integral identities in Ω_2 using the chain rule and the change of variables in the integral. We have

$$\int_{\Omega_1} \sum_{i,j} a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_j} + \sum_i b^i(x) \frac{\partial u}{\partial x_i}\, \eta + c(x) u \eta \, dx = \int_{\Omega_1} f \eta \, dx$$

for all $\eta \in W_0^{1,2}(\Omega_1)$. Denote

$$v(y) = u(\Psi(y)) \quad \text{and} \quad \zeta(y) = \eta(\Psi(y)).$$

Then $v \in W^{1,2}(\Omega_2)$ and $\zeta \in W_0^{1,2}(\Omega_2)$ by Theorem 2.46. The chain rule and the change of variables give

$$\begin{aligned}
&\int_{\Omega_1} \sum_{i,j} a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_j} \, dx \\
&= \int_{\Omega_1} \sum_{i,j} a^{ij}(x) \left(\sum_k \frac{\partial v}{\partial y_k}(\Phi(x)) \frac{\partial \Phi^k}{\partial x_i}(x) \right) \left(\sum_\ell \frac{\partial \zeta}{\partial y_\ell}(\Phi(x)) \frac{\partial \Phi^\ell}{\partial x_j}(x) \right) \\
&= \int_{\Omega_2} \sum_{k,\ell} \left(\sum_{i,j} a^{ij}(\Phi(y)) \frac{\partial \Phi^k}{\partial x_i}(\Psi(y)) \frac{\partial \Phi^\ell}{\partial x_j}(\Psi(y)) \right) |J_\Psi(y)| \, \frac{\partial v}{\partial y_k} \frac{\partial \zeta}{\partial y_\ell} \, dy \\
&= \int_{\Omega_2} \sum_{k,\ell} \tilde{a}^{k\ell}(y) \, \frac{\partial v}{\partial y_k} \frac{\partial \zeta}{\partial y_\ell} \, dy,
\end{aligned}$$

where J_Ψ is the Jacobian of the diffeomorphism Ψ and the matrix

$$\tilde{A}(y) = [\tilde{a}^{k,\ell}]$$

is defined by[15]

$$\tilde{A}(y) = \left(\frac{D\Phi \bullet A \bullet (D\Phi)^T}{|J_\Phi|}\right)(\Psi(y))\,.$$

Here $\bullet$ denotes the matrix multiplication. Similarly

$$\int_{\Omega_1} \sum_i b^i(x)\frac{\partial u}{\partial x_i}\,\eta = \int_{\Omega_2} \sum_k \tilde{b}^k(y)\frac{\partial v}{\partial y_k}\,\zeta\,dy,$$

where

$$\tilde{b}^k(y) = \left(\sum_i b^i(\Psi(y))\frac{\partial \Phi^k}{\partial x_i}(\Psi(y))\right)|J_\Psi(y)|\,,$$

i.e. the vector $\tilde{B} = [\tilde{b}^k]$ is obtained from the vector $B = [b^i]$ by the formula

$$\tilde{B}(y) = \left(\frac{D\Phi \bullet B}{|J_\Phi|}\right)(\Psi(y))\,.$$

Finally

$$\int_{\Omega_1} cu\eta\,dx = \int_{\Omega_2} \underbrace{c(\Psi(y))|J_\psi|}_{\tilde{c}}\,v\zeta\,dy = \int_{\Omega_2} \tilde{c}v\zeta\,dy,$$

and

$$\int_{\Omega_1} f\eta\,dx = \int_{\Omega_2} \underbrace{f(\Psi(y))|J_\Psi|}_{\tilde{f}}\,\zeta\,dy = \int_{\Omega_2} \tilde{f}\zeta\,dy.$$

Thus

$$\int_{\Omega_2} \sum_{k,\ell} \tilde{a}^{k,\ell}(y)\frac{\partial v}{\partial y_k}\frac{\partial \zeta}{\partial y_\ell} + \sum_k \tilde{b}^k(y)\frac{\partial v}{\partial y_k}\zeta + \tilde{c}v\zeta\,dy = \int_{\Omega_2} \tilde{f}\zeta\,dy \tag{3.26}$$

for all[16] $\zeta \in W_0^{1,2}(\Omega_2)$, i.e.

$$\tilde{L}v = \tilde{f},$$

where the operator $\tilde{L}$ is associated with (3.26).

Observe that the operator $\tilde{L}$ is uniformly elliptic. Indeed, $|I_\Phi|$ is bounded from below and below and above and

$$\begin{aligned}\langle \tilde{A}\xi,\xi\rangle &= |J_\Phi|^{-1}\langle D\Phi A(D\Phi)^T\xi,\xi\rangle = |J_\Phi|^{-1}\langle A(D\Phi)^T\xi,(D\Phi)^T\xi\rangle \\ &\geq \theta|J_\Phi|^{-1}|(D\Phi)^T\xi|^2 \approx \theta|\xi|^2,\end{aligned}$$

because the matrix $(D\Phi)^T$ is non-degenerate.

If

$$a^{ij} \in C^1(\overline{\Omega}_1),\ b^i, c \in L^\infty(\Omega_1),\ f \in L^2(\Omega_1),$$

then

$$\tilde{a}^{k,\ell} \in C^1(\overline{\Omega}_2),\ \tilde{b}^k, c \in L^\infty(\Omega_2),\ \tilde{f} \in L^2(\Omega_2) \tag{3.27}$$

[15] We use the fact that $|J_\Psi(y)| = |J_\Phi(\Psi(y))|^{-1}$.

[16] Indeed, any such ζ can be represented as $\eta \circ \Psi$ for some $\eta \in W_0^{1,2}(\Omega_1)$.

Indeed, $\Phi \in C^2$, so $D\Phi, |J_\Phi| \in C^1$ and (3.27) follows from formulas for the coefficients of $\tilde{L}$.

Now we can complete the proof of the theorem. Let Ω_1 be a neighborhood of a point on the boundary $\partial\Omega$. Since $\partial\Omega \in C^2$, there is a C^2 diffeomorphism

$$\Phi : \Omega_1 \to B(0,1) \cap \mathbb{R}^n_+ := U.$$

Let $u \in W^{1,2}_0(\Omega)$ be a solution to (3.21). Then $v = u \circ \Psi$ is a solution of

$$\tilde{L}v = \tilde{f} \quad \text{in } B(0,1) \cap \mathbb{R}^n_+$$

and v vanishes along $\{x_n = 0\}$. Let $V = B(0,1/2) \cap \mathbb{R}^n_+$. It follows from the first part of the proof that

$$\|v\|_{W^{2,2}(V)} \leq C \left(\|\tilde{f}\|_{L^2(U)} + \|v\|_{W^{1,2}(U)} \right)$$

and hence

$$\|u\|_{W^{2,2}(\Psi(V))} \leq C \left(\|f\|_{L^2(\Omega_1)} + \|u\|_{W^{1,2}(\Omega_1)} \right) \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)} \right)$$

because $W^{2,2}$ space is invariant under C^2 diffeomorphisms. Since $\partial\Omega$ is compact, it can be covered by a finite number of neighborhoods like Ω_1 and hence

$$\|u\|_{W^{2,2}(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{W^{1,2}(\Omega)} \right).$$

Now it suffices to apply Proposition 3.17 to get the estimate (3.22). The proof is complete. □

Theorem 3.23 (Higher boundary regularity)**.** *Let m be a nonnegative integer. Assume that Ω is a bounded domain with C^{m+2} boundary,*

$$a^{ij}, b^i, c \in C^{m+1}(\overline{\Omega}),$$

and

$$f \in W^{m,2}(\Omega).$$

If $u \in W^{1,2}_0(\Omega)$ is a weak solution of the boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u \in W^{1,2}_0(\Omega), \end{cases}$$

then $u \in W^{m+2,2}(\Omega)$ and

$$\|u\|_{W^{m+2,2}(\Omega)} \leq C \left(\|f\|_{W^{m,2}(\Omega)} + \|u\|_{L^2(\Omega)} \right),$$

where the constant C depends on Ω, m and the coefficients of L only.

Sketch of the proof. First we prove the result by induction in the case in which

$$\Omega = B^n(0,1) \cap \mathbb{R}^n_+.$$

This part of the proof is an adaptation of the induction argument used in the proof of Theorem 3.19 to the current situation. Then the general case follows by the change of variables that straightens the boundary. □

As a corollary we obtain

Theorem 3.24 (C^∞ boundary regularity)**.** *Assume that* Ω *is a bounded domain with* C^∞ *boundary,*

$$a^{ij}, b^i, c \in C^\infty(\overline{\Omega}),$$

and

$$f \in C^\infty(\overline{\Omega}).$$

If $u \in W_0^{1,2}(\Omega)$ *is a weak solution of the boundary value problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

then $u \in C^\infty(\overline{\Omega})$.

4. Simple nonlinear problems: variational approach

At the beginning we proved the existence of a weak solution of the Dichchlet problem for the Laplace equation using variational approach. Later we realized that for this problem and for more general linear elliptic equations, instead of the variational approach, we could use the Lax-Milgram theorem. It turns out, however, that the variational approach gives more flexibility as it also applies to nonlinear problems for which the Lax-Milgram theorem is of no use. In this section we will describe some elementary nonlinear problems that will illustrate the variational method.

4.1. Euler Lagrange Equations.

The Dirichlet principle asserts that finding a solution to the Dirichlet problem for the Laplace equation is equivalent to finding a minimizer of the Dirichlet integral I. The Laplace equation was derived by taking directional derivatives of the functional I in all the directions $\varphi \in C_0^\infty$. We took directional derivatives in the infinite dimensional space of functions. Thus roughly speaking we can say that u is a minimizer of the functional I if the "derivative" of I at the point u is zero. We will generalize this observation to the abstract setting of functionals on Banach spaces.

Definition 4.1. Let X be a Banach space and let $I : X \to \mathbb{R}$ be a functional.[17] The *directional derivative* of I at $u \in X$ in the direction $h \in X$, $h \neq 0$ is defined as

$$D_h I(u) = \lim_{t \to 0} \frac{I(u+th) - I(u)}{t}\,.$$

We say that I is *differentiable in the sense of Gateaux* at a point $u \in X$ if for every $h \in X$, $h \neq 0$ the directional derivative $D_h I(u)$ exists and the function $h \mapsto D_h I(u)$ is linear and continuous. This defines functional $DI(u) \in X^*$ by the formula $\langle DI(u), h\rangle = D_h I(u)$. We call $DI(u)$ the *Gateaux differential.*

[17] Nonlinear — as usual.

Proposition 4.2. *If $I : X \to \mathbb{R}$ is Gateaux differentiable and $I(\overline{u}) = \inf_{u\in X} I(u)$, then $DI(\overline{u}) = 0$.* □

Later we will see that this is an abstract statement of the so called Euler–Lagrange equations. For example we will see that when we differentiate the functional $I(u) = \int |\nabla u|^2$, we obtain the equation $\Delta u = 0$ in the weak form. This looks like the Dirichlet principle.

Theorem 4.3. *Let $I : X \to \mathbb{R}$ be Gateaux differentiable. Then the following conditions are equivalent.*

(1) *I is convex,*
(2) *$I(v) - I(u) \geq \langle DI(u), v - u\rangle$ for all $u, v \in X$,*
(3) *$\langle DI(v) - DI(u), v - u\rangle \geq 0$ for all $u, v \in X$.*

Proof. (1)⇒(2). Convexity implies that

$$\frac{I(u + t(v - u)) - I(u)}{t} \leq I(v) - I(u)$$

for $t \in (0, 1)$ and hence the claim follows by passing to the limit as $t \to 0$.

(2)⇒(3). It follows directly from the assumption that $\langle -DI(u), v - u\rangle \geq I(u) - I(v)$ and $\langle DI(v), v - u\rangle \geq I(v) - I(u)$. Adding both inequalities we obtain the desired inequality.

(3)⇒(1) We have to prove that for any two points $u, v \in X$, the function $f(t) = I(u + t(v - u))$ is convex. To this end it suffices to prove that $f'(t)$ is increasing. This follows easily from the assumed inequality and the formula for f'. □

In the case of convex functionals the necessary condition given in Proposition 4.2 is also sufficient.

Proposition 4.4. *If $I : X \to \mathbb{R}$ is convex and Gateaux differentiable, then $I(\overline{u}) = \inf_{u\in X} I(u)$ if and only if $DI(\overline{u}) = 0$.*

Proof. It remains to prove the implication ⇐. By the convexity and the above theorem $I(u) - I(\overline{u}) \geq \langle DI(\overline{u}), u - \overline{u}\rangle = 0$, hence $I(u) \geq I(\overline{u})$. □

Note that we have similar situation in the case of the Dirichlet principle: The condition $\Delta u = 0$ is necessary and sufficient for u to be the minimizer.

Below we give some important examples of functionals differentiable in the Gateaux sense.

Lemma 4.5. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in $x \in \Omega$ (for every $u \in \mathbb{R}$) and C^1 in $u \in \mathbb{R}$ (for almost every $x \in \Omega$). Moreover assume the following growth conditions*

$$|f(x, u)| \leq a(x) + C|u|^p, \qquad |f'_u(x, u)| \leq b(x) + C|u|^{p-1}$$

where $a \in L^1(\Omega)$, $b \in L^{p/(p-1)}(\Omega)$ *and* $1 < p < \infty$. *Then the functional* $I(u) = \int_\Omega f(x, u(x))\, dx$ *is Gateaux differentiable as defined on* $L^p(\Omega)$ *and*

$$\langle DI(u), v\rangle = \int_\Omega f'_u(x, u)v\, dx$$

Proof. The growth condition implies that I is defined and finite on $L^p(\Omega)$. We have

$$\frac{I(u+tv) - I(u)}{t} = \int_\Omega \frac{f(x, u+tv) - f(x, u)}{t} = \int_\Omega \frac{1}{t}\int_0^t f'_u(x, u+sv)v\, ds\, dx. \tag{4.1}$$

Observe that

$$\frac{1}{t}\int_0^t f'_u(x, u+sv)v\, ds \to f'_u(x, u)v \quad \text{as } t \to 0 \text{ for a.e. } x,$$

and that by the growth condition

$$\left|\frac{1}{t}\int_0^t f'_u(x, u+sv)v\, ds\right| \le C\left(|b|^{p/(p-1)} + |u|^p + |v|^p\right) \in L^1(\Omega)\,,$$

where C does not depend on t. Hence we may pass to the limit[18] in (4.1) and we get

$$\lim_{t\to 0}\frac{I(u+tv) - I(u)}{t} = \int_\Omega f'_u(x, u)v\, dx.$$

The proof is complete. □

It is also easy to prove the following

Lemma 4.6. *For* $1 < p < \infty$ *the functional* $I_p = \int_\Omega |\nabla u|^p$ *is Gateaux differentiable on* $W^{1,p}(\Omega, \mathbb{R}^m)$ *and*

$$\langle DI_p(u), v\rangle = p\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla v\rangle.$$

Since the functional I_p is convex we get that

$$I_p(\overline{u}) = \inf_{u\in W^{1,p}_w(\Omega,\mathbb{R}^m)} I_p(u)\,, \tag{4.2}$$

if and only if $DI_p(\overline{u}) = 0$, i.e. if and only if $\overline{u}$ is a weak solution to the following system

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u_i) = 0 \qquad u_i - w_i \in W^{1,p}_0(\Omega)\,, \qquad i = 1, 2, \ldots, m\,. \tag{4.3}$$

Here $w = (w_1, w_2, \ldots, w_m) \in W^{1,p}(\Omega, \mathbb{R}^m)$. Moreover the strict convexity guarantees uniqueness of the minimizer or equivalently, uniqueness of the solution to the system.

[18]Dominated convergence theorem.

This is a version of the Dirichlet principle. Thus the problem of solving (4.3) reduces to finding the minimizer of (4.2), which is easy, see Corollary 1.10. Equations (4.3) are called Euler–Lagrange system for the minimizer of I_p. The variational approach easily generalizes to more complicated elliptic equations or systems.

Now for the simplicity of the notation we will be concerned with the case $m = 1$.

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lischitz boundary and $1 < p < \infty$. Assume that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \Omega$, C^1 in $u \in \mathbb{R}$ and satisfies*

$$|f(x,u)| \le a(x) + C|u|^q \qquad |f'_u(x,u)| \le b(x) + C|u|^{q-1}$$

where $q = np/(n-p)$ if $p < n$ and $q < \infty$ is any exponent if $p \ge n$. Then the functional

$$I(u) = \int_\Omega \frac{1}{p}|\nabla u|^p + f(x,u) \tag{4.4}$$

is Gateaux differentiable on the space $W^{1,p}(\Omega)$ and

$$\langle DI(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla v\rangle + \int_\Omega f'_u(x,u)v \tag{4.5}$$

for every $v \in W^{1,p}(\Omega)$.

Proof. It easily follows from Lemma 4.5, Lemma 4.6 and the Sobolev embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$. □

Remark. If we define I on $W^{1,p}_0(\Omega)$ only, Ω can be an arbitrary open and bounded set, because then we still have $W^{1,p}_0(\Omega) \subset L^q(\Omega)$. In this case (4.5) holds for $v \in W^{1,p}_0(\Omega)$.

Fix $w \in W^{1,p}(\Omega)$. If u is a minimizer of (4.4) in $W^{1,p}_w(\Omega)$, then u solves the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f'_u(x,u)\,, \qquad u - w \in W^{1,p}_0(\Omega)\,. \tag{4.6}$$

Indeed, if $J(v) = I(v+w)$, then $v = u - w$ is a minimizer of J on $W^{1,p}_0(\Omega)$ and hence $DJ(v) = 0$ i.e.,

$$0 = \langle DJ(v), \varphi\rangle = \langle DI(\underbrace{v+w}_{u}), \varphi\rangle \qquad \forall \varphi \in W^{1,p}_0(\Omega)\,,$$

i.e. (4.6) holds. Equation (4.6) is so called Euler–Lagrange equation for (4.4). Conversely, one can prove existence of a solution to (4.6) by proving existence of a minimizer to (4.4). This is our next aim.

Recall that if $1 < p < n$, then $p^* = np/(n-p)$ is the Sobolev exponent. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Assume that a function g :

$\Omega\times\mathbb{R}\to\mathbb{R}$ is measurable in $x\in\Omega$ and continuous in $u\in\mathbb{R}$. If the following growth condition holds

$$|g(x,u)|\leq C(1+|u|^q) \tag{4.7}$$

where $q\leq p^*-1$ when $1<p<n$ and $q<\infty$ when $n\leq p<\infty$, then the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)=g(x,u)\,,\qquad u-w\in W_0^{1,p}(\Omega) \tag{4.8}$$

is the Euler–Lagrange equation for the minimizer of the functional

$$I(u)=\int_\Omega \frac{1}{p}|\nabla u|^p+G(x,u)\,, \tag{4.9}$$

where $G(x,u)=\int_0^u g(x,t)\,dt$, defined on $W_w^{1,p}(\Omega)$. This follows from the estimate $|G(x,u)|\leq C(1+|u|^{q+1})$ and from Theorem 4.7.

In order to prove the existence of a minimizer of I we need find assumptions that would guarantee that I is coercive and SWLSC.

Observe that if $g(x,u)u\geq 0$ for all u, then $G(x,u)\geq 0$ and hence I is coercive. This condition for g is too strong. We will relax it now.

Assume that

$$g(x,u)\frac{u}{|u|}\geq -\gamma|u|^{p-1} \tag{4.10}$$

for all $|u|\geq M$ and a.e. $x\in\Omega$. Here M and γ are constants such that $M\geq 0$ and $\gamma<\mu_1^{-1}$, where μ_1 is the best (the least) constant in the Poincaré inequality

$$\int_\Omega |u|^p\leq \mu_1\int_\Omega|\nabla u|^p\qquad \forall u\in C_0^\infty(\Omega)\,.$$

We claim that under conditions (4.7) and (4.10) the functional I is coercive on $W_w^{1,p}(\Omega)$. Indeed, both the conditions imply that

$$G(x,u)\geq -C-\frac{\gamma}{p}|u|^p$$

for all $u\in\mathbb{R}$. Hence

$$I(u)\geq\frac{1}{p}\int_\Omega|\nabla u|^p-\frac{\gamma}{p}\int_\Omega|u|^p-C\geq\frac{1}{p}(1-\gamma\mu_1)\int_\Omega|\nabla u|^p-C\to\infty$$

as $\|u\|_{1,p}\to\infty$ and $u\in W_w^{1,p}(\Omega)$.

Condition (4.10) is optimal. Namely later in this section we will show that the functional constructed for $g(x,u)=-\mu_1^{-1}u|u|^{p-2}$ is not coercive on $W_0^{1,p}(\Omega)$.

Now in order to have the SWLC condition we need slightly relax condition (4.7).

Theorem 4.8. *Let g satisfies* (4.7) *and* (4.10) *and let the functional I be defined on $W^{1,p}_w(\Omega)$ a above. Then the functional is coercive. If in addition $q < p^* - 1$ when $1 < p < n$ (in the case $p \geq n$ we do not change the assumption) then the functional I is SWLSC and hence it assumes the minimum which solves the Dirichlet problem* (4.8)*.*

Proof. We have already proved coercivity. Now assume that $q < p^* - 1$ when $1 < p < n$ and $q < \infty$ when $p \geq n$.

Let $v_k \rightharpoonup v$ weakly in $W^{1,p}_0(\Omega)$. We have to prove that

$$I(v + w) \leq \liminf_{k\to\infty} I(v_k + w). \tag{4.11}$$

Under the additional assumption about q, the embedding $W^{1,p}_0(\Omega) \subset L^{q+1}(\Omega)$ is compact and hence $v_k + w \to v + w$ in L^{q+1}. Since $|G(x, u)| \leq C(1 + |u|^{q+1})$ we conclude that

$$\int_\Omega G(x, v_k + w) \to \int_\Omega G(x, v + w)\,. \tag{4.12}$$

This follows easily form the following version of the dominated convergence theorem.

Lemma 4.9. *Let $|f_k| \leq g_k$, $g_k \to g$ in L^1 and $f_k \to f$ a.e. Then $\int f_k \to \int f$.*
□

Observe that

$$\int_\Omega |\nabla(v + w)|^p \leq \liminf_{k\to\infty} \int_\Omega |\nabla(v_k + w)|^p. \tag{4.13}$$

This is a direct consequence of Theorem 1.8.[19] Now (4.12) and (4.13) imply the SWLSC property (4.11) and then the theorem follows directly form Theorem 1.6. □

We leave as an exercise the proof of the following variant of the above result.

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 < p < \infty$. Then for every $f \in (W^{1,p}_0(\Omega, \mathbb{R}^m))^*$ there exists the unique solution to the following system*

$$\mathrm{div}\,(|\nabla u|^{p-2}\nabla u_i) = f \qquad u_i \in W^{1,p}_0(\Omega)\,, \qquad i = 1, 2, \ldots, m\,.$$

We already proved regularity results for linear elliptic equations. Now we show a tricky and powerful method of proving regularity results for weak solutions to nonlinear equations of the form

$$-\Delta u = g(x, u), \qquad u \in W^{1,2}_{\mathrm{loc}}(\Omega)\,. \tag{4.14}$$

[19] It also follows from the lower semicontinuity of the norm (by the Hahn-Banach theorem).

The method is called *bootstrap*. We will need the following result which is a special case of Theorem 3.19.

Lemma 4.11. *If* $u \in W^{1,2}_{\rm loc}(\Omega)$ *is a weak solution to* $-\Delta u = f$*, where* $f \in W^{m,2}_{\rm loc}(\Omega)$*, then* $u \in W^{m+2,2}_{\rm loc}(\Omega)$*.*

Assume now that $g \in C^\infty(\Omega \times \mathbb{R})$. Let u be a solution to (4.14). Then $g(x,u) \in W^{1,2}_{\rm loc}$ and hence by Theorem 4.13 $u \in W^{3,2}_{\rm loc}$. This implies in turn that $g(x,u) \in W^{3,2}_{\rm loc}$ and then $u \in W^{5,2}_{\rm loc} \ldots$ Iterating this argument yields $u \in W^{k,2}_{\rm loc}$ for any k. Hence by the Sobolev embedding theorem $u \in C^\infty$. Thus we have proved

Theorem 4.12. *If* u *is a weak solution to* (4.14) *with* $g \in C^\infty(\Omega \times \mathbb{R})$*, then* $u \in C^\infty(\Omega)$*.*

For the next result we will need a deep generalization of Lemma 4.11 due to Calcerón and Zygmund.

Theorem 4.13 (Calderón–Zygmund)**.** *If* $u \in W^{1,2}_{\rm loc}(\Omega)$ *is a weak solution to* $-\Delta u = f$*, where* $f \in W^{m,p}_{\rm loc}(\Omega)$*,* $1 < p < \infty$*, then* $u \in W^{m+2,p}_{\rm loc}(\Omega)$*.*

The proof in the case $p \neq 2$ is much more difficult and is based on a deep tool from harmonic analysis: the Calderón-Zygmund theory of singular integrals.

Assume now that g is measurable in x, continuous in u and that it satisfies the following growth condition

$$|g(x,u)| \leq C(1+|u|^q), \qquad 1 \leq q < 2^* - 1 = \frac{n+2}{n-2}, \quad n \geq 3. \tag{4.15}$$

Theorem 4.14. *If* u *is a solution to* (4.14) *with the growth condition* (4.15)*, then* $u \in C^{1,\alpha}_{\rm loc}$ *for any* $0 < \alpha < 1$*.*

Proof. Since $u \in W^{1,2}_{\rm loc}$, Sobolev embedding yields $u \in L^{2n/(n-2)}_{\rm loc}$ and hence $g(x,u) \in L^{p_1}_{\rm loc}$, where $p_1 = \frac{2n}{(n-2)q} > 1$. Now by Theorem 4.13 $u \in W^{2,p_1}_{\rm loc}$ and again by Sobolev embedding $g(x,u) \in L^{p_2}_{\rm loc}$, $p_2 = \frac{2n}{((n-2)q-4)q} > p_1$, $u \in W^{2,p_2}_{\rm loc}$. The inequality $p_2 > p_1$ follows from the fact that $q < (n+2)/(n-1)$. Iterating this argument finitely many times yields $2p_k > n$, so $u \in W^{2,p_k}_{\rm loc} \subset L^\infty_{\rm loc}$, $g(x,u) \in L^\infty_{\rm loc} \subset L^r_{\rm loc}$ for any $r < \infty$ and hence $u \in W^{2,r}_{\rm loc}$ for any $r < \infty$. Thus the Sobolev embedding theorem yields $u \in C^{1,\alpha}_{\rm loc}$ for all $\alpha < 1$. The proof is complete. □

Eigenvalue problem. We start with considering the following two problems.

1. The best constant in the Poincaré inequality.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p < \infty$. Find the smallest constant μ such that the inequality

$$\int_\Omega |u|^p \leq \mu \int_\Omega |\nabla u|^p \tag{4.16}$$

holds for every $u \in W_0^{1,p}(\Omega)$.

Of course such a constant exists and is positive. This follows from the Poincaré inequality. In what follows, the smallest constant (called the best constant) in the Poincaré inequality and it will be denoted by μ_1.

The second problem looks much different.

2. The first eigenvalue of the p-Laplace operator.

We say that λ is an *eigenvalue* of the p-Laplace operator $-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$ on $W_0^{1,p}(\Omega)$, where $1 < p < \infty$, if there exists $0 \neq u \in W_0^{1,p}(\Omega)$ such that

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda u|u|^{p-2}. \tag{4.17}$$

Such a function u is called *eigenfunction.* If $p = 2$, then we have the classical eigenvalue problem for the Laplace operator.

Observe that every eigenvalue is positive. Indeed, by the definition of the weak solution, (4.17) means that

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla v\rangle = \lambda \int_\Omega |u|^{p-2}uv$$

for every $v \in C_0^\infty(\Omega)$ and hence for every $v \in W_0^{1,p}(\Omega)$. Taking $v = u$ we obtain

$$\int_\Omega |\nabla u|^p = \lambda \int_\Omega |u|^p\,. \tag{4.18}$$

Thus $\lambda > 0$. Note that inequality (4.16) implies that $\lambda \geq \mu_1^{-1}$. This gives the lower bound for the eigenvalues. The following theorem says much more.

Theorem 4.15. *There exists the smallest eigenvalue λ_1 of the problem* (4.17) *and it satisfies $\lambda_1 = \mu_1^{-1}$, where μ_1 is the best constant in the Poincaré inequality* (4.16)*.*

The eigenvalue λ_1 is called *the first eigenvalue.*

Note that (4.17) is the Euler–Lagrange equation of the functional

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda |u|^p) \tag{4.19}$$

The similar situation was in Theorem 4.8, there are, however, two essential differences. First of all Theorem 4.8 guarantees the existence of a solution but it does not say anything about the properties of the solution. We already know that a solution of (4.17) exits — a function constant equal to zero. However, we are not interested in that solution, we are looking for a non-zero solution. In such a situation Theorem 4.8 cannot help. The second difference is that that functional defined by (4.19) is not coercive.

Proof of Theorem 4.15. We know that $\lambda \geq \mu_1^{-1}$. It remains to prove that there exists $0 \neq u_0 \in W_0^{1,p}(\Omega)$ such that

$$-\mathrm{div}\,(|\nabla u_0|^{p-2}\nabla u_0) = \mu_1^{-1} u_0 |u_0|^{p-2}. \tag{4.20}$$

First note that $u_0 \in W_0^{1,p}(\Omega)$ satisfies (4.20) if and only if

$$\int_\Omega |u_0|^p = \mu_1 \int_\Omega |\nabla u_0|^p. \tag{4.21}$$

Assume that u_0 satisfies (4.20). Integrating both sides of (4.20) against the test function u_0 we obtain (4.21). In the opposite direction, if u_0 satisfies (4.21), then u_0 is a minimizer of the functional

$$E(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \mu_1^{-1}|u|^p\right).$$

Indeed, $E(u_0) = 0$ and always $E(u) \geq 0$ because of the Poincaré inequality. Hence u_0 satisfies Euler–Lagrange equations (4.20).

Thus it remains to prove that there exists a nontrivial minimizer $0 \neq u_0 \in W_0^{1,p}(\Omega)$ of the functional E.

Let M consists of all $u \in W_0^{1,p}(\Omega)$ such that $\int_\Omega |u|^p = 1$. Minimize the functional $I_p(u) = \int_\Omega |\nabla u|^p$ over M. Let $u_k \in M$, $I_p(u_k) \to \inf_M I_p = \mu_1^{-1}$. By reflexivity of the space $W_0^{1,p}(\Omega)$ we can select a weakly convergent subsequence $u_k \rightharpoonup u_0 \in W_0^{1,p}(\Omega)$. Then $\int_\Omega |\nabla u_0|^p \leq \mu_1^{-1}$. Since the imbedding $W_0^{1,p}(\Omega) \subset L^p(\Omega)$ is compact we conclude that $u_0 \in M$. Hence $E(u_0) = 0$ and thus u_0 is the desired minimizer. This completes the proof. □

Observe that the functional $E(u)$ is not coercive. Indeed, if u_0 is as in the above proof then $E(tu_0) = 0$, while $\|tu_0\|_{1,p} \to \infty$. This also shows that condition (4.10) in Theorem 4.8 are optimal: here $g(x,u) = -\mu_1^{-1} u|u|^{p-2}$ satisfies (4.7) and (4.10) with $\gamma = -\mu_1^{-1}$ which is not allowed.

Observe also that the functional E has infinitely many minimizers — each of the functions tu_0.

There are many open problems concerning the eigenvalues of the problem (4.17). In the linear case $p = 2$ the eigenvalues form an infinite discrete set $0 < \lambda_1 < \lambda_2 < \ldots$, $\lambda_i \to \infty$. No such result is known for $p \neq 2$. It is easy to show that the set of eigenvalues of (4.17) is closed. Moreover one can prove that the set of eigenvalues is infinite, unbounded and that the first eigenvalue is isolated.

5. General variational problems: existence of minimizers

5.1. **Hilbert problems.** In this section we will discuss general variational integrals of the form

$$I(u) = \int_\Omega F(x, u(x), \nabla u(x))\, dx \tag{5.1}$$

and we will investigate conditions that guarantee existence of minimizers in a class of functions with prescribed boundary data and also we will study associated Euler-Lagrange equations.

The development of the general theory of minimizers of such variational problem was initiated by Hilbert. He formulated famous 23 problems in his address on the International Congress of Mathematicians at Paris in 1900. The essential part of his 19th and 20th problem read as follows.

19th problem: *Are the solutions of regular problems in the calculus of variations always necessarily analytic?*

20th problem: *Has not every regular variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also if need be that the notion of a solution shall be suitably extended?*

Hilbert was aware that in order to prove existence of a minimizer of a variational problem, one needs to consider a suitable notion of a weak solution. We have already seen that the right setting for the solution of the 20th problem is the theory of Sobolev spaces. In Section 4 we proved existence of Sobolev minimizers for certain non-linear variational problems and in this section we will treat a quite general case. This problem turned out to be quite elementary. The 19th problem has also been solved in the positive, but this theory is much more difficult and goes far beyond the scope of our presentation.

5.2. **Euler-Lagrange equations.** Here $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given function. Let

$$A(x, u, \xi) = \nabla_\xi F(x, u, \xi), \quad B(x, u, \xi) = F'_u(x, u, \xi).$$

This requires some regularity from F, but for a moment we will do formal calculations, assuming that F allows for such computations.

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, $w \in W^{1,p}(\Omega)$ and

$$I(u_0) = \inf_{u \in W^{1,p}_w(\Omega)} I(u). \tag{5.2}$$

Then for every $v \in W_0^{1,p}(\Omega)$, the function $g(t) = I(u_0+tv)$ attains minimum at $t = 0$. Hence

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} g(t) = \frac{d}{dt}\Big|_{t=0} \int_\Omega F(x, u_0 + tv, \nabla u_0 + t\nabla v)\, dx \\ &= \int_\Omega F'_u(x, u_0, \nabla u_0)v + \nabla_\xi F(x, u_0, \nabla u_0) \cdot \nabla v\, dx, \end{aligned}$$

so

$$-\int_\Omega A(x, u_0, \nabla u_0) \cdot \nabla v = \int_\Omega B(x, u_0, \nabla u_0) v \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

This, however, means that u_0 is a weak solution of the equation

$$\left\{ \begin{array}{l} \operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \Omega, \\ \qquad u \in W_w^{1,p}(\Omega). \end{array} \right. \tag{5.3}$$

Clearly (5.3) is the Euler-Lagrange equation associated with the functional I.

Let us now check carefully what assumptions about F will guarantee validity of the above calculation. We need to make sure that the functional I is well defined on $W^{1,p}(\Omega)$ and that it is Gateaux differentiable — the Gateaux derivative can be computed by differentiating under the sign of the integral. This is exactly the same situation as in Theorem 4.7, so, not surprisingly, the assumptions will be similar. For simplicity we will consider the case $1 \le p < n$ only. The interested reader will have no difficulty to figure out what assumptions are sufficient in the case $p \ge n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $1 \le p < n$ and let $F(x, u, \xi)$ be measurable in x for all u, ξ and C^1 in u and ξ for a.e. x. Assume also the growth conditions

$$|F(x, u, \xi)| \le a(x) + C\Big(|u|^{p^*} + |\xi|^p\Big),$$

$$|F'_u(x, u, \xi)| \le b(x) + C\Big(|u|^{p^*-1} + |\xi|^{p-1+\frac{p}{n}}\Big),$$

$$|\nabla_\xi F(x, u, \xi)| \le c(x) + C\Big(|u|^{\frac{n(p-1)}{n-p}} + |\xi|^{p-1}\Big),$$

where $p^* = np/(n-p)$ and $a(x) \in L^1(\Omega)$, $b(x) \in L^{np/(np-n+p)}(\Omega)$, $c(x) \in L^{p/(p-1)}$.

Using exactly the same arguments as in the proof of Theorem 4.7 one can prove

Theorem 5.1. *Under the above assumptions, the functional*

$$I(u) = \int_\Omega F(x, u, \nabla u)\, dx$$

is Gateaux differentiable on $W^{1,p}(\Omega)$ *and*

$$\langle DI(u), v\rangle = \int_\Omega F'_u(x,u,\nabla u)v + \nabla_\xi F(x,u,\nabla u)\cdot \nabla v\, dx$$

for all $v \in W^{1,p}(\Omega)$.

In particular minimizers of (5.2) satisfy the Euler-Lagrange equation (5.3).

Remark 5.2. One can prove that if $F \in C^\infty$ satisfies certain growth conditions, then minimizers of the variational functional I are C^∞ and if F is real analytic, then the minimizers are real analytic, too. This can be regarded as a positive solution to the 19th Hilbert problem. Many mathematicians contributed to the solution of this problem and the final result was obtained by De Giorgi in 1957. This was a culmination of an enormous effort in the research in partial differential equations in the first half of the XXth century.

5.3. **Existence of minimizers.** The existence of minimizers of the variational problem (5.2) can be obtained by a direct method of the calculus of variations as outlined in Section 1. We make the following assumptions:[20]

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let

$$F = F(x,u,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

be a function such that

- F is continuous;
- $\nabla_\xi F$ is continuous;
- $\xi \mapsto F(x,u,\xi)$ is convex for every $(x,u) \in \Omega \times \mathbb{R}$.

We will call it *standard assumptions.*

Theorem 5.3. *Suppose that* F *satisfies the standard assumptions and that*

$$F(x,u,\xi) \geq a(x)$$

for some $a \in L^1(\Omega)$. *Then for any* $1 \leq p < \infty$, *the functional*

$$I(u) = \int_\Omega F(x,u,\nabla u)\, dx$$

is SWLSC on $W^{1,p}(\Omega)$.[21]

[20] They are by far not optimal, but under such assumptions the proofs are much easier.

[21] We do not assume that $I(u) < \infty$ for all $u \in W^{1,p}(\Omega)$, so $I : W^{1,p}(\Omega) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. Even if the infinite values are allowed, the definition of the sequential weak lower semicontinuity remains the same.

Proof. It suffices to consider the case $p = 1$.[22] Choose any weakly convergent sequence

$$u_k \rightharpoonup u \quad \text{in } W^{1,1}(\Omega). \tag{5.4}$$

We must show that

$$L := \liminf_{k\to\infty} I(u_k) \geq I(u).$$

By passing to a subsequence we may further assume that

$$\lim_{k\to\infty} I(u_k) = L.$$

Since the embedding $W^{1,1}(\Omega) \Subset L^1(\Omega)$ is compact, we have $u_k \to u$ in $L^1(\Omega)$ and after passing to a subsequence $u_k \to u$ a.e. Egorov's theorem implies that there is a compact set E_ε such that

$$u_k \rightrightarrows u \text{ uniformly on } E_\varepsilon \quad \text{and} \quad |\Omega \setminus E_\varepsilon| < \varepsilon/2.$$

Since

$$|\{x \in \Omega : \, |u(x)| + |\nabla u(x)| > t\}| \to 0 \quad \text{as } t \to \infty$$

we can find compact sets $F_\varepsilon \subset \Omega$ such that

$$|u(x)| + |\nabla u(x)| \text{ is bounded on } F_\varepsilon \quad \text{and} \quad |\Omega \setminus F_\varepsilon| < \varepsilon/2.$$

Let $K_\varepsilon = E_\varepsilon \cap F_\varepsilon$. Then

$$|\Omega \setminus K_\varepsilon| < \varepsilon.$$

We can also assume that K_ε is an increasing family of sets.

Convexity of F is ξ and Theorem 4.3 imply

$$F(x, u_k, \nabla u_k) - F(x, u_k, \nabla u) \geq \nabla_\xi F(x, u_k, \nabla u) \cdot (\nabla u_k - \nabla u).$$

This and the inequality $F \geq a$ yield

$$\begin{aligned} I(u_k) &= \int_\Omega F(x, u_k, \nabla u_k)\, dx \\ &\geq \int_{K_\varepsilon} F(x, u_k, \nabla u_k) + \int_{\Omega\setminus K_\varepsilon} a(x)\, dx \\ &\geq \int_{K_\varepsilon} F(x, u_k, \nabla u)\, dx \\ &+ \int_{K_\varepsilon} \nabla_\xi F(x, u_k, \nabla u) \cdot (\nabla u_k - \nabla u)\, dx + \int_{\Omega\setminus K_\varepsilon} a(x)\, dx. \end{aligned} \tag{5.5}$$

∇u is bounded on K_ε, u_k is uniformly bounded on K_ε and $u_k \rightrightarrows u$ uniformly on K_ε. Since F is uniformly continuous on compact sets in $\Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$F(x, u_k, \nabla u) \rightrightarrows F(x, u, \nabla u) \quad \text{uniformly on } K_\varepsilon.$$

Thus

$$\lim_{k\to\infty} \int_{K_\varepsilon} F(x, u_k, \nabla u)\, dx = \int_{K_\varepsilon} F(x, u, \nabla u)\, dx. \tag{5.6}$$

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[22] Indeed, weak convergence in $W^{1,p}$ implies weak convergence in $W^{1,1}$.

Similarly $\nabla_\xi F(x,u_k,\nabla u) \rightrightarrows \nabla_\xi F(x,u,\nabla u)$ uniformly on K_ε. Since $Du_k \rightharpoonup Du$ weakly in L^1 we have[23]

$$\lim_{k\to\infty}\int_{K_\varepsilon} \nabla_\xi F(x,u_k,\nabla u)\cdot(\nabla u_k-\nabla u)\,dx = 0 \tag{5.7}$$

Now (5.5), (5.6) and (5.7) yield

$$\begin{aligned} L &= \lim_{k\to\infty} I(u_k) \geq \int_{K_\varepsilon} F(x,u,\nabla u)\,dx + \int_{\Omega\setminus K_\varepsilon} a(x)\,dx \\ &= \int_{K_\varepsilon} F(x,u,\nabla u) - a(x)\,dx + \int_\Omega a(x)\,dx. \end{aligned}$$

Since $F - a \geq 0$ letting $\varepsilon \to 0$ the monotone convergence theorem gives

$$L \geq \int_\Omega F(x,u,\nabla u) - a(x)\,dx + \int_\Omega a(x)\,dx = \int_\Omega F(x,u,\nabla u)\,dx = I(u).$$

The proof is complete. □

In order to prove existence of a minimizer we still need to prove coercivity of the functional. This will be done in the next theorem.

Theorem 5.4. *Let $F:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfy standard assumptions defined above. Suppose also that*

$$F(x,u,\xi) \geq C|\xi|^p + a(x) \tag{5.8}$$

for some $C>0$, $a\in L^1(\Omega)$ and $1<p<\infty$. Finally let $w\in W^{1,p}(\Omega)$. Then there exists $\bar{u}\in W^{1,p}_w(\Omega)$ such that

$$I(\bar{u}) = \inf_{u\in W^{1,p}_w(\Omega)} I(u).$$

Proof. Let

$$m = \inf_{u\in W^{1,p}_w(\Omega)} I(u).$$

If $m=\infty$, the result is obvious, so we may assume that $m<\infty$. Let $\{u_k\}\subset W^{1,p}_w(\Omega)$ be a minimizing sequence, i.e.

$$\lim_{k\to\infty} I(u_k) = m.$$

The condition (5.8), called the *coercivity condition* implies boundedness of u_k in $W^{1,p}(\Omega)$. Indeed,

$$I(u_k) + \|a\|_{L^1(\Omega)} \geq C\int_\Omega |\nabla u_k|^p\,dx.$$

Hence

$$\sup_k \|\nabla u_k\|_{L^p(\Omega)} < \infty.$$

[23]To prove (5.7) we need to observe that $Du_k - Du$ is bounded in L^1. This is a consequence of a general fact: if $x_n \rightharpoonup x$ in a normed space, then $\sup_n \|x\| < \infty$. This result follows from the Banach-Steinhaus theorem.

Since $u_k - w \in W_0^{1,p}(\Omega)$, The Poincaré inequality (Lemma 1.12) yields

$$\|u_k\|_{L^p(\Omega)} \leq \|u_k - w\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)} \leq C\|\nabla(u_k - w)\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)}$$

and the right hand side is bounded by a constant independent of k. Hence

$$\sup_k \|u_k\|_{W^{1,p}(\Omega)} < \infty.$$

Reflexivity of $W^{1,p}(\Omega)$ implies existence of a subsequence $\{u_{k_j}\}$ weakly convergent in $W^{1,p}(\Omega)$

$$u_{k_j} \rightharpoonup \bar{u} \quad \text{in } W^{1,p}(\Omega).$$

It follows from Mazur's lemma (Lemma 1.9) that a sequence of convex combinations of u_{k_j} (which belongs to $W_w^{1,p}(\Omega)$) converges to $\bar{u}$ in norm, so $\bar{u} \in W_w^{1,p}(\Omega)$. Now, Theorem 5.3 implies

$$I(\bar{u}) \leq \liminf_{j\to\infty} I(u_{k_j}) = \inf_{u \in W_w^{1,p}(\Omega)} I(u).$$

The proof is complete. □

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