Calculus 3

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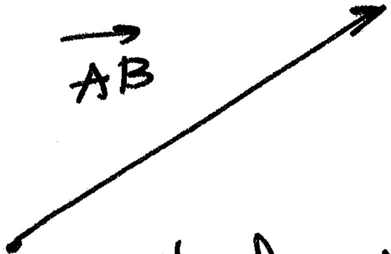
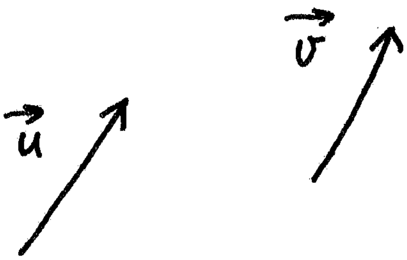
Vectors

B - terminal point



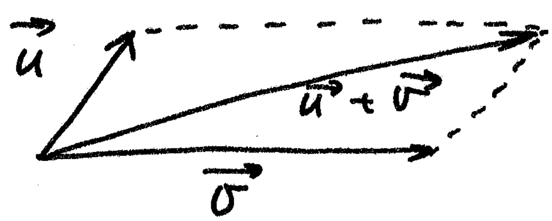
A - initial point

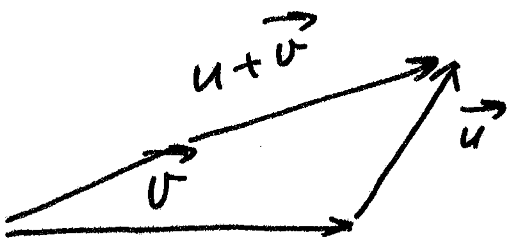


$\vec{u} = \vec{v}$
if the vectors are parallel, have the same length and point in the same direction

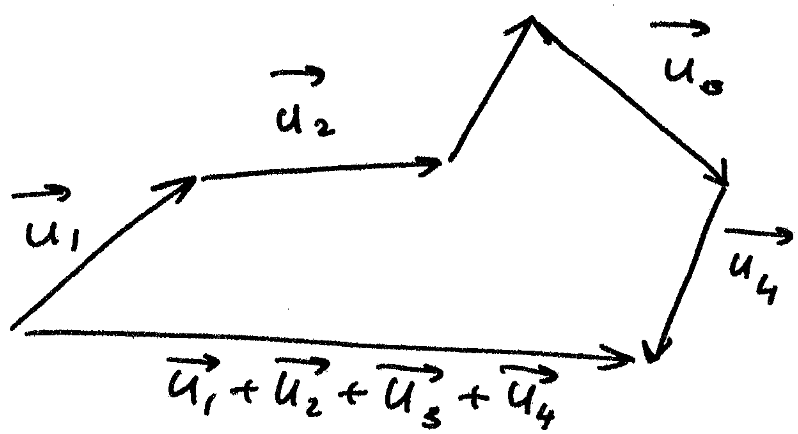
$\vec{0}$ - zero vector of zero length.

How to add vectors



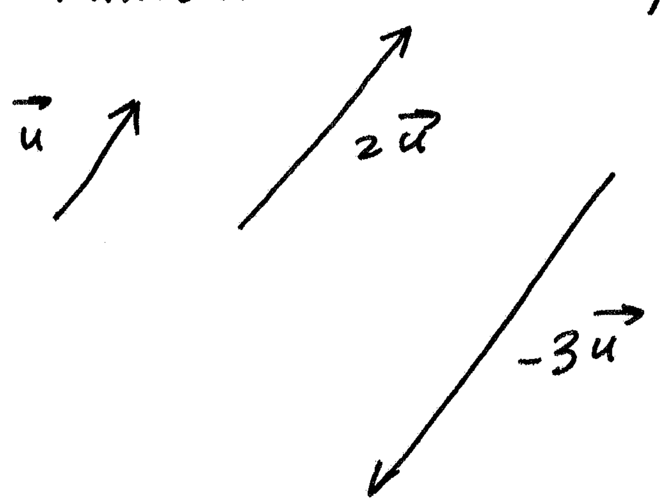


$\vec{u} + \vec{v} = \vec{v} + \vec{u}$



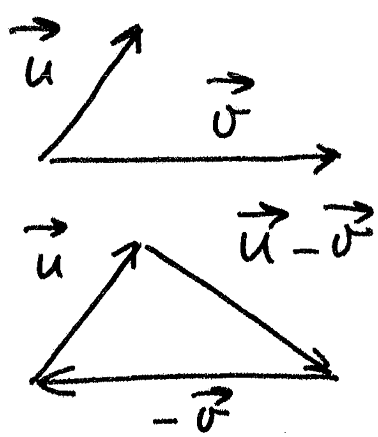
Multiplication by a scalar

c - number $\quad \vec{u}$ - vector, $\quad c\vec{u}$ - vector

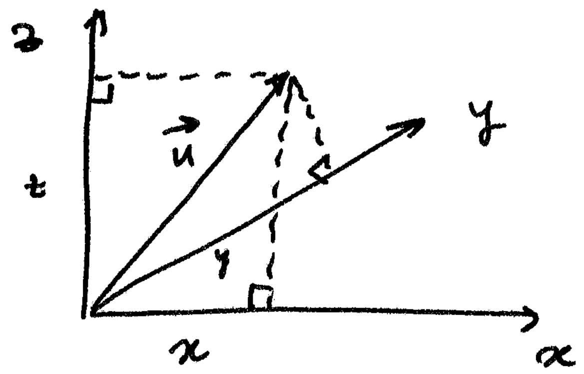


Difference

$$\vec{u} - \vec{v} = \vec{u} + (-1)\cdot\vec{v}$$



Components

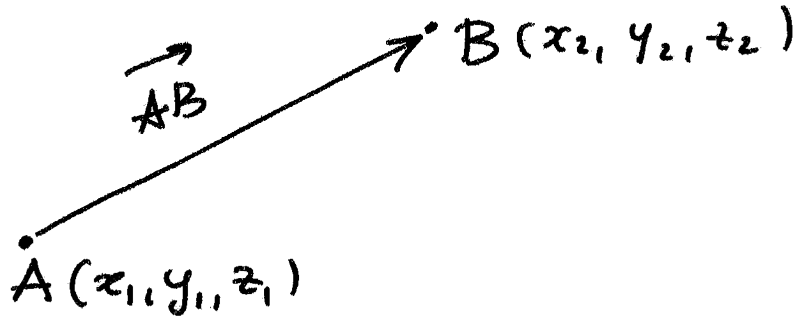


$$\vec{u} = \langle x, y, z \rangle$$

$$c \langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1+b_1, a_2+b_2, a_3+b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1-b_1, a_2-b_2, a_3-b_3 \rangle$$



$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Length

$$\vec{u} = \langle a, b, c \rangle$$

$$|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$$

Length of the vector $\overrightarrow{AB}$ connecting points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ equals

$$|\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

i.e. it equals the distance between the points A and B.

The vectors

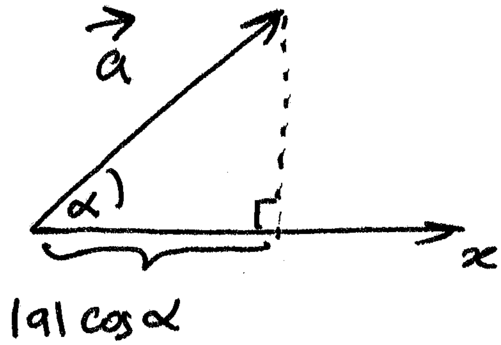
$$\vec{i} = \langle 1, 0, 0 \rangle, \ \vec{j} = \langle 0, 1, 0 \rangle, \ \vec{k} = \langle 0, 0, 1 \rangle$$

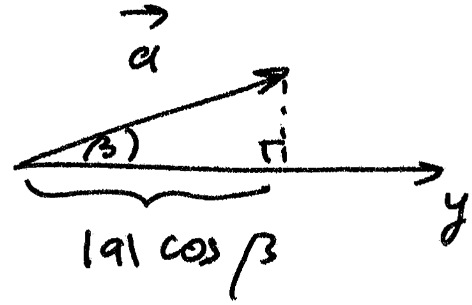
have unit length and they are in the direction of the x, y and z axis respectively.

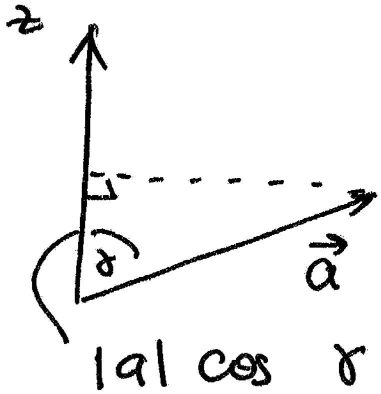
If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ then

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$$
$$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

Suppose a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ forms angles α, β, γ with the axes x, y, z







Thus the x, y and z components of $\vec{a}$ are

$|a|\cos\alpha$, $|a|\cos\beta$, $|a|\cos\gamma$

That means

$$\vec{a} = \langle |\vec{a}|\cos\alpha,\ |\vec{a}|\cos\beta,\ |\vec{a}|\cos\gamma \rangle$$

Hence the length of the vector equals

$$|\vec{a}| = \sqrt{(|\vec{a}|\cos\alpha)^2 + (|\vec{a}|\cos\beta)^2 + (|\vec{a}|\cos\gamma)^2}$$

$$= |\vec{a}|\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}$$

Dividing both sides by $|\vec{a}|$ yields

$$\boxed{\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1.}$$

Dot product (scalar product)

For $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$
we define

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Basic properties:

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

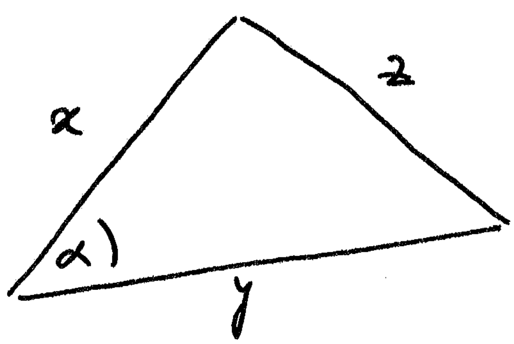
$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$\vec{0} \cdot \vec{a} = 0$$

$$(c\vec{a}) \cdot \vec{b} = \vec{a} \cdot (c\vec{b}) = c(\vec{a} \cdot \vec{b})$$

Recall the Law of Cosines

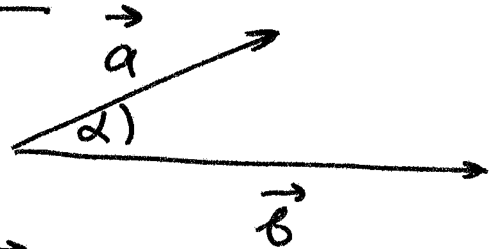


$$z^2 = x^2 + y^2 - 2xy \cos\alpha$$

If $\alpha = \pi/2$, then $\cos\alpha = 0$ and the Law of Cosines reduces to the Phytagorean theorem

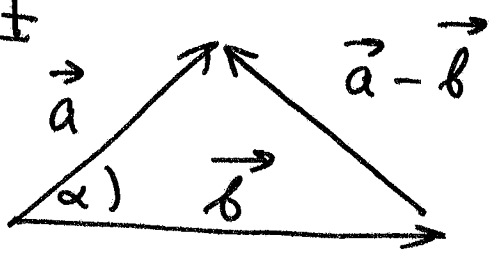
$$z^2 = x^2 + y^2 .$$

Theorem



$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\alpha$$

Proof



From the Law of Cosines we have

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}| \cos\alpha$$

Hence

$$|\vec{a}||\vec{b}|\cos\alpha = \frac{1}{2}\Big(|\vec{a}|^2 + |\vec{b}|^2 - \underbrace{|\vec{a}-\vec{b}|^2}_{(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})}\Big) = \heartsuit$$

$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$$

Thus

$$\heartsuit = \frac{1}{2}\Big(\cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - \big(\cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2\,\vec{a}\cdot\vec{b}\big)\Big)$$

$$= \vec{a}\cdot\vec{b}$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\alpha \qquad \square$$

The angle between vectors can be computed from

$$\cos\alpha = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\alpha = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

If on an exam you are asked to find the angle, it is enough if you provide $\cos\alpha$.

Example Find the angle between the vectors

$$\vec{a} = \langle 3, 5, 2 \rangle, \quad \vec{b} = \langle 1, -1, 1 \rangle$$

Solution

$$|\vec{a}||\vec{b}| \cos\alpha = \vec{a} \cdot \vec{b} = 3 \cdot 1 + 5 \cdot (-1) + 2 \cdot 1 = 0$$

$$\cos\alpha = 0$$

$$\alpha = \pi/2 .$$

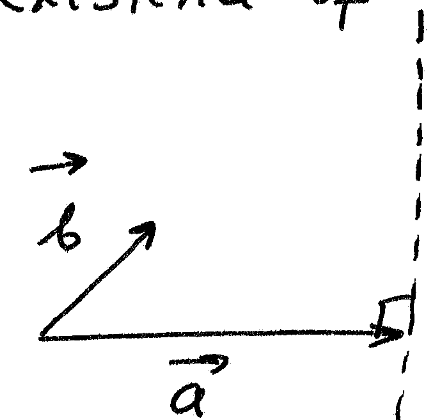
Definition Two vectors $\vec{a}, \vec{b}$ are called orthogonal (perpendicular) if $\vec{a} \cdot \vec{b} = 0$. In other words, if the angle between the vectors is $\pi/2$ (or if one of the vectors is a zero vector).

Example $\vec{i}, \vec{j}, \vec{k}$ are ortogonal to each other

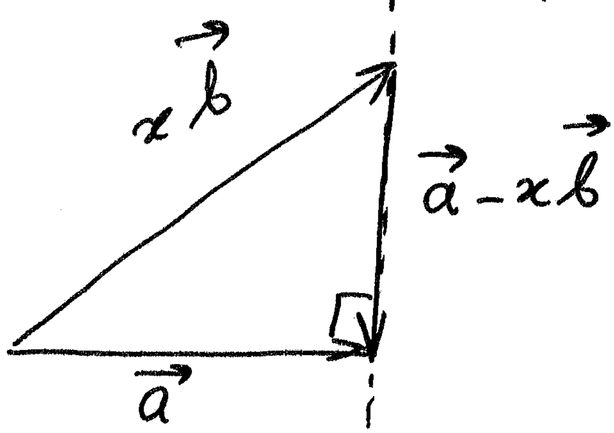
Example The vectors $\vec{a} = \langle 3, 5, 2 \rangle$, $\vec{b} = \langle 1, -1, 1 \rangle$ considered above are orthogonal.

Example Suppose that the vectors $\vec{a}$ and $\vec{b}$ are <u>not</u> orthogonal. Find x such that the vectors $\vec{a}$ and $\vec{a} - x\vec{b}$ are orthogonal.

Solution Geometrically speaking existence of such x is obvious



We extend the vector $\vec{b}$ so that it intersects the dotted line. The extension of $\vec{b}$ is a vector of the form $x\vec{b}$



From the picture we see that the vector $\vec{a} - x\vec{b}$ is orthogonal to $\vec{a}$

Now we will find x.

$$\vec{a}\cdot(\vec{a}-x\vec{b})=0$$

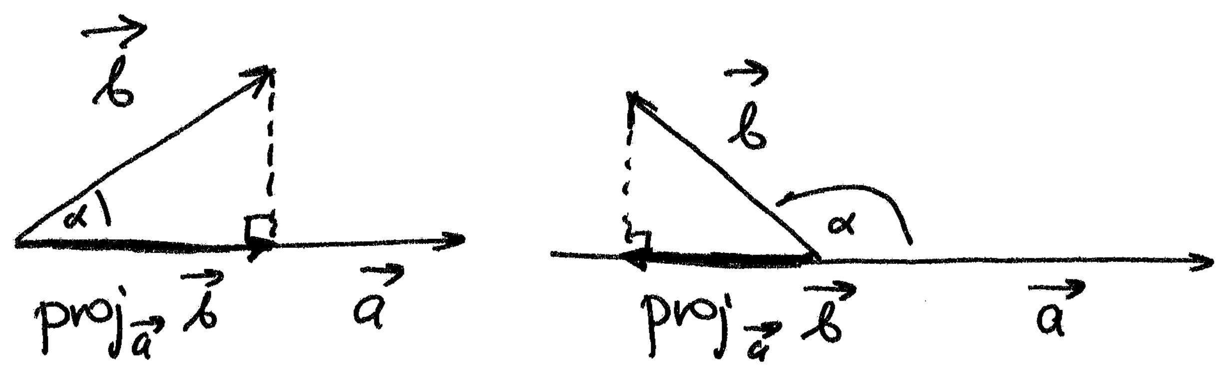
$$\vec{a}\cdot\vec{a}-x\,\vec{a}\cdot\vec{b}=0$$

$$|\vec{a}|^2=x\,\vec{a}\cdot\vec{b}$$

$$x=\frac{|\vec{a}|^2}{\vec{a}\cdot\vec{b}}.$$

We assumed that the vectors $\vec{a}$ and $\vec{b}$ are not orthogonal, so $\vec{a}\cdot\vec{b}\neq 0$.

<u>Vector projection of $\vec{b}$ onto $\vec{a}$</u>



From the definition of $\cos\alpha$ we see that

$|\vec{b}| \cos\alpha$

equals the length of the projection $\text{proj}_{\vec{a}} \vec{b}$ and it is taken with "+" sign if $\alpha < \pi/2$, so the projection points in the same direction as $\vec{a}$ and with the "−" sign if $\alpha > \pi/2$, so the projection points in the direction opposite to $\vec{a}$.

$$\text{comp}_{\vec{a}} \vec{b} = |\vec{b}| \cos\alpha$$

is called <u>scalar projection</u> of $\vec{b}$ onto $\vec{a}$ or component of $\vec{b}$ along $\vec{a}$.

We have

$$\text{comp}_{\vec{a}} \vec{b} = |\vec{b}| \cos\alpha = \frac{|\vec{a}||\vec{b}| \cos\alpha}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}.$$

<u>Unit vector</u> - any vector of length 1.

$\dfrac{\vec{a}}{|\vec{a}|}$ - unit vector parallel to $\vec{a}$.

$\text{proj}_{\vec{a}}\vec{b}$ is parallel to $\frac{\vec{a}}{|\vec{a}|}$ and (13)
it has signed length equal

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}.$$

Thus

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \frac{(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2}$$

We proved

$$\boxed{\begin{aligned} \text{comp}_{\vec{a}}\vec{b} &= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \\ \text{proj}_{\vec{a}}\vec{b} &= \frac{(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2} \end{aligned}}$$

Cross product

We need to recall how to compute determinants of 2×2 and 3×3 matrices

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_2(b_1c_2 - b_2c_1)$$

<u>Definition</u> For $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ we define

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and we compute the determinant as abore

$$\vec{a} \times \vec{b} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle.$$

Theorem If α is the angle between the vectors $\vec{a}$ and $\vec{b}$ then

$\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$

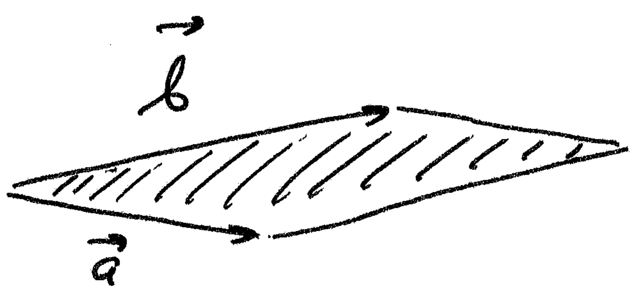
$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\alpha$$

The orientation of the vector $\vec{a} \times \vec{b}$ is determined through the right hand rule.

The length of the vector $\vec{a} \times \vec{b}$, i.e.

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\alpha$$

equals the area of the paralleogram with sides $\vec{a}$ and $\vec{b}$.



Example Find $\vec{a} \times \vec{b}$ where

$\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle 3, 2, 1 \rangle$

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} =$$

$$\vec{i}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} =$$

$$\vec{i}(2-6) - \vec{j}(1-9) + \vec{k}(2-6) =$$

$$\langle -4, 8, -4 \rangle.$$

<u>Example</u> Find the area of the triangle with vertices
$A(1,2,3)$, $B(1,1,2)$, $C(2,2,2)$

<u>Solution</u> The area equals half of the area of the parallelogram with sides

$$\vec{AB} \text{ and } \vec{AC}$$

$$\vec{AB} = \langle 1-1, 1-2, 2-3 \rangle = \langle 0, -1, -1 \rangle$$
$$\vec{AC} = \langle 2-1, 2-2, 2-3 \rangle = \langle 1, 0, -1 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix} =$$

$$\vec{i}(1-0) - \vec{j}(0+1) + \vec{k}(0+1) = \langle 1, -1, 1 \rangle$$

The area of the triangle equals

$$\frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{1^2 + (-1)^2 + 1^2} = \frac{\sqrt{3}}{2}.$$

Example Find a vector perpendicular to the plane that passes through the points

$A(1,2,3)$, $B(1,1,2)$, $C(2,2,2)$

Solution The vector

$$\vec{AB} \times \vec{AC} = \langle 1,-1,1 \rangle$$

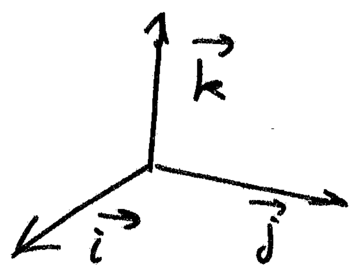
is perpendicular to that plane

Properties

$$\vec{i} \times \vec{j} = \vec{k}$$

$$\vec{i} \times \vec{k} = -\vec{j}$$

$$\vec{j} \times \vec{k} = \vec{i}$$



Theorem

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b}) = c(\vec{a} \times \vec{b})$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Important In general it may happen that

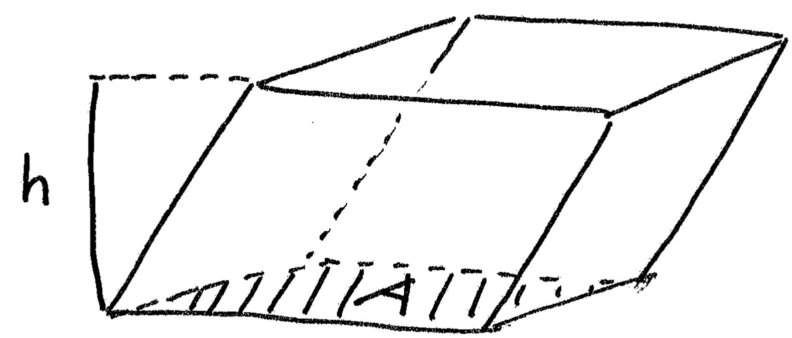
$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$$

Indeed, here is an example of such a situation

$$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$$

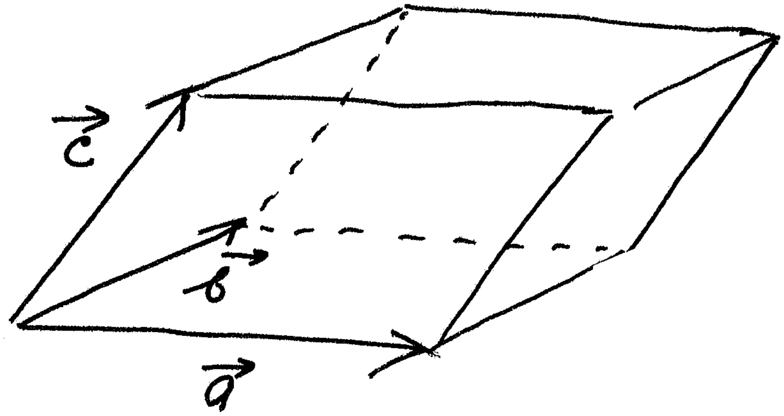
$$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}.$$

Volume of a parallelpiped

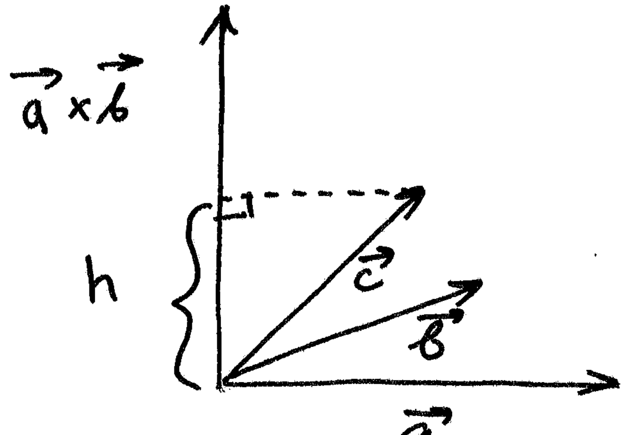


$$Vol = A \cdot h$$

Now assume that the edges are vectors



Thus $A = |\vec{a} \times \vec{b}|$ and h equals the length of the projection of $\vec{c}$ onto $\vec{a} \times \vec{b}$, because the vector $\vec{a} \times \vec{b}$ is orthogonal to the base



Hence

$$h = \left| \operatorname{comp}_{\vec{a} \times \vec{b}} \vec{c} \right| = \left| \frac{(\vec{a} \times \vec{b}) \cdot \vec{c}}{|\vec{a} \times \vec{b}|} \right|$$

We put absolute value, because as we know "comp" may have positive or negative value. Thus

$$Vol = A \cdot h = |\vec{a} \times \vec{b}| \left| \frac{(\vec{a} \times \vec{b}) \cdot \vec{c}}{|\vec{a} \times \vec{b}|} \right|$$

$$= |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

As we know

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$$

Thus we also have

$$\text{Vol} = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

We proved

Theorem $\text{vol} = |\vec{a} \cdot (\vec{b} \times \vec{c})| = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$

Definition The product $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the scalar triple product of vectors $\vec{a}, \vec{b}, \vec{c}$.

Problems

Exercise 1 Find the unit vector in the direction of $\vec{i} + \vec{j} + \vec{k}$

Solution By the definition the unit vector has length 1. If $\vec{a}$ is a non-zero vector, then $\vec{a}/|\vec{a}|$ is the unit vector in the direction of $\vec{a}$.
In our situation $|\vec{i} + \vec{j} + \vec{k}| = \sqrt{3}$ so the answer is

$$\frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}.$$

Exercise 2 Find the vector of length 7 in the same direction as

$$\sqrt{5}\,\vec{i} - 4\vec{j} + 2\vec{k}$$

Solution The given vector has length

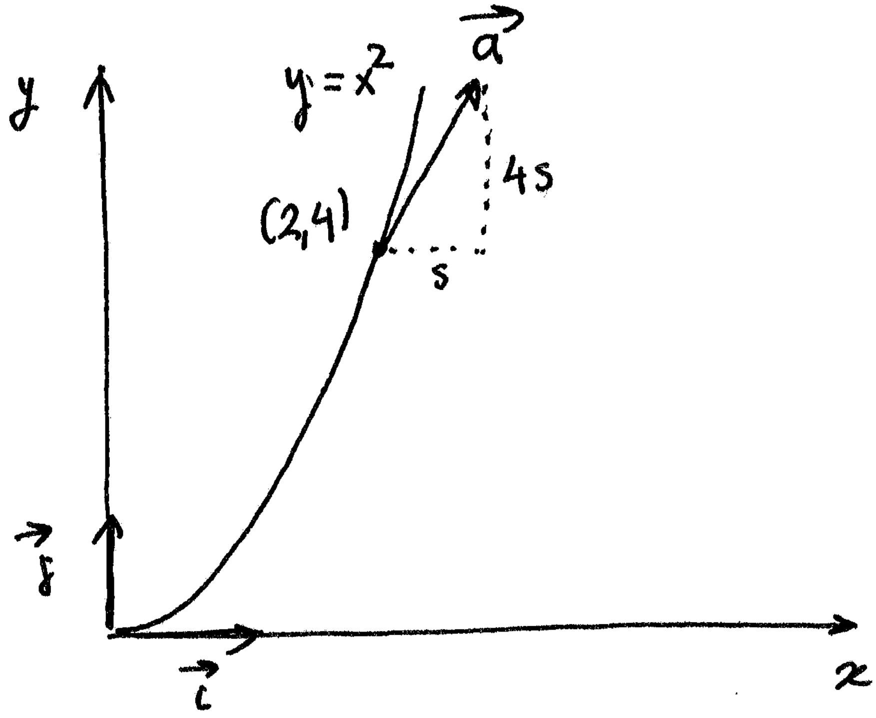
$$\sqrt{\sqrt{5}^2 + (-4)^2 + 2^2} = \sqrt{25} = 5$$

Thus we need to multiply our vector by the factor $7/5$, so the answer is

$$\frac{7}{5}\left(\sqrt{5}\,\vec{i} - 4\vec{j} + 2\vec{k}\right) = \frac{7\sqrt{5}}{5}\vec{i} - \frac{28}{5}\vec{j} + \frac{14}{5}\vec{k}.$$

Exercise 3 Find all unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2,4)$.

Solution Let $f(x) = x^2$. Denote by $\vec{a}$ the unit tangent vector as on the picture



The slope of the tangent line at $x=2$ equals $f'(2) = 4$. Thus the $\vec{j}$ component of $\vec{a}$ is 4 times the $\vec{i}$ component of $\vec{a}$

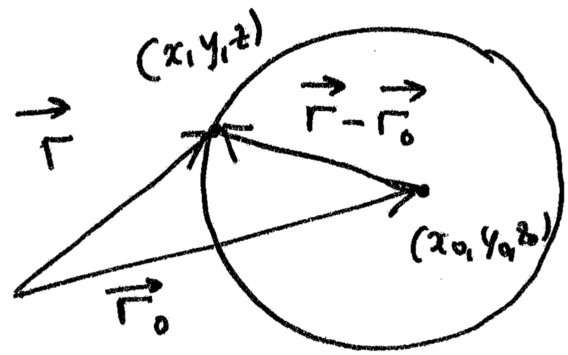
Hence
$$\vec{a} = s\vec{i} + 4s\vec{j}$$
for some s. We have
$$1 = |\vec{a}| = \sqrt{s^2 + (4s)^2} = \sqrt{17}\, s$$
$$s = \frac{1}{\sqrt{17}}$$
$$\vec{a} = \frac{\vec{i} + 4\vec{j}}{\sqrt{17}}$$
but also
$$-\vec{a} = -\frac{\vec{i} + 4\vec{j}}{\sqrt{17}}$$
is unit and parallel to the tangent line but with the opposite orientation

Hence the answer is

$$\pm \frac{\vec{i} + 4\vec{j}}{\sqrt{17}}$$

Exercise 4 If $\vec{r} = \langle x, y, z \rangle$ and $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, describe the set of all points (x, y, z) such that $|\vec{r} - \vec{r}_0| = 1$.

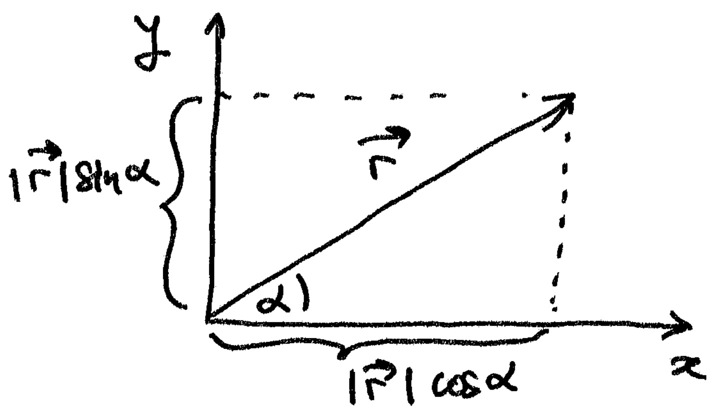
Solution



Thus the set is the sphere of radius 1 centred at (x_0, y_0, z_0)

Exercise 5 Find the planar vector of length 2 that makes an angle of $\pi/3$ with the x-axis.

Solution Recall that



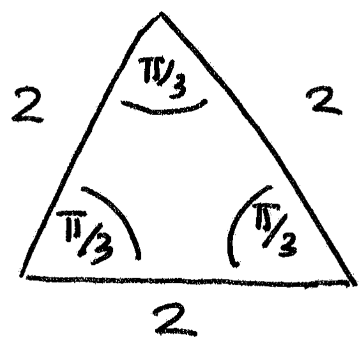
$$\vec{r} = |\vec{r}|\left(\cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}\right)$$

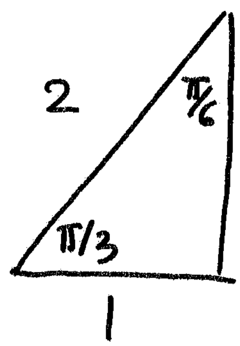
In our situation

$$\vec{r} = 2\left(\cos\frac{\pi}{3}\,\vec{i} + \sin\frac{\pi}{3}\,\vec{j}\right) =$$

$$2\left(\frac{1}{2}\,\vec{i} + \frac{\sqrt{3}}{2}\,\vec{j}\right) = \vec{i} + \sqrt{3}\,\vec{j}$$

<u>Trigonometric functions to remember:</u>

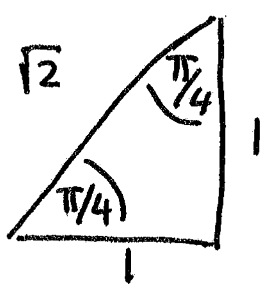




(Phytagorean thm)

$$\cos\frac{\pi}{3} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{6} = \frac{1}{2}$$



$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Exercise 6 Prove that $|\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}|$.

Solution $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\alpha$

$$|\vec{a}\cdot\vec{b}| = |\vec{a}||\vec{b}||\cos\alpha| \leq |\vec{a}||\vec{b}|.$$

Exercise 7 Prove the triangle inequality

$$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$$

Proof $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) =$

$\vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} =$

$|\vec{a}|^2 + 2\,\vec{a}\cdot\vec{b} + |\vec{b}|^2 \underset{\uparrow \text{Ex.6}}{\leq} |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2$

$= (|\vec{a}| + |\vec{b}|)^2$

$$|\vec{a}+\vec{b}|^2 \leq (|\vec{a}| + |\vec{b}|)^2$$

$$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|.$$

Exercise 8 Find the angle between the vectors $\vec{a} = \langle 4,3 \rangle$ and $\vec{b} = \langle 2,-1 \rangle$

<u>Solution</u> $\vec{a}\cdot\vec{b} = 4\cdot 2 + 3\cdot(-1) = 5$

$$|\vec{a}| = \sqrt{4^2+3^2} = \sqrt{25} = 5$$

$$|\vec{b}| = \sqrt{2^2+(-1)^2} = \sqrt{5}$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\alpha$$

$$\cos\alpha = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{5\cdot\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\alpha = \arccos\left(\frac{\sqrt{5}}{5}\right).$$

<u>Exercise 9</u> Find the angle between a diagonal of a cube and one of its edges.

<u>Solution</u> Diagonal $\vec{a} = \langle 1,1,1\rangle$, edge $\vec{b} = \langle 1,0,0\rangle$

$$\cos\alpha = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{\sqrt{3}}$$

$$\alpha = \arccos\left(\frac{1}{\sqrt{3}}\right).$$

Exercise 10 Find the scalar and vector projections of $\vec{b} = \langle 4, 6 \rangle$ onto $\vec{a} = \langle -5, 12 \rangle$

Solution Recall:

Scalar projection

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

vector projection

$$\text{proj}_{\vec{a}} \vec{b} = \frac{(\vec{a} \cdot \vec{b})\,\vec{a}}{|\vec{a}|^2} = \left(\text{comp}_{\vec{a}} \vec{b}\right) \frac{\vec{a}}{|\vec{a}|}.$$

$$\vec{a} \cdot \vec{b} = 4 \cdot (-5) + 6 \cdot 12 = 52$$

$$|\vec{a}| = \sqrt{(-5)^2 + 12^2} = 13$$

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{52}{13} = 4$$

$$\text{proj}_{\vec{a}} \vec{b} = \left(\text{comp}_{\vec{a}} \vec{b}\right) \frac{\vec{a}}{|\vec{a}|} =$$

$$4 \cdot \frac{\langle -5, 12 \rangle}{13} = \left\langle -\frac{20}{13}, \frac{48}{13} \right\rangle.$$

Exercise 1) Show that the vector $\langle a, b\rangle$ is orthogonal to the line $ax+by+c=0$.

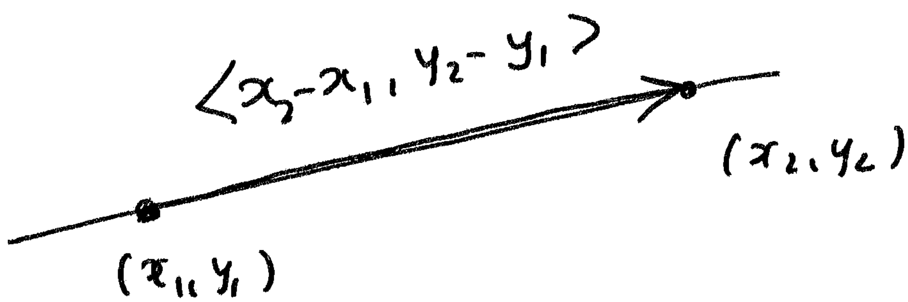
Proof Suppose that (x_1, y_1), (x_2, y_2) are two distinct points on the line. We have

$$a x_2 + b y_2 + c = 0$$
$$a x_1 + b y_1 + c = 0$$

Subtracting the second equality from the first one yields

$$a(x_2-x_1) + b(y_2-y_1) = 0$$
$$\langle a, b\rangle \cdot \langle x_2-x_1, y_2-y_1\rangle = 0$$

but the vector $\langle x_2-x_1, y_2-y_1\rangle$ is parallel to the line



Thus $\langle a, b\rangle$ is orthogonal to the line.

Exercise 12 Show that if the vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal, then $\vec{u}$ and $\vec{v}$ must have the same length.

Proof $0 = (\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) =$

$$\vec{u}\cdot\vec{u} - \cancel{\vec{u}\cdot\vec{v}} + \cancel{\vec{v}\cdot\vec{u}} - \vec{v}\cdot\vec{v} =$$

$$|\vec{u}|^2 - |\vec{v}|^2$$

$$|\vec{u}|^2 = |\vec{v}|^2$$

$$|\vec{u}| = |\vec{v}|.$$

Exercise 13 Show that

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

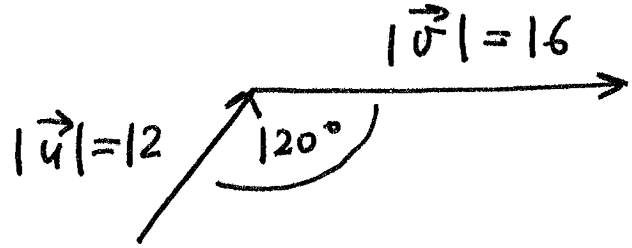
Proof.

$$|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}||\vec{b}|\cos\alpha)^2 =$$

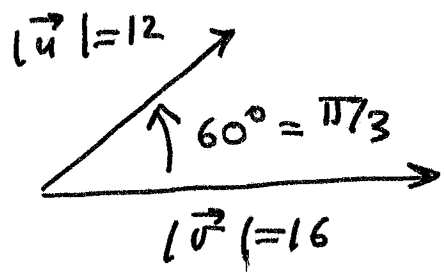
$$|\vec{a}|^2|\vec{b}|^2(1-\cos^2\alpha) = |\vec{a}|^2|\vec{b}|^2\sin^2\alpha =$$

$$(|\vec{a}||\vec{b}|\sin\alpha)^2 = |\vec{a}\times\vec{b}|^2.$$

Exercise 14 Find $|\vec{u}\times\vec{v}|$ and determine whether $\vec{u}\times\vec{v}$ is directed into the page or out of the page where



Solution To compute the cross product vectors need to be attached to the same point, so the actual situation is as follows



Now

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin \pi/3 = 12\cdot 16\cdot \frac{\sqrt{3}}{2} = 96\sqrt{3}.$$

The right hand rule tells us that $\vec{u}\times\vec{v}$ is oriented into the page.

Exercise 15 Find the volume of the parallelpiped with adjacent edges PQ, PR, PS, where
$P(-2,1,0)$, $Q(2,3,2)$, $R(1,4,-1)$, $S(3,6,1)$

Solution

$$Vol = |\vec{a}\cdot(\vec{b}\times\vec{c})| = |\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS})|$$

$$= |\langle 4,2,2\rangle\cdot(\langle 3,3,-1\rangle\times\langle 5,5,1\rangle)|$$

$$\langle 3,3,-1\rangle\times\langle 5,5,1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix} =$$

$$= 8\vec{i} - 8\vec{j} = \langle 8,-8,0\rangle$$

$$|\langle 4,2,2\rangle\cdot\langle 8,-8,0\rangle| = |32-16| = 16.$$

Exercise 16 Show that the vectors $\vec{u} = \vec{i} + 5\vec{j} - 2\vec{k}$, $\vec{v} = 3\vec{i} - \vec{j}$ and $\vec{w} = 5\vec{i} + 9\vec{j} - 4\vec{k}$ are coplanar.

Solution It suffices to show that the volume of the parallelpiped with adjacent edges $\vec{u}$, $\vec{v}$ and $\vec{w}$ equals zero. (32)

$$|\vec{u}\cdot(\vec{v}\times\vec{w})| = |\langle 1,5,-2\rangle\cdot(\langle 3,-1,0\rangle\times\langle 5,9,-4\rangle)|$$

$$\langle 3,-1,0\rangle\times\langle 5,9,-4\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 0 \\ 9 & -4 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & -1 \\ 5 & 9 \end{vmatrix} =$$

$$= 4\vec{i} + 12\vec{j} + 32\vec{k} = \langle 4,12,32\rangle$$

$$\langle 1,5,-2\rangle\cdot\langle 4,12,32\rangle = 4+60-64 = 0.$$

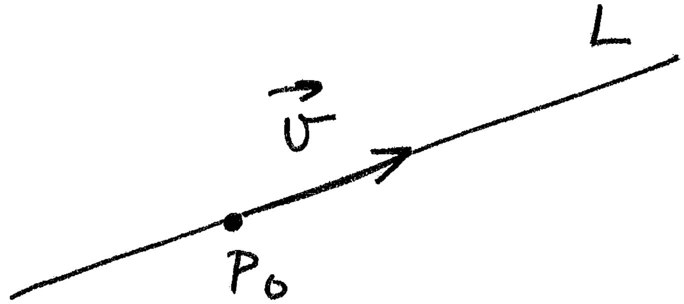
Exercise 17 Explain why there is no vector $\vec{v}$ such that

$$\langle 1,2,1\rangle\times\vec{v} = \langle 3,1,5\rangle$$

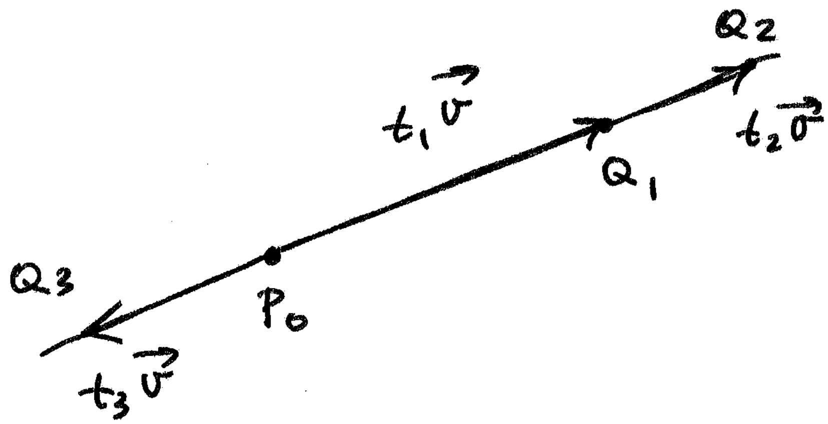
Solution The cross product would have to be orthogonal to $\langle 1,2,1\rangle$, but the vector $\langle 3,1,5\rangle$ is not.

Equations of lines and planes

To describe a line L in $\mathbb{R}^3$ we need a point $P(x_0, y_0, z_0)$ on the line and a vector $\vec{v}$ parallel to that line



The terminal points Q of the vectors $t\vec{v}$ attached to P_0 will cover the entire line L when we take all real numbers t

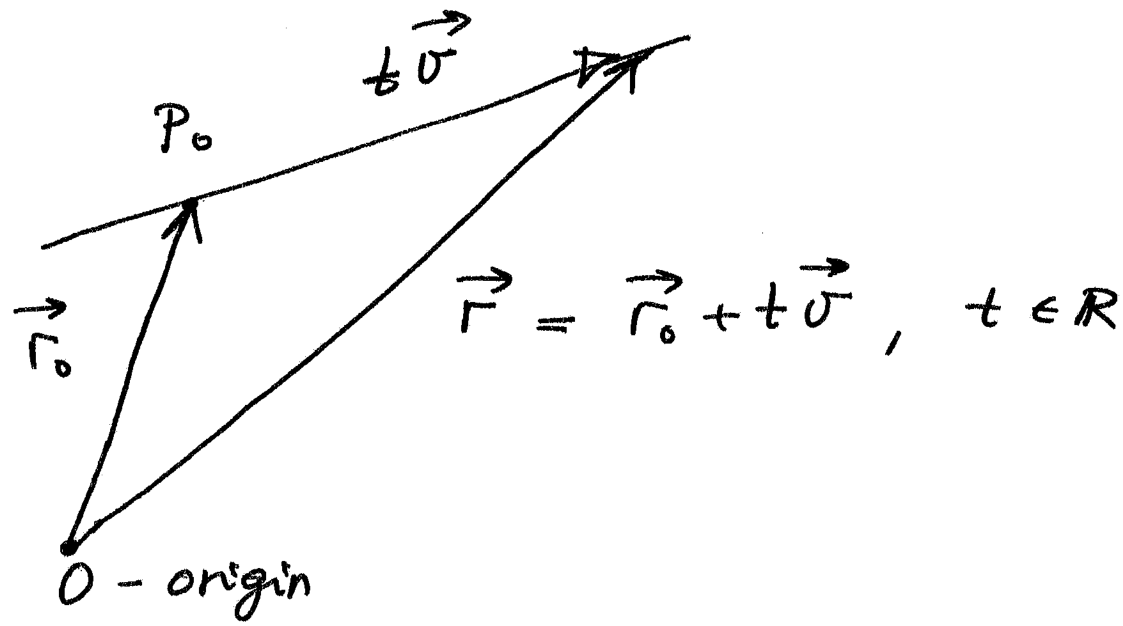


If $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ is the position vector of the point P_0, i.e. $\vec{r}_0 = \overrightarrow{OP_0}$, then

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

is the position vector of a point on

the line L.



For this reason

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$$

is called a vector equation of the line L.

If $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\vec{r} = \langle x, y, z \rangle$ and $\vec{v} = \langle a, b, c \rangle$, the equation becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

or

$$\boxed{x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc}$$

which are called <u>parametric equations</u>

of the line L through the point P_0 and parallel to $\vec{v}$.

Each value of the parameter t describes a point (x, y, z) on the line L.

We can also eliminate t from the equations as follows:

$$t = \frac{x - x_0}{a}, \quad t = \frac{y - y_0}{b}, \quad t = \frac{z - z_0}{c}$$

So

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

which are called symmetric equations of L.

<u>Example</u> Find parametric equation of the line passing through $P_1\ (2, 3, 1)$ and $P_2\ (3, 5, 6)$. Find the point of intersection of the line with the xy plane.

<u>Solution</u> $\vec{v} = \overrightarrow{P_1P_2} = \langle 3-2, 5-3, 6-1 \rangle$
$= \langle 1, 2, 5 \rangle.$

$\vec{r}_0 = \overrightarrow{OP_1} = \langle 2, 3, 1 \rangle$

$\vec{r} = \vec{r}_0 + t\vec{v}$

$\langle x, y, z \rangle = \langle 2+t, 3+2t, 1+5t \rangle$

Thus the parametric equations are

$x = 2+t, \quad y = 3+2t, \quad z = 1+5t.$

At the point of intersection with the xy plane we have $z=0$, so

$1+5t = 0$

$t = -\frac{1}{5}$

and hence

$x = 2 - \frac{1}{5} = \frac{9}{5}, \quad y = 3 + 2\left(-\frac{1}{5}\right) = \frac{13}{5}$

Thus the point of intersection is

$\left(\frac{9}{5}, \frac{13}{5}, 0\right).$

Remark Instead of P_1 we could also take P_2 as a point on the line and the same vector $\vec{v}$. This would give us the equation

$$\langle x, y, z \rangle = \langle 3+t, 5+2t, 6+5t \rangle$$

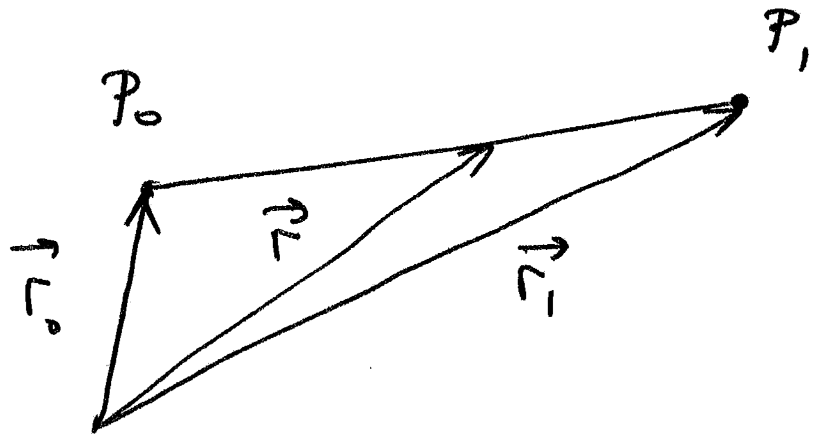
i.e.

$$x = 3+t, \quad y = 5+2t, \quad z = 6+5t$$

These equations look differently than the equations we obtained previously, but they describe the same line.

Segment

We can also describe the segment connecting given two points P_0 and P_1



The vector equation describing points on that segment is

$$\boxed{\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1 , \quad 0 \le t \le 1}$$

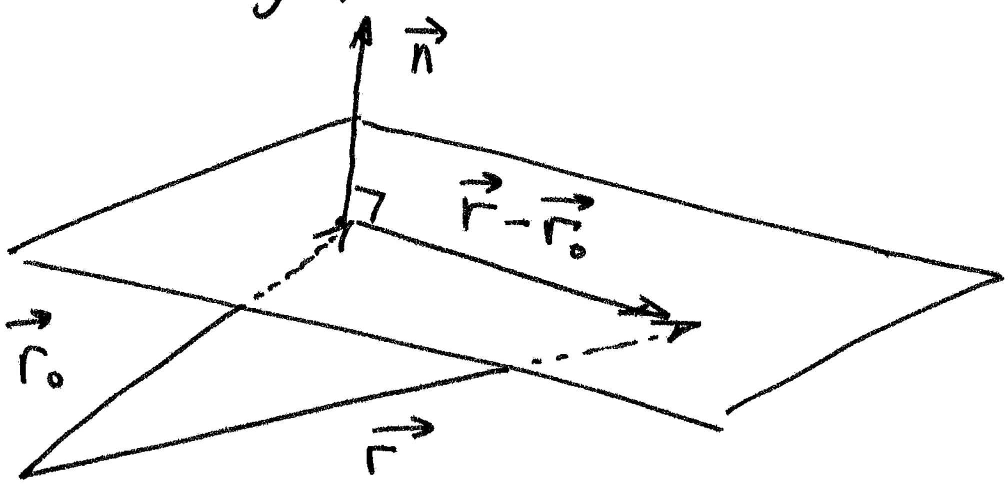
If $t=0$, $\vec{r} = \vec{r}_0$ describes the endpoint P_0,
if $t=1$, $\vec{r} = \vec{r}_1$ describes the endpoint P_1
and if $0<t<1$, $\vec{r}$ describes points inside the interval.

Planes

Now we will find equation that describe a plane in $\mathbb{R}^3$. To describe a plane we a point on it and a vector which is normal to the plane.

Let $P_0(x_0, y_0, z_0)$ be a point on the plane and let $\vec{n} = \langle a, b, c \rangle$ be a normal (i.e. orthogonal) vector to that plane.

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ describes position of the point P_0. Let $\vec{r} = \langle x, y, z \rangle$ be a vector that describes position of an arbitrary point on the plane.



The vector $\vec{r} - \vec{r}_0$ is parallel to the plane and hence it is orthogonal to the vector $\vec{n}$. Thus

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0}$$

or equivalently

$$\boxed{\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0}$$

Each of the two equations is called a vector equation of the plane

Suppose now that $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\vec{n} = \langle a, b, c \rangle$ and $\vec{r} = \langle x, y, z \rangle$.
Then the first of the two equations becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

i.e.

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \qquad (*)$$

which is called the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with the normal $\langle a, b, c \rangle$.

This equation can be rewritten as

$$ax + by + cz + \underbrace{(-ax_0 - by_0 - cz_0)}_{d} = 0$$

$$\boxed{ax + by + cz + d = 0} \qquad (**)$$

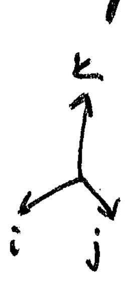
It is called a linear equation of the plane

From the equation (*) we know right away that (x_0, y_0, z_0) is a point on the plane, but (**) does not show us right away any point on the plane. However in both cases (*) and (**) we know that the vector $\langle a, b, c \rangle$ formed by the coefficients at x, y, z is normal to the plane.

Problems

Exercise 18 Find parametric and symmetric equations of the line through $(2,1,0)$ and perpendicular to both $\vec{i}+\vec{j}$ and $\vec{j}+\vec{k}$

Solution Since the line is perpendicular to both $\vec{i}+\vec{j}$ and $\vec{j}+\vec{k}$, it is parallel to the vector



$$\vec{v} = (\vec{i}+\vec{j}) \times (\vec{j}+\vec{k}) = \vec{i}\times\vec{j} + \vec{i}\times\vec{k} + \vec{j}\times\vec{j} + \vec{j}\times\vec{k}$$

$$= \vec{k} - \vec{j} + \vec{0} + \vec{i} = \langle 1, -1, 1 \rangle$$

$$\vec{r} = \vec{r_0} + t\vec{v}$$

$$\langle x, y, z\rangle = \langle 2, 1, 0\rangle + t\langle 1, -1, 1\rangle$$

$$\langle x, y, z\rangle = \langle 2+t, 1-t, t\rangle$$

Parametric equations:

$$x = 2+t, \quad y = 1-t, \quad z = t$$

Now

$$t = x-2, \quad t = 1-y, \quad t = z$$

So the symmetric equations are

$$x-2 = 1-y = z$$

Exercise 19 Find symmetric equations of the line through the points $(1, 2, 3)$, $(5, 1, 7)$.

Solution $\vec{r_0} = \langle 1, 2, 3\rangle$,

$$\vec{v} = \langle 5-1, 1-2, 7-3\rangle = \langle 4, -1, 4\rangle$$

$$\vec{r} = \vec{r_0} + t\vec{v}$$

$$\langle x, y, z\rangle = \langle 1+4t, 2-t, 3+4t\rangle$$

$$x = 1+4t, \quad y = 2-t, \quad z = 3+4t$$

$$t = \frac{x-1}{4}, \quad t = 2-y, \quad t = \frac{z-3}{4}$$

Answer

$$\frac{x-1}{4} = 2-y = \frac{z-3}{4}.$$

Exercise 20 Find the line through the point $(1,0,6)$ and perpendicular to the plane $x+3y+z=5$

Solution The line is parallel to the normal vector to the plane

$$\vec{n} = \langle 1,3,1 \rangle$$

So the vector equation is

$$\vec{r} = \vec{r_0} + t\,\vec{n}$$

$$\langle x,y,z \rangle = \langle 1,0,6 \rangle + t\langle 1,3,1 \rangle$$

$$\langle x,y,z \rangle = \langle 1+t,\ 3t,\ 6+t \rangle$$

Exercise 21 Prove that the planes

$x + 2y - z = 7$ and $2x - 5 + 4(y-3) = 2z$

are parallel.

Solution The planes are parallel if the normal vectors are parallel. The normal vector to $x + 2y - z = 7$ is $\vec{n}_1 = \langle 1, 2, -1 \rangle$. To see the normal vector for the second plane we move all variables to the left hand side

$$2x - 5 + 4(y-3) - 2z = 0$$

Hence the vector $\vec{n}_2 = \langle 2, 4, -2 \rangle$ is normal to the second plane. Since

$$\vec{n}_2 = 2\vec{n}_1$$

the normal vectors are parallel and hence the planes are parallel too.

Exercise 22 Prove that the planes $3x+7y-z=2$ and $6y-2z=5-3x$ are not parallel.

Solution The normal vectors are $\langle 3,7,-1\rangle$ and $\langle 3,6,-2\rangle$. They are not proportional, so they are not parallel. Hence the planes are not parallel.

Exercise 23 Find the angle between the planes $3x+7y-z=2$ and $3x+6y-2z=5$

Solution The angle between two planes equals the angle between the normal vectors

$$\vec{n}_1=\langle 3,7,-1\rangle \text{ and } \vec{n}_2=\langle 3,6,-2\rangle$$

$$\cos\alpha=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}=\frac{53}{\sqrt{59}\sqrt{49}}=\frac{53}{7\sqrt{59}}$$

$$\alpha=\arccos\left(\frac{53}{7\sqrt{59}}\right).$$

Exercise 24 Find a scalar equation of the plane $3x+7y-z=2$.

Solution. We know the normal vector $\vec{n} = \langle 3, 7, -1 \rangle$, but we still need to find a point on the plane. It suffices to guess a solution. For example

$$(x_0, y_0, z_0) = (1, 1, 8)$$

solves the equation, i.e. $P_0(1,1,8)$ is a point on the plane. Hence

$$3(x-1)+7(y-1)-(z-8) = 0$$

is the scalar equation.

Exercise 25 Find an equation of the plane that passes through the points $P(1,1,1)$, $Q(1,2,3)$, $R(3,2,1)$

Solution We know now a point on the plane (even three points), but we need to find a normal vector.

Note that vectors

$\overrightarrow{PQ} = \langle 0, 1, 2 \rangle$ and $\overrightarrow{PR} = \langle 2, 1, 0 \rangle$

are parallel to the plane, so their cross product is normal

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix} =$$

$$\vec{i} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} =$$

$$-2\vec{i} + 4\vec{j} - 2\vec{k} = \langle -2, 4, -2 \rangle.$$

Now using the point $P(1,1,1)$ we get the equation

$$-2(x-1) + 4(y-1) - 2(z-1) = 0$$

or

$$-2x + 2 + 4y - 4 - 2z + 2 = 0$$

$$-2x + 4y - 2z = 0.$$

Remark Using the point $Q(1,2,3)$ we will get an equation which looks differently

$$-2(x-1) + 4(y-2) - 2(z-3) = 0$$

But this equation is equivalent to the previous one, because it can be rewritten as

$$-2x+2+4y-8-2z+6=0$$

$$-2x+4y-2z=0$$

which is the same equation as before.

Exercise 26 Find symmetric equations of the line of intersection of the planes

$$x+y+z=1 \quad \text{and} \quad x-2y+3z=1$$

Solution. The line of intersection lies on the first plane, so the normal vector

$$\vec{n}_1 = \langle 1,1,1 \rangle$$

is orthogonal to that line. But the line of intersection also lies on the second plane, so the normal vector to the second plane

$$\vec{n}_2 = \langle 1,-2,3 \rangle$$

is also orthogonal to that line. Thus the line of intersection is orthogonal to both vectors $\vec{n}_1$ and $\vec{n}_2$.

Hence it is parallel to their cross product

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5, -2, -3 \rangle.$$

We still need to find a point on the line of intersection. Coordinates x, y, z of such point must solve equations of both planes

$$\begin{cases} x + y + z = 1 \\ x - 2y + 3z = 1 \end{cases} \qquad (*)$$

There are two equations and three unknowns, so we can put an arbitrary value to one of variables. Let $z = 0$. Then we have

$$\begin{cases} x + y = 1 \\ x - 2y = 1 \end{cases}$$

and it is easy to see that $x = 1$ and $y = 0$ is the solution. Thus

$$x = 1, \; y = 0, \; z = 0$$

solves the system $(*)$, i.e. the point

$(x_0, y_0, z_0) = (1, 0, 0)$ belongs to the line of intersection. Now we can find the equation

$$\vec{r} = \vec{r_0} + t\vec{v}$$

$$\langle x, y, z \rangle = \langle 1, 0, 0 \rangle + t \langle 5, -2, -3 \rangle$$

$$x = 1 + 5t, \quad y = -2t, \quad z = -3t$$

$$t = \frac{x-1}{5}, \quad t = -\frac{y}{2}, \quad t = -\frac{z}{3}$$

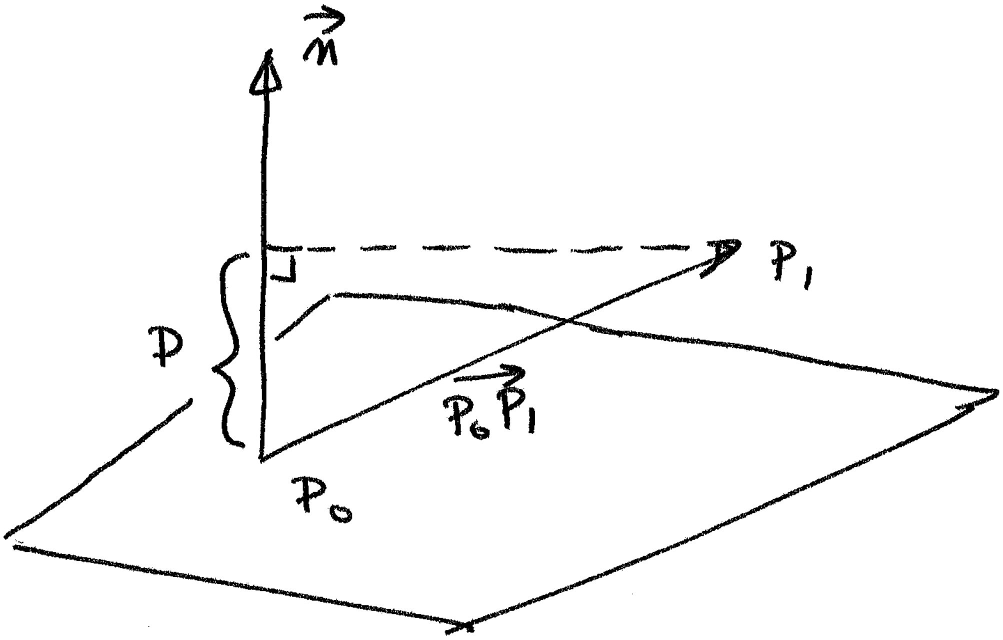
and the symmetric equations are

$$\frac{x-1}{5} = -\frac{y}{2} = -\frac{z}{3}.$$

<u>Exercise 27</u> Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$

<u>Solution</u> Let $P_0(x_0, y_0, z_0)$ be any point on the plane. The distance D equals the length of the projection of the vector $\overrightarrow{P_0P_1}$ on the normal vector $\vec{n}$. The reason is explained

on the picture



However, $D = \left| \text{comp}_{\vec{n}} \overrightarrow{P_0P_1} \right|$.
We need to take the absolute value because "comp" may have negative values. We have

$$\overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

$$\vec{n} = \langle a, b, c \rangle$$

$$D = \left| \text{comp}_{\vec{n}} \overrightarrow{P_0P_1} \right| = \frac{\left| \vec{n} \cdot \overrightarrow{P_0P_1} \right|}{|\vec{n}|} =$$

$$\frac{\left| a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) \right|}{\sqrt{a^2 + b^2 + c^2}} =$$

$$\frac{|(ax_1+by_1+cz_1)-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}} = \boxed{\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

because

$$ax_0+by_0+cz_0+d=0$$

so

$$-(ax_0+by_0+cz_0)=d.$$

<u>Exercise 28</u> Find the equation of the plane through the point $(-1,2,1)$ that contains the line of intersection of the planes $x+y-z=2$ and $2x-y+3z=1$.

<u>Solution</u> Vectors $\langle 1,1,-1\rangle$ and $\langle 2,-1,3\rangle$ are orthogonal to the line of intersection so the vector

$\langle 1,1,-1\rangle \times \langle 2,-1,3\rangle = \langle 2,-5,-3\rangle$

is parallel to that line and hence it is also parallel to the plane we want to find. To find a point on the line of intersection we need to solve both equations

$$\begin{cases} x+y-z=2 \\ 2x-y+3z=1. \end{cases}$$

Taking $z=0$ we find that

$$(x,y,z) = (1,1,0)$$

is a solution. Hence the points $(1,1,0)$ and $(-1,2,1)$ are on that plane. Thus the vector connecting these two points, i.e.

$\langle -2,1,1\rangle$ is parallel to the plane.

Now we know two vectors that are parallel to the plane:

$\langle 2,-5,-3\rangle$ and $\langle -2,1,1\rangle$.

Hence their cross product is normal

$\vec{n} = \langle 2,-5,-3 \rangle \times \langle -2,1,1 \rangle = \langle -2,4,-8 \rangle.$

Hence we get the equation

$$\boxed{-2(x+1) + 4(y-2) - 8(z-1) = 0.}$$

Cylinders and quadratic surfaces

In general an equation of the form

$$F(x,y) = 0 \qquad (*)$$

describes a curve in $\mathbb{R}^2$. We are looking here for all points (x,y) that satisfy the equation $(*)$. For example

$$x^2+y^2=4$$

or equivalently

$$\underbrace{x^2+y^2-4}_{F(x,y)} = 0$$

describes a circle and

$$y = x^2$$

or

$$\underbrace{y-x^2}_{F(x,y)} = 0$$

describes a parabola.

Equation

$$x^2+y^2 = 4x+2y-1$$

seems more complicated, but we may complete the squares

$$x^2-4x+4+y^2-2y+1 = 4$$

$$(x-2)^2+(y-1)^2 = 4$$

and we see that this is still a circle (center $(2,1)$, radius 2).

In the three dimensional space an equation of the form

$$F(x,y,z) = 0$$

describes a surface. For example

$$x^2+y^2+z^2 = 4$$

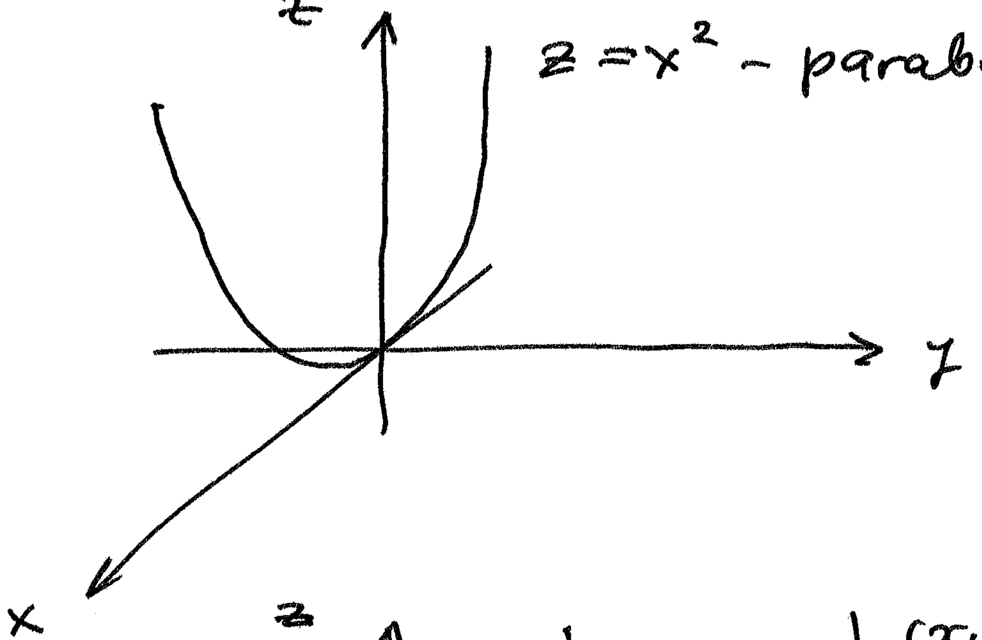
or

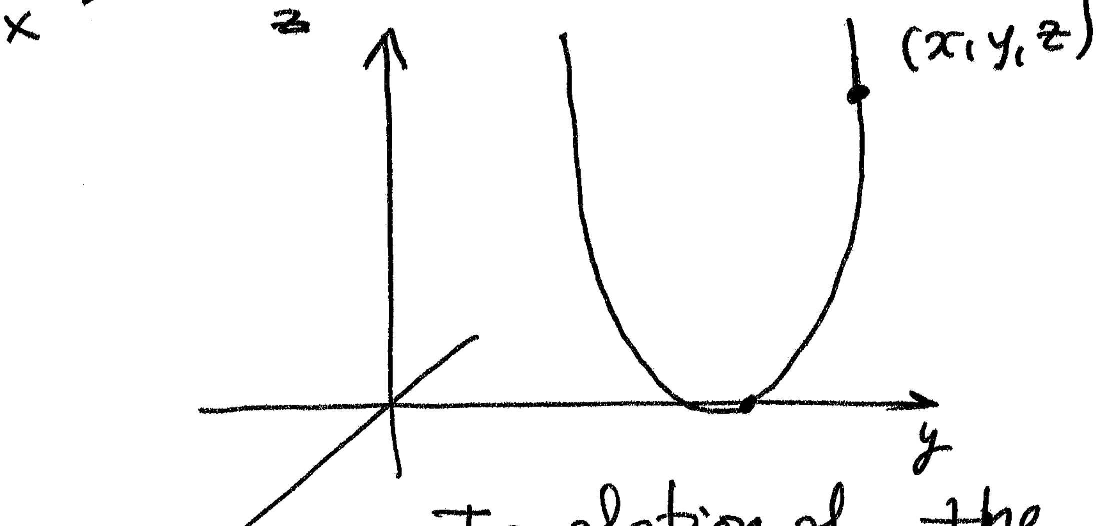
$$\underbrace{x^2+y^2+z^2-4}_{F(x,y,z)} = 0$$

describes the sphere of radius 2 centered at the origin. Now we will see more examples.

Example Sketch the surface $z=x^2$. (56)

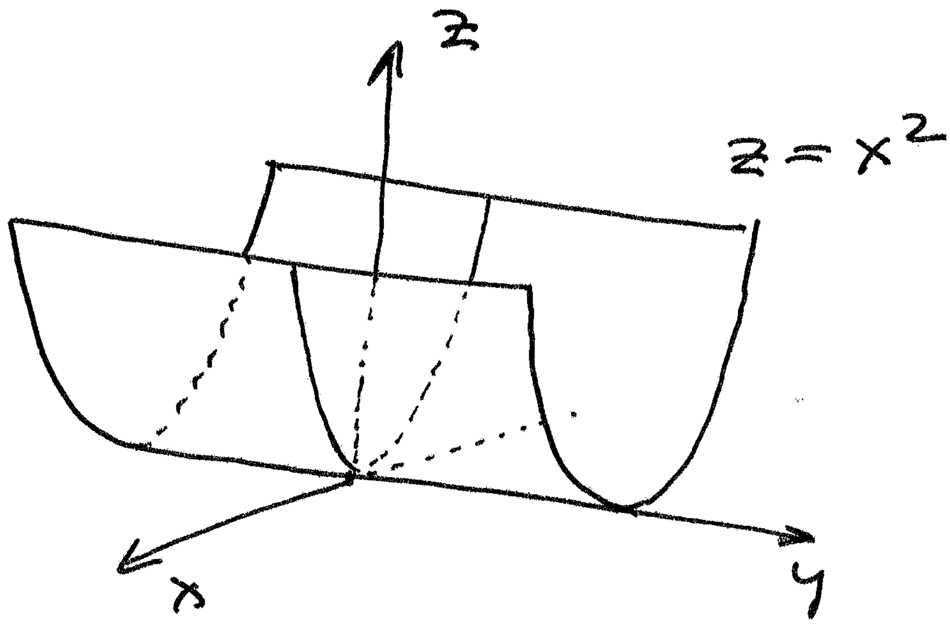
Solution It looks like the equation of a parabola in the xz plane. There is no variable y, so how can it be a surface? It is a surface - since y is not present, if x and z satisfy $z=x^2$, then we can choose any value for y.





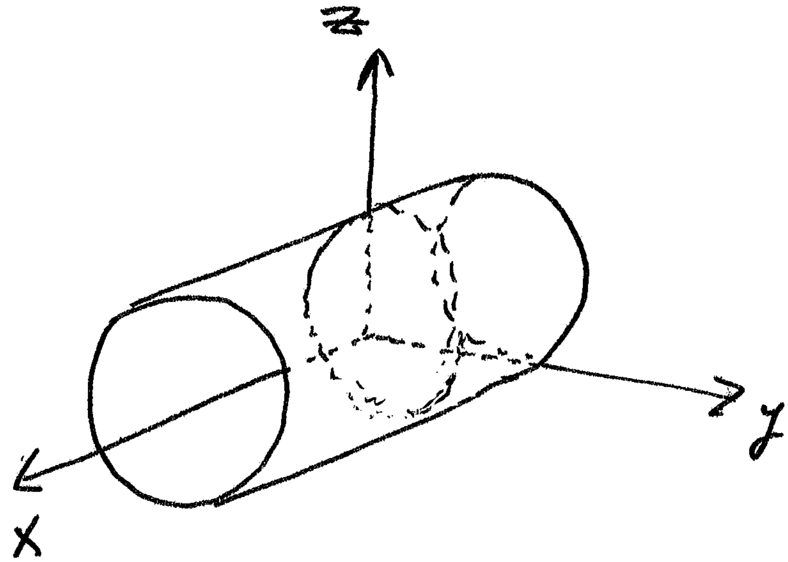
Translation of the parabola along the y-axis

If (x, y, z) is a point on the translated parabola, then the equation $z = x^2$ is still satisfied. Thus the surface consists of all such translated parabolas



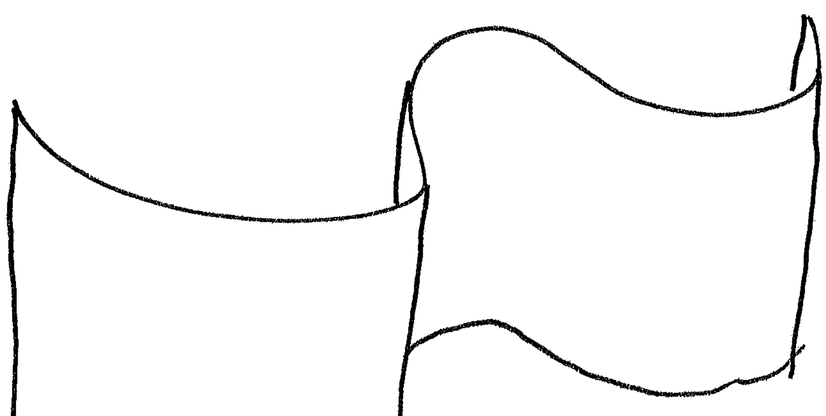
<u>Example</u> Sketch the surface $y^2+z^2=1$.

<u>Solution</u> In the yz plane the equation describes a circle. Since x is not present in the equation we can translate the circle along the x-axis to any position.
All such circles form a cylinder



Both surfaces discussed above are called cylinders. The surface in the first example is called a parabolic cylinder.

More generally a cylinder is a surface that consists of lines parallel to a given direction that pass through a given planar curve



This is also a cylinder.

The above two examples are special cases of so called quadratic surfaces which can be described by a quadratic function in three variables

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+$$
$$Gx+Hy+Iz+J=0.$$

From such a formula one cannot guess what is the shape of the surface, so one needs to simplify it for example by completing the squares

Example Describe the surface
$$2x^2+2y^2+2z^2-4x-6=0.$$

Solution Divide by 2
$$x^2+y^2+z^2-2x-3=0$$
$$x^2-2x+1+y^2+z^2=4$$
$$(x-1)^2+y^2+z^2=4$$

Sphere of radius 2 centered at $(1,0,0)$.

Example Sketch the surface

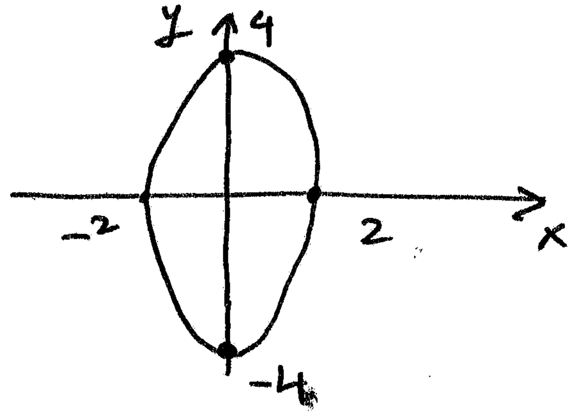
$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

Solution The idea is to fix one variable and see what is the shape of the resulting curve. Such curves are called traces.

If $z = 0$, then

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

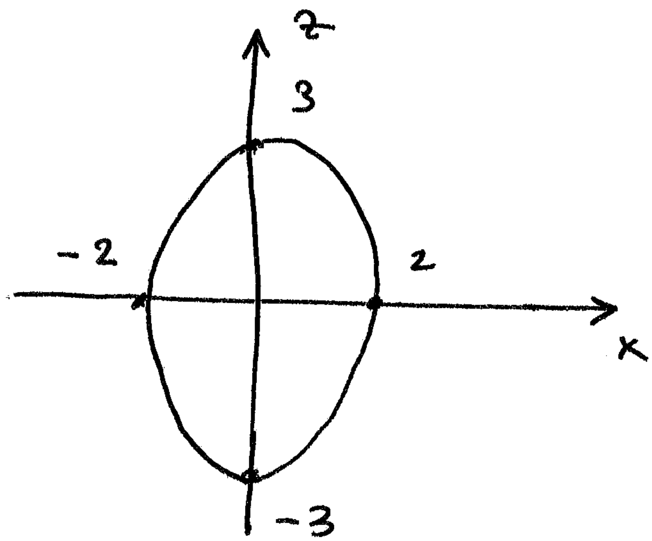
is an ellipse in the xy plane



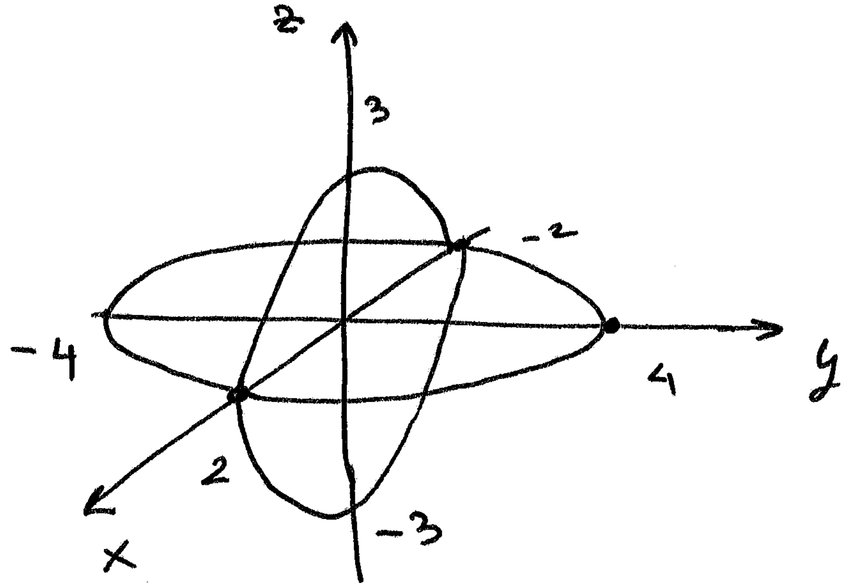
If $y = 0$, then

$$\frac{x^2}{4} + \frac{z^2}{9} = 1$$

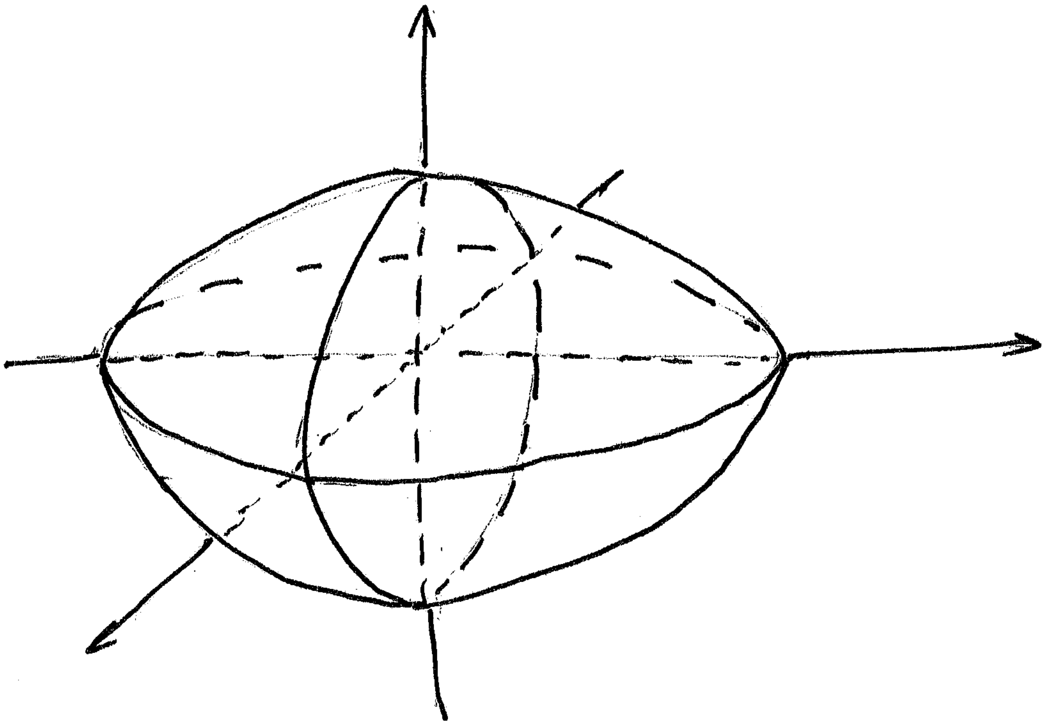
is an ellipse in the xz plane



Thus the following two ellipses are on the surface



and the surface is an ellipsoid



Example Sketch the surface

$$z = 4x^2 + y^2$$

Solution If $x = 0$ we obtain a parabola in the xy plane

$$z = y^2$$

If $y = 0$ we obtain parabola

$$z = 4x^2$$

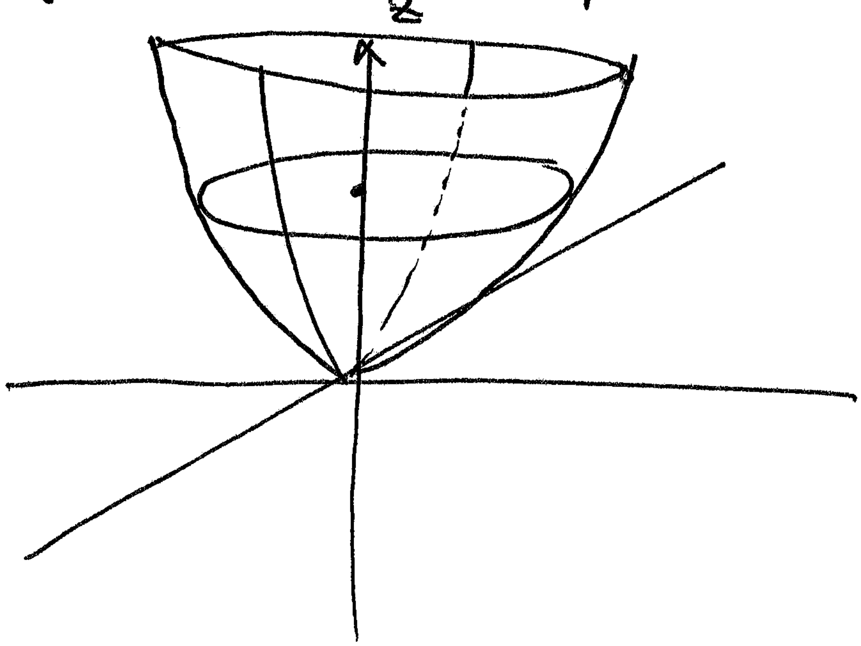
in the xz plane.

Note that z cannot be negative because $4x^2 + y^2 \geq 0$.

If $z = 1$ we obtain an ellipse

$$1 = 4x^2 + y^2$$

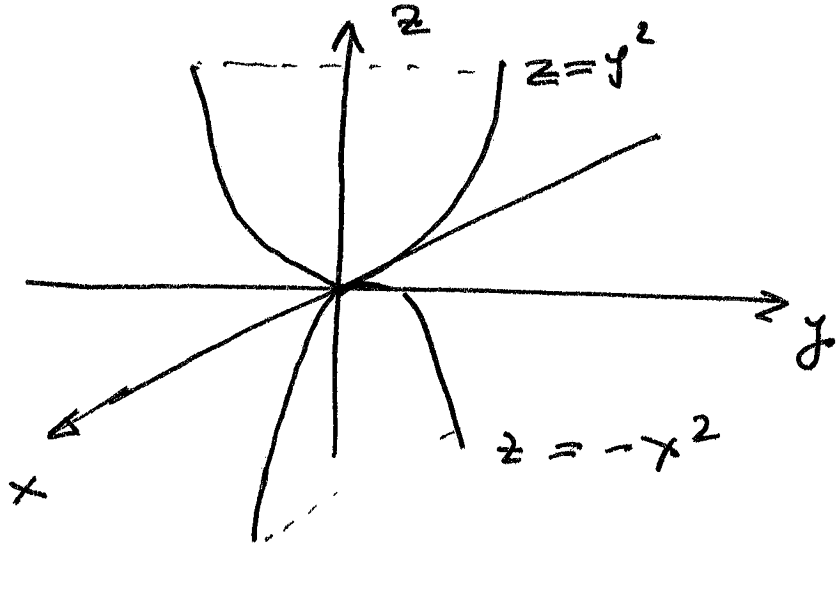
in the plane $z=1$. Thus the surface looks as follows



and it is called elliptic paraboloid

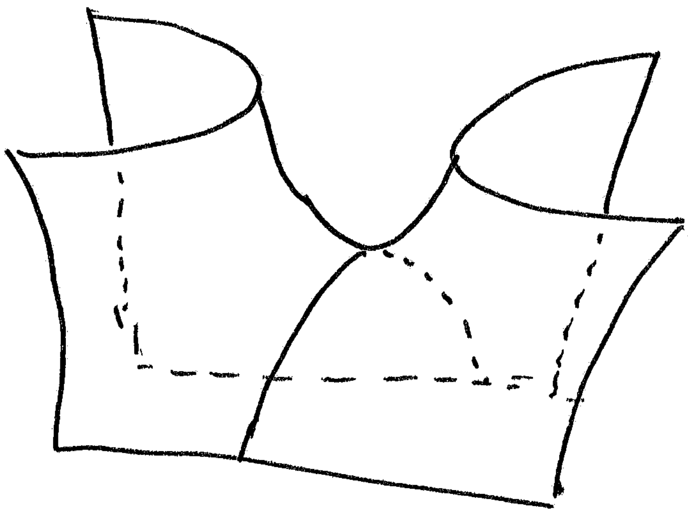
Example Sketch the surface
$z = y^2 - x^2$.

Solution $x=0$ gives a parabola $z=y^2$;
$y=0$ gives a parabola $z=-x^2$



That is quite strange. It is difficult to imagine a surface with two such parabolas on it.

If $z=1$ we get a hyperbola $1=y^2-x^2$ and the surface is



which is called hyperbolic paraboloid

Here is a classification of surfaces (pictures are in the book)

<u>Ellipsoid</u>

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

<u>Elliptic paraboloid</u>

$$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Hyperbolic paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Cone

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Hyperboloid of two sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All the equations shown above describe surfaces "centered" at the origin.
For example

$$\frac{(x-1)^2}{3^2} + \frac{(y-2)^2}{2^2} + \frac{(z-5)^2}{4^2} = 1$$

is still an ellipsoid, but it is centered at $(1, 2, 5)$.

Problems

Problem 29 Classify the surface
$$4x^2+y^2+4z^2-4y-24z+36=0$$

Solution We need to complete the squares

$$4x^2+(y^2-4y+4)+4(z^2-6z+9)=4$$
$$4x^2+(y-2)^2+4(z-3)^2=4$$
$$x^2+\frac{(y-2)^2}{4}+(z-3)^2=1$$

Ellipsoid centered at $(0,2,3)$.

Problem 30 Find an equation of the surface consisting of all points that are equidistant from the point $(-1,0,0)$ and the plane $x=1$. Identify the surface

Solution Distance of (x,y,z) to the plane $x=1$ equals $|x-1|$ (why?)

Distance of (x,y,z) to $(-1,0,0)$ equal
$$\sqrt{(x+1)^2+y^2+z^2}$$

Thus

$$|x-1| = \sqrt{(x+1)^2+y^2+z^2}$$

To simplify the situation we square the equation

$$(x-1)^2 = (x+1)^2 + y^2 + z^2$$

$$x^2 - 2x+1 = x^2+2x+1+y^2+z^2$$

$$-4x = y^2+z^2.$$

The surface is a paraboloid (traces parallel to the yz plane are circles)

<u>Problem 31</u> Show that the curve of intersection of the surfaces
$x^2+2y^2-z^2+3x=1$ and
$2x^2+4y^2-2z^2-5y=0$
lies in a plane.

<u>Solution</u> The curve of intersection must solve both equations at the same time

$$\begin{cases} x^2+2y^2-z^2+3z=1 \\ 2x^2+4y^2-2z^2-5y=0 \end{cases}$$

Multiply the first equation by 2 and subtract the second equation from it

$$
\begin{array}{rl}
 & 2x^2+4y^2-2z^2+6z=2 \\
- & 2x^2+4y^2-2z^2-5y=0 \\
\hline
 & 6z+5y=2
\end{array}
$$

Thus the points of the curve must satisfy the equation $6z+5y=2$ and hence they must lie in the plane $6z+5y=2$.

Problem 32 Find an equation of the surface consisting of all points such that the distance to the x-axis is twice the distance to the yz-plane. Identify the surface

Solution Distance of (x,y,z) to the x-axis is $\sqrt{y^2+z^2}$. Distance of (x,y,z) to the yz-plane is $|x|$. Thus we have

$$\sqrt{y^2+z^2} = 2|x|$$

$$y^2+z^2=4x^2$$

$$x^2=\frac{y^2}{4}+\frac{z^2}{4}.$$

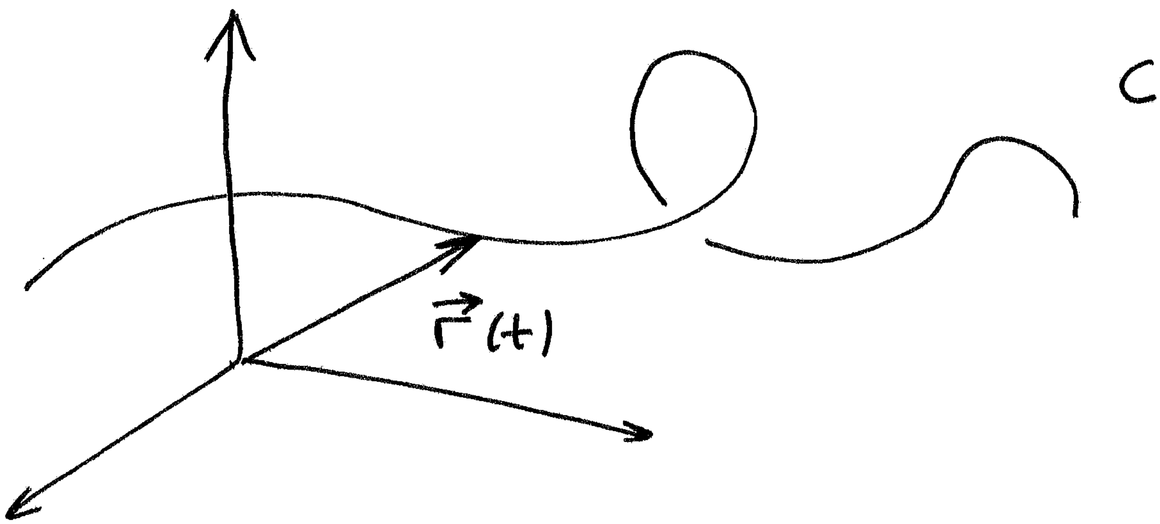
The surface is a cone.

Vector functions and space curves

A vector function is

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, \quad t \in I \text{ (interval)}$$

If t changes, position of the vector changes and it slides along a curve



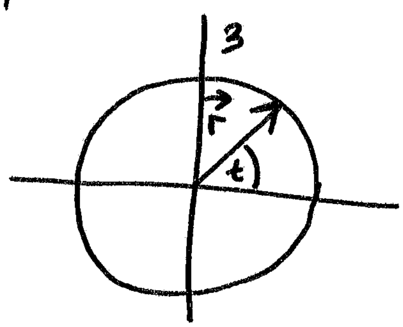
We often interpret t as time and in such a case the vector function describes a motion. The curve C along which $\vec{r}$ slides consists of points (x, y, z) such that

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I.$$

These are called parametric equations of the curve C.

__Example__ The vector function
$\vec{r}(t) = \langle 1+2t, 3-t, 2t \rangle$
describes the line
$$x = 1+2t,\ y = 3-t,\ z = 2t.$$

__Example__ The curve
$$\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle,\ t \in [0, 2\pi]$$
describes the circle centered at $(0,0)$ of radius 3. As parameter t changes from 0 to 2π, $\vec{r}$ goes one time around the circle in the counterclockwise direction



$$\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle, \quad t \in [0, 4\pi]$$
also describes the circle, but this time $\vec{r}(t)$ goes twice around the circle.

Example Find a vector function that connects the points $(2,3,4)$ and $(5,6,7)$ along the segment.

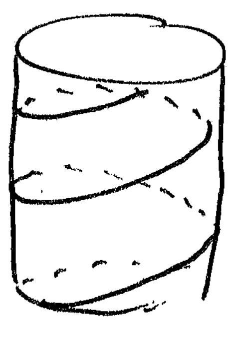
Solution $\vec{r}(t) = (1-t)\langle 2,3,4\rangle + t\langle 5,6,7\rangle$ where $0 \le t \le 1$. At $t=0$ it starts at $(2,3,4)$ and at $t=1$ it stops at $(5,6,7)$.

Example Describe the curve given by $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$

Solution Observe that

$$x = \cos t, \quad y = \sin t$$

describes the unit circle $x^2+y^2=1$. Hence the curve is on the surface of the cylinder $x^2+y^2=1$ and $z=t$, the height above the xy plane increases at the constant rate



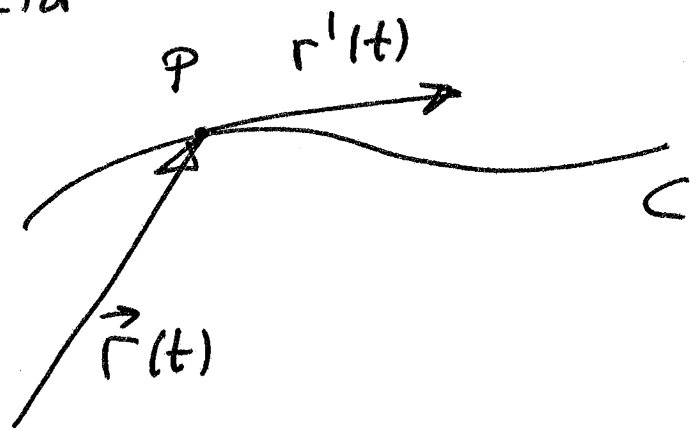
Such a curve is called a helix.

Derivative The derivative of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is defined as follows

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

The vector $\vec{r}\,'(t)$ is tangent to the curve C described by $\vec{r}(t)$ and it can be interpreted as a velocity vector



The tangent line to C at P is defined as the line through P, parallel to $\vec{r}\,'(t)$. This however requires $\vec{r}\,'(t) \neq \vec{0}$. If $\vec{r}\,'(t) \neq 0$

We also define the unit tangent vector (73)

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}.$$

Example Find parametric equations of the tangent line to the curve

$$\vec{r}(t) = \langle e^t, \cos t, t-\pi \rangle$$

at the point $(e^\pi, -1, 0)$.

Solution For $t = \pi$, $\vec{r}(\pi) = \langle e^\pi, -1, 0 \rangle$ so the given point is on the curve and it corresponds to $t = \pi$. We have

$$\vec{r}\,'(t) = \langle e^t, -\sin t, 1 \rangle$$

$$\vec{r}\,'(\pi) = \langle e^\pi, 0, 1 \rangle.$$

Now we need to find parametric equations of the line through $(e^\pi, -1, 0)$ that is parallel to $\langle e^\pi, 0, 1 \rangle$.

$$x = e^\pi + t e^\pi, \quad y = -1, \quad z = t.$$

Theorem

$$(\vec{u}(t)+\vec{v}(t))' = \vec{u}'(t) + \vec{v}'(t)$$

$$(c\,\vec{u}(t))' = c\,\vec{u}'(t)$$

$$(f(t)\,\vec{u}(t))' = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$$

$$(\vec{u}(t)\cdot\vec{v}(t))' = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

$$(\vec{u}(t)\times\vec{v}(t))' = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$$

$$(\vec{u}(f(t)))' = f'(t)\,\vec{u}'(f(t))$$

For a vector function we also consider the second derivative $\vec{r}''(t)$. While $\vec{r}'(t)$ can be interpreted as velocity, $\vec{r}''(t)$ can be interpreted as acceleration.

We can also integrate vector functions.

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

then

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle,$$

i.e. we integrate each component.

Length

The length of the curve

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $t \in [a,b]$

is defined by

$$L = \int_a^b |\vec{r}\,'(t)|\,dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt$$

If we interpret $\vec{r}\,'(t)$ as velocity,

$|\vec{r}\,'(t)|$ should be interpreted as speed, and then the length of the curve is the total distance traveled.

Example Find the length of the curve

$$\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t \in [0, 2\pi]$$

Solution $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle,$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$$

$$L = \int_0^{2\pi} 1 \, dt = 2\pi$$

Thus our answer is consistent with the fact that the length of the unit circle is 2π.

Example Find the length of the curve

$$\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t \in [0, 4\pi]$$

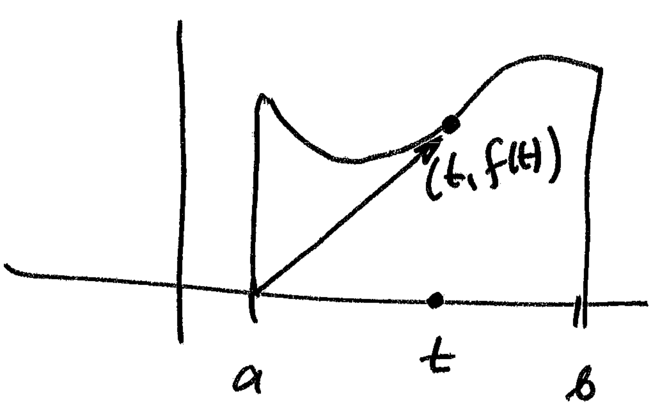
Solution As above we have $|\vec{r}\,'(t)| = 1$

and hence

$$L = \int_0^{4\pi} 1 \, dt = 4\pi$$

which is twice the length of the circle. The reason is that now the vector function describes the motion twice along the circle so the length, i.e. the distance traveled is twice the length of the circle.

<u>Example</u> Graph of a function $y = f(x)$, $a \le x \le b$ can be regarded as a planar curve



If we choose $a \le t \le b$, then the point on the graph has coordinates $(t, f(t))$. Thus

$$\vec{r}(t) = \langle t, f(t) \rangle, \quad a \le t \le b$$

is a vector function that describes the graph of L. Hence the length of the graph equals the length of the curve, i.e. it equals

$$L = \int_a^b |\vec{r}\,'(t)|\,dt = \int_a^b \sqrt{1 + (f'(t))^2}\,dt$$

which is a well known formula

Problems

Exercise 33 Show that the curve

$\vec{r}(t) = \langle \cos t \sin t, \cos^2 t, \sin t \rangle$

lies on the unit sphere.

Solution We need to show that the components

$x = \cos t \sin t, \quad y = \cos^2 t, \quad z = \sin t$
satisfy the equation

$$x^2 + y^2 + z^2 = 1.$$

We have

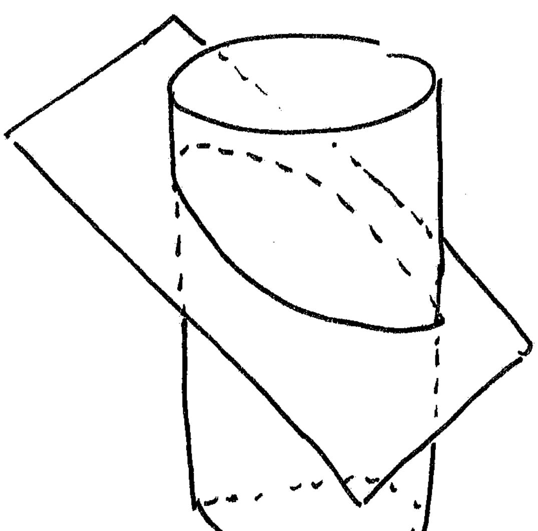
$$x^2 + y^2 + z^2 = \cos^2 t \sin^2 t + \cos^4 t + \sin^2 t$$

$$= \cos^2 t \,(\sin^2 t + \cos^2 t) + \sin^2 t =$$

$$= \cos^2 t + \sin^2 t = 1.$$

<u>Exercise 34</u> Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

<u>Solution</u>



The curve of intersection is an ellipse and we need to find a vector function that describes that curve

The projection of the curve on the xy plane is the circle $x^2+y^2=1$, because the curve is on the cylinder with the same equation. Hence we can parametrize the x and y components as

$$x=\cos t,\quad y=\sin t,\quad 0\le t\le 2\pi,$$

because this is a paremetrization of the circle $x^2+y^2=1$. Now we find z from the equation of the plane

$$y+z=2$$

$$z=2-y=2-\sin t.$$

Hence

$$\vec{r}\,(t)=\langle \cos t,\ \sin t,\ 2-t\rangle,\quad t\in[0,2\pi]$$

is the parametrization of the curve of intersection.

<u>Exercise 35</u> Show that the tangent line to the curve

$$\vec{r}(t) = \langle e^t, \cos t, t-\pi \rangle$$

at $(e^\pi, -1, 0)$ passes through the point $(0,-1,-1)$.

<u>Solution</u> $\vec{r}(\pi) = \langle e^\pi, -1, 0 \rangle$.

Hence the point $(e^\pi, -1, 0)$ is on the curve. The tangent vector is

$$\vec{r}\,'(t) = \langle e^t, -\sin t, 1 \rangle$$

$$\vec{r}\,'(\pi) = \langle e^\pi, 0, 1 \rangle.$$

Thus the equation of the tangent line is

$$\langle x, y, z \rangle = \langle e^\pi, -1, 0 \rangle + t \langle e^\pi, 0, 1 \rangle$$

$$x = e^\pi + te^\pi, \quad y = -1, \quad z = t$$

In order to show that this tangent line passes through $(0,-1,0)$ we need to show that for some value of

t we have

$$e^{\pi} + te^{\pi} = 0, \quad -1 = -1, \quad -1 = t.$$

Clearly, if $t=-1$ these equations are satisfied.

Exercise 36 Let $\vec{r}(t)$ be a curve of constant speed. Prove that acceleration is orthogonal to velocity.

Solution Constant speed means that

$$|\vec{r}\,'(t)| = c \quad \text{(constant)}$$

Then

$$\vec{r}\,'(t) \cdot \vec{r}\,'(t) = |\vec{r}\,'(t)|^2 = c^2$$

Recall that

$$(\vec{u}(t) \cdot \vec{v}(t))' = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$$

Hence

$$0 = (c^2)' = (\vec{r}\,'(t) \cdot \vec{r}\,'(t))' =$$

$$\vec{r}\,''(t) \cdot \vec{r}\,'(t) + \vec{r}\,'(t) \cdot \vec{r}\,''(t) =$$

$$2\vec{r}\,'(t) \cdot \vec{r}\,''(t)$$

$$\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0.$$

Exercise 37 Show that the curve

$$\vec{r}(t) = t\,\vec{i} + \frac{1+t}{t}\,\vec{j} + \frac{1-t^2}{t}\,\vec{k}, \quad t>0$$

lies in the plane $x-y+z+1=0$.

Solution It suffices to show that components of the curve

$$x=t, \quad y=\frac{1+t}{t}, \quad z=\frac{1-t^2}{t}$$

satisfy the equation of the plane. We have

$$x-y+z+1 = t - \frac{1+t}{t} + \frac{1-t^2}{t} + 1 =$$

$$t - \frac{1}{t} - \frac{t}{t} + \frac{1}{t} - \frac{t^2}{t} + 1 =$$

$$t - \frac{1}{t} - 1 + \frac{1}{t} - t + 1 = 0.$$

Exercise 38 Find the domain of

$$\vec{r}(t) = \langle \ln|t-1|,\ e^t,\ \sqrt{t}\,\rangle.$$

Solution We are asked to find all numbers t for which $\vec{r}(t)$ is well defined.

$\ln|t-1|$ is well defined if $|t-1|>0$,
i.e. if $t \neq 1$.
e^t is well defined for all values of t
$\sqrt{t}$ is well defined for $t \geq 0$.

Putting all conditions together we obtain that the domain of $\vec{r}(t)$ is

$$[0,1) \cup (1, \infty),$$

i.e. it consists of all t such that

$$0 \leq t < 1 \quad \text{or} \quad t > 1.$$

Exercise 39 Find $\int_0^1 \vec{r}(t)\,dt$, where
$\vec{r}(t) = t^2\vec{i} + e^t\vec{j} - (2\cos\pi t)\vec{k}$.

Solution.

$$\int_0^1 \vec{r}(t)\,dt = \left(\int_0^1 t^2\,dt\right)\vec{i} + \left(\int_0^1 e^t\,dt\right)\vec{j} - \left(\int_0^1 2\cos\pi t\,dt\right)\vec{k}$$

$$= \frac{t^3}{3}\Big|_0^1 \vec{i} + e^t\Big|_0^1 \vec{j} - \frac{2}{\pi}\sin\pi t\Big|_0^1 \vec{k} =$$

$$\frac{1}{3}\vec{i} + (e^t-1)\vec{j}.$$

Exercise 40 Find $\vec{r}(t)$ given that $\vec{r}\,'(t) = \langle 3, 2t\rangle$ and $\vec{r}(1) = \langle 2, 5\rangle$.

Solution Integrating $\vec{r}\,'(t)$ we obtain

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \left\langle \int 3\,dt, \int 2t\,dt \right\rangle$$

$$= \left\langle 3t + c_1, t^2 + c_2 \right\rangle$$

$$\langle 2, 5\rangle = \vec{r}(1) = \langle 3 + c_1, 1 + c_2\rangle$$

$$2 = 3 + c_1, \qquad c_1 = -1$$

$$5 = 1 + c_2, \qquad c_2 = 4$$

$$\vec{r}(t) = \langle 3t - 1, t^2 + 4\rangle$$

Exercise 41 Let

$$\vec{r}_1(t) = (\arctan t)\vec{i} + (\sin t)\vec{j} + t^2\vec{k}$$

$$\vec{r}_2(t) = (t^2 - t)\vec{i} + (2t - 2)\vec{j} + (\ln t)\vec{k}.$$

Both curves intersect at the origin. Find the angele at which the curves intersect at the origin.

Solution If two curves intersect at a point, then by definition the angle between curves is the angle between tangent lines.

The curve $\vec{r}_1(t)$ passes through the origin at $t=0$ with the tangent vector

$$\vec{r}_1{}'(0) = \left\langle \frac{1}{1+t^2},\ \cos t,\ 2t \right\rangle\Big|_{t=0} = \langle 1,1,0\rangle$$

The curve $\vec{r}_2(t)$ passes through the origin at $t=1$ with the tangent vector

$$\vec{r}_2{}'(1) = \left\langle 2t-1, 2, \frac{1}{t} \right\rangle\Big|_{t=1} = \langle 1,2,1\rangle.$$

The angle between the vectors $\vec{r}_1{}'(0)$ and $\vec{r}_2{}'(1)$ is given by

$$\cos\alpha = \frac{\vec{r}_1(0)\cdot\vec{r}_2(1)}{|\vec{r}_1(0)|\,|\vec{r}_2(1)|} = \frac{\langle 1,1,0\rangle\cdot\langle 1,2,1\rangle}{\sqrt{2}\ \sqrt{6}}$$

$$= \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\alpha = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

Exercise 42 An object is moving with acceleration $\vec{a}(t) = -(3\cos t)\vec{i} - (3\sin t)\vec{j} + 2\vec{k}$. Its initial position at $t=0$ was $(3,0,0)$ and the velocity at $t=0$ was $\vec{v}(0) = 3j$. Find the object's position as a function of time t.

Solution

$$\vec{a} = \vec{r}\,'' = -(3\cos t)\vec{i} - (3\sin t)\vec{j} + 2\vec{k}$$

After integration we obtain

$$\vec{r}\,'(t) = -(3\sin t)\vec{i} + (3\cos t)\vec{j} + 2t\vec{k} + \vec{C}_1$$

where $\vec{C}_1 = \langle a, b, c\rangle$ is a constant vector.

$$3\vec{j} = \vec{v}(0) = \vec{r}\,'(0) = \langle 0, 3, 0\rangle + \langle a, b, c\rangle$$

from which we conclude that

$$a = 0, \quad b = 0, \quad c = 0.$$

$$\vec{r}\,'(t) = -(3\sin t)\vec{i} + (3\cos t)\vec{j} + 2t\vec{k}$$

Integrating again yields

$$\vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{k} + \vec{C}_2$$

where $\vec{C}_2 = \langle d, e, f\rangle$

is a constant vector.

$$\langle 3,0,0\rangle = \vec{r}(0) = \langle 3,0,0\rangle + \vec{C_2},$$

$$\vec{C_2} = \vec{0}$$

Thus the answer is

$$\vec{r}(t) = \langle 3\cos t, 3\sin t, t^2\rangle.$$

<u>Exercise</u> 43 Find the length of the helix

$$\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle, \quad 0 \le t \le 2\pi.$$

<u>Solution.</u> We have

$$\vec{r}\,'(t) = \langle -a\sin t, a\cos t, b\rangle$$

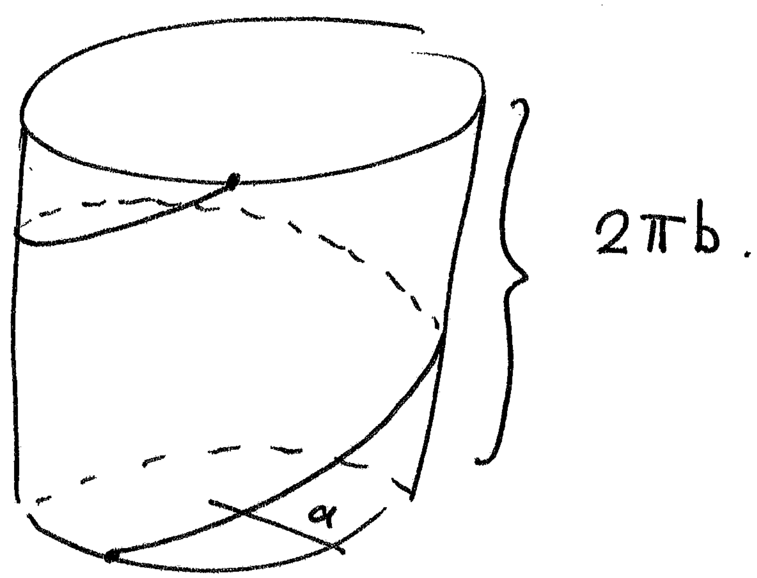
$$|\vec{r}\,'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}$$

$$= \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

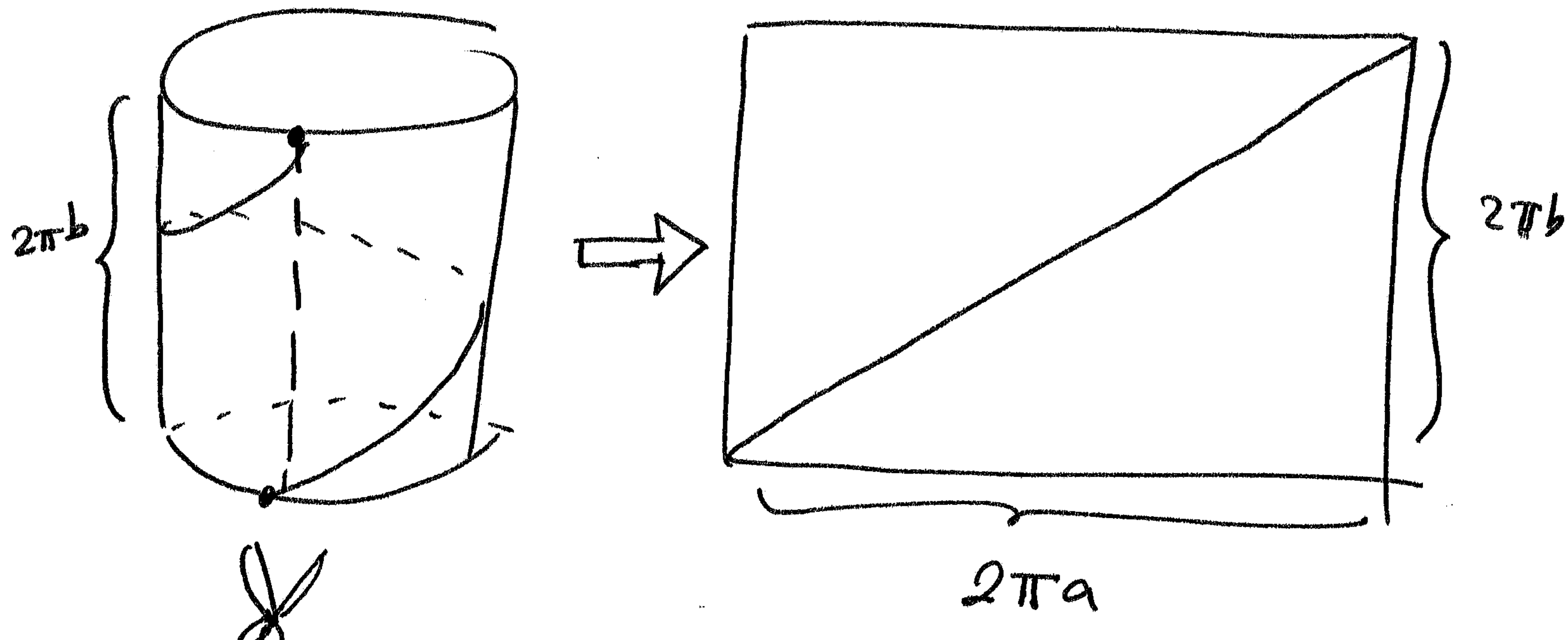
$$L = \int_0^{2\pi} \sqrt{a^2+b^2}\,dt = 2\pi\sqrt{a^2+b^2}.$$

There is another, purely geometric solution to this problem. The helix is on the cylinder $x^2+y^2=a^2$

and it makes one revolution climbing from the height $z=0$ (at $t=0$) to $z=2\pi b$ (at $t=2\pi$)



Imagine that the cylinder is made of a folded sheet of paper. If we unfold the paper, the helix will look as follows



It will become a straight line, because it climbs up at a constant speed.

Thus its length can be computed from the Phytagorean theorem

$$L = \sqrt{(2\pi a)^2 + (2\pi b)^2} = 2\pi\sqrt{a^2+b^2}.$$

Arc length and curvature

Consider two curves

$$\vec{r}_1(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \le t \le 2\pi$$

and

$$\vec{r}_2(u) = \langle \cos(u^2), \sin(u^2), u^2 \rangle, \quad 0 \le u \le \sqrt{2\pi}$$

Observe that for $u = \sqrt{t}$,

$$\vec{r}_1(t) = \vec{r}_2(u)$$

Hence both of the vector functions describe the same curve - the helix.

$\vec{r}_1{}'(t) = \langle -\sin t, \cos t, 1 \rangle$, $|\vec{r}_1{}'(t)| = \sqrt{2}$

$\vec{r}_2{}'(u) = \langle (-\sin u^2)2u, \cos(u^2)2u, 2u \rangle$, $|\vec{r}_2{}'(u)| = 2u\sqrt{2}$

Thus $\vec{r}_1(t)$ and $\vec{r}_2(u)$ are two different <u>parametrizations</u> of the same curve.

For example the lengths of $\vec{r}_1$ and $\vec{r}_2$ are the same. Indeed,

$$L_1 = \int_0^{2\pi} |\vec{r}_1{}'(t)|dt = \int_0^{2\pi} \sqrt{2}\, dt =$$

$$\left(\begin{array}{l} \text{substitution } t = u^2, \ dt = 2u\,du \\ 0 \le t \le 2\pi, \quad 0 \le u \le \sqrt{2\pi} \end{array} \right)$$

$$= \int_0^{\sqrt{2\pi}} \sqrt{2} \cdot 2u\,du = \int_0^{\sqrt{2\pi}} |\vec{r}_2{}'(u)|du = L_2.$$

In many situations it is important to find a parametrization of the curve with speed equal 1. For example since the speed of $\vec{r}_1$ is $\sqrt{2}$, if we take

$$s = \sqrt{2}\, t, \quad 0 \le s \le 2\sqrt{2}\,\pi$$

we have $t = s/\sqrt{2}$ and hence

$$\vec{r}_3(s) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle$$

$$\vec{r}_3{}'(s) = \left\langle \frac{-\sin s/\sqrt{2}}{\sqrt{2}}, \frac{\cos s/\sqrt{2}}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$|\vec{r}_3{}'(s)| = \frac{1}{\sqrt{2}} \sqrt{\left(-\sin \frac{s}{\sqrt{2}}\right)^2 + \left(\cos \frac{s}{\sqrt{2}}\right)^2 + 1^2}$$

$$= \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

We were lucky - in this case it was very easy to find a parametrization with unit speed.

There is, however a general method how to find a parametrization that gives unit speed.

Given a curve

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, \quad a \le t \le b$$

we define the <u>arc length function</u>

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}\,du$$

i.e. $s(t)$ is the length of the curve from a to t. From the

Fundamental Theorem of Calculus we have

$$(*) \quad \frac{ds}{dt} = \frac{d}{dt}\int_0^t |\vec{r}\,'(u)|\,du = |\vec{r}\,'(t)|.$$

Now we solve the equation $s = s(t)$ for t, i.e. we represent t as a function of s

$$t = t(s)$$

(i.e. $t = t(s)$ is the inverse function of $s = s(t)$) and

$$\vec{r}_1(s) = \vec{r}(t(s))$$

which is called <u>parametrization of the curve with respect to arc length</u> has unit speed. Indeed,

$$t = t(s(t))$$

(property of the inverse function) and the chain rule give.

$$1 = \frac{d}{dt}\, t(s(t)) = \frac{dt}{ds}(s(t))\,\frac{ds}{dt}(t)$$

$$\frac{dt}{ds}(s(t)) = \frac{1}{\frac{ds}{dt}(t)} = \frac{1}{|\vec{r}\,'(t)|}.$$

Hence

$$|\vec{r}_1\,'(s)| = \left|\frac{d}{ds}\vec{r}(t(s))\right| = \left|\frac{d\vec{r}}{dt}(t(s))\,\frac{dt}{ds}(s)\right|$$

$$= \left|\frac{d\vec{r}}{dt}\right|\left|\frac{dt}{ds}\right| = |\vec{r}\,'|\frac{1}{|\vec{r}\,'|} = 1.$$

That looks very complicated and very abstract, but we will see that in practice that is not so difficult.

Example Parametrize the curve

$$\vec{r}(t) = \langle e^t \cos t,\ e^t \sin t,\ e^t \rangle,\quad t \in \mathbb{R}$$

with respect to arc-length.

Solution First we find the arc length function

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du =$$

$$\int_0^t |\langle e^u\cos u - e^u\sin u,\ e^u\sin u + e^u\cos u,\ e^u\rangle|\,du =$$

$$\int_0^t e^u\sqrt{(\cos u - \sin u)^2 + (\sin u + \cos u)^2 + 1^2}\,du =$$

$$\int_0^t e^u\sqrt{\cos^2 - 2\cancel{\cos\sin} + \sin^2 + \sin^2 + 2\cancel{\sin\cos} + \cos^2 + 1}\,du =$$

$$\int_0^t e^u\sqrt{2(\sin^2 u + \cos^2 u) + 1}\,du =$$

$$\int_0^t e^u\sqrt{3}\,du = e^u\sqrt{3}\Big|_0^t = \sqrt{3}(e^t - 1)$$

$$s(t) = \sqrt{3}(e^t - 1)$$

Now we solve the equation

$$s = \sqrt{3}(e^t - 1)$$

for t.

$$e^t = \frac{s}{\sqrt{3}} + 1$$

$$t = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$$

while $-\infty < t < \infty$ can be any number, $s/\sqrt{3} + 1 > 0$, $s > -\sqrt{3}$, so the domain of the new parametrization is

$$s > -\sqrt{3}$$

and

$$\vec{r}_1(s) = \vec{r}(t(s)) = \langle e^{t(s)} \cos t(s), e^{t(s)} \sin t(s), e^{t(s)} \rangle$$

$$= \left\langle e^{\ln\left(\frac{s}{\sqrt{3}}+1\right)} \cos \ln\left(\frac{s}{\sqrt{3}}+1\right), e^{\ln\left(\frac{s}{\sqrt{3}}+1\right)} \sin \ln\left(\frac{s}{\sqrt{3}}+1\right), e^{\ln\left(\frac{s}{\sqrt{3}}+1\right)} \right\rangle$$

$$= \left\langle \left(\frac{s}{\sqrt{3}}+1\right) \cos \ln\left(\frac{s}{\sqrt{3}}+1\right), \left(\frac{s}{\sqrt{3}}+1\right) \sin \ln\left(\frac{s}{\sqrt{3}}+1\right), \frac{s}{\sqrt{3}}+1 \right\rangle$$

$$-\sqrt{3} < s < \infty$$

is the arc-length parametrization of the curve.

while it is not necessary, the reader should check that the speed of the parametrization $\vec{r}_1(s)$ equals 1.

Curvature

A parametrization $\vec{r}(t)$ of a curve is said to be smooth if $\vec{r}\,'(t) \neq \vec{0}$ for all t. In this situation we can define the unit tangent vector

$$T(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

for all t and hence we can define the tangent line at every point of the curve.

Example Consider the curve

$$\vec{r}(t) = \langle t^3, t^2 \rangle, \quad t \in \mathbb{R}.$$

Observe that

$$\vec{r}\,'(t) = \langle 3t^2, 2t \rangle$$

$$\vec{r}\,'(0) = \langle 0, 0 \rangle$$

so the parametrization $\vec{r}(t)$ of the curve is not smooth at $t = 0$, and as we will see, despite the fact that the functions t^3 and t^2 are smooth for all t, the resulting curve is not smooth at $t = 0$.

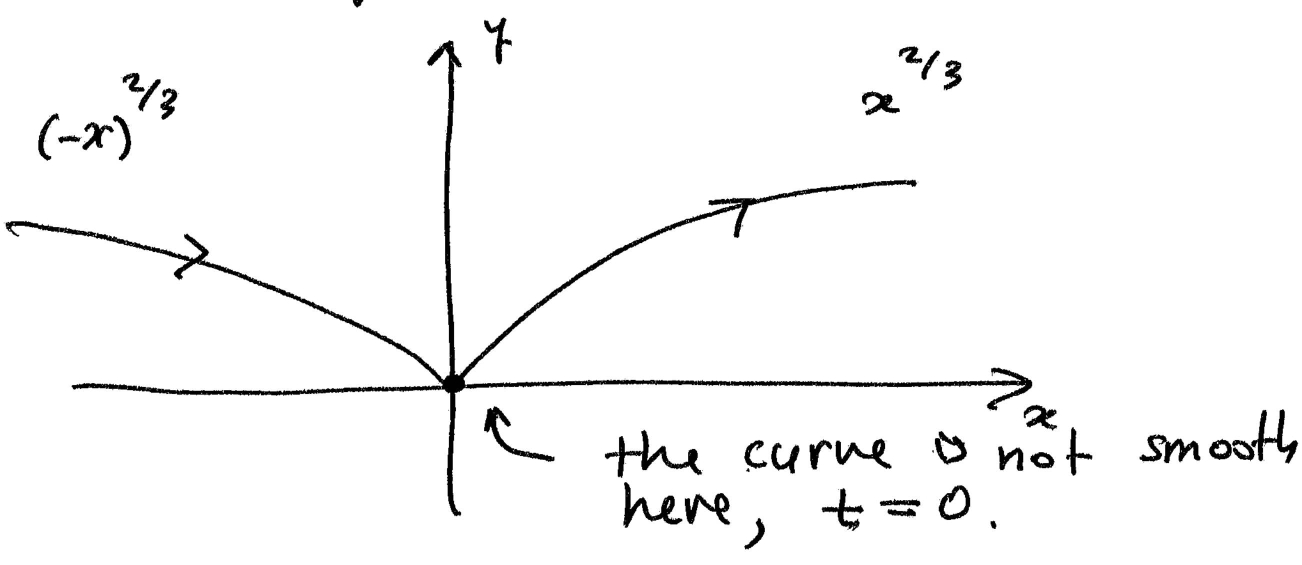
We have

$$x = t^3, \quad y = t^2$$

$$y^3 = t^6 = x^2$$

$$y^{3/2} = |x|$$

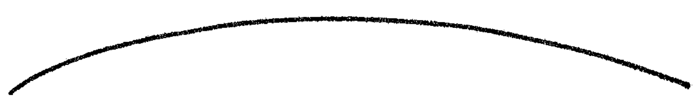
$$y = |x|^{2/3}$$



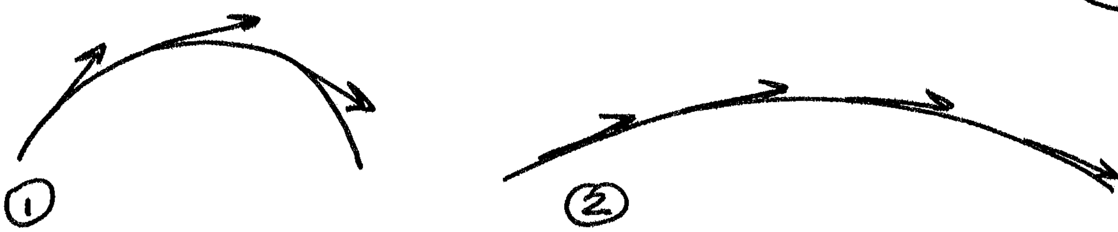
Such a situation cannot happen if the parametrization is smooth i.e. $\vec{r}\,(t) \neq 0$ for all t.

The curvature tells us how much the curve is curved. For example the curvature of

is larger than the curvature of



How can we compute the curvature. Consider the arc-length parametrization of the curve, i.e. the parametrization with unit speed. The length of the tangent vector $\vec{T}(s)$ equals 1.



As we move along the curve with speed 1, the tangent vector rotates and it rotates more quickly in the case of the more curved curve. In the situation presented on the picture the unit tangent vector rotates faster on the curve 1.

Thus it is natural to define the curvature as the rate at which the unit tangent vector rotates.

<u>Definition</u> The curvature of a smooth curve is defined by

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| .$$

Remember, that we consider here the arc-length parametrization. Observe that $\vec{T}(s) = \vec{r}\,'(s)$ (unit speed) so $\kappa = |\vec{T}\,'(s)| = |\vec{r}\,''(s)|$.

But we can also compute the curvature for any given parametrization. We have

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\,\frac{ds}{dt}$$

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{|d\vec{T}/dt|}{|ds/dt|}$$

We proved ((*) p. 93) that

$$\frac{ds}{dt} = |\vec{r}\,'(t)|.$$

Hence,

$$\boxed{\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}.}$$

<u>Example</u> Prove that the curvature of a circle of radius a equals $\frac{1}{a}$.

<u>Remark</u> The smaller radius a, the larger curvature $\frac{1}{a}$. That is consistent with our intuition. Small circles are more curved.

Proof We can parametrize the circle of radius a as

$$\vec{r}(t) = \langle a\cos t,\ a\sin t \rangle.$$

We have

$$\vec{r}\,'(t) = \langle -a\sin t,\ a\cos t \rangle$$

$$|\vec{r}\,'(t)| = a$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin t,\ \cos t \rangle$$

$$\vec{T}\,'(t) = \langle -\cos t,\ -\sin t \rangle$$

$$|\vec{T}\,'(t)| = 1$$

$$\kappa = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{a}.$$

There is yet another formula to compute the curvature

Theorem

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}.$$

Example Find the curvature of the helix

$$\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle, \quad a, b > 0.$$

Solution 1

$$\vec{r}'(t) = \langle -a\sin t, a\cos t, b \rangle$$

$$|\vec{r}'(t)| = \sqrt{a^2+b^2}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -a\sin t, a\cos t, b \rangle}{\sqrt{a^2+b^2}}$$

$$\vec{T}'(t) = \frac{\langle -a\cos t, -a\sin t, 0 \rangle}{\sqrt{a^2+b^2}}$$

$$|\vec{T}'(t)| = \frac{a}{\sqrt{a^2+b^2}}$$

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{a}{a^2+b^2}.$$

Solution 2 We will use formula

$$\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}.$$

$$\vec{r}\,' = \langle -a\sin t,\ a\cos t,\ b \rangle$$

$$|\vec{r}\,'| = \sqrt{a^2+b^2}$$

$$\vec{r}\,'' = \langle -a\cos t,\ -a\sin t,\ 0 \rangle$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} =$$

$$\langle ab\sin t,\ -ab\cos t,\ a^2\sin^2 t + a^2\cos^2 t \rangle =$$

$$\langle ab\sin t,\ -ab\cos t,\ a^2 \rangle$$

$$|\vec{r}\,' \times \vec{r}\,''| = \sqrt{(ab)^2 + a^4} = a\sqrt{b^2+a^2}$$

$$k = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{a\sqrt{a^2+b^2}}{(\sqrt{a^2+b^2})^3} = \frac{a}{a^2+b^2}.$$

In the examples discussed above the curvature was constant, but in general it may depend on t, i.e. on the position of the point on the curve.

Normal and binormal vectors

Vectors $\vec{T}(t)$ and $\vec{T}'(t)$ are orthogonal. Indeed,

$$|\vec{T}(t)|=1, \qquad \vec{T}(t)\cdot\vec{T}(t) = |\vec{T}(t)|^2 = 1.$$

Taking the derivative we obtain

$$0 = (\vec{T}(t)\cdot\vec{T}(t))' = \vec{T}'(t)\cdot\vec{T}(t) + \vec{T}(t)\cdot\vec{T}'(t)$$

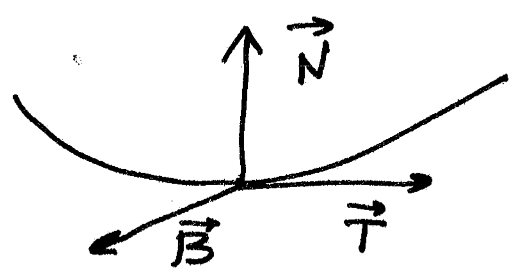
$$0 = 2\,\vec{T}'(t)\cdot\vec{T}(t)$$

$$\vec{T}'(t)\cdot\vec{T}(t) = 0$$

i.e. the vectors are orthogonal.

For a given smooth curve $\vec{r}(t)$ we define normal and binormal vectors by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}, \qquad \vec{B}(t) = \vec{T}(t)\times\vec{N}(t)$$



The vectors $\vec{T}, \vec{N}, \vec{B}$ are orthogonal to each other.

Problems

Exercise 44 Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the curvature of C at the point $(6, 18, 36)$

Solution First we need to find a parametrization of that curve. Points of the curve must satisfy both of the equations

$$\begin{cases} x^2 = 2y \\ 3z = xy \end{cases}$$

If we fix x, we find

$$y = \frac{x^2}{2}, \quad z = \frac{xy}{3} = \frac{x \, x^2/2}{3} = \frac{x^3}{6}.$$

Thus points in the intersection are of the form

$$\left(x, \frac{x^2}{2}, \frac{x^3}{6}\right)$$

Hence the intersection of the

surfaces is a curve with the parametrization

$$\vec{r}(t) = \left\langle t, \frac{t^2}{2}, \frac{t^3}{6} \right\rangle.$$

If $t=6$, then

$$\vec{r}(6) = \langle 6, 18, 36 \rangle,$$

so we need to find the curvature at $t=6$. We will use the formula

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}.$$

We have

$$\vec{r}\,'(t) = \left\langle 1, t, \frac{t^2}{2} \right\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1 + t^2 + \left(\frac{t^2}{2}\right)^2} = \sqrt{\left(1+\frac{t^2}{2}\right)^2} = 1+\frac{t^2}{2}.$$

$$\vec{r}\,''(t) = \langle 0, 1, t \rangle$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & t & t^2/2 \\ 0 & 1 & t \end{vmatrix} =$$

$$= \left\langle t^2 - \frac{t^2}{2}, -t, 1 \right\rangle = \left\langle \frac{t^2}{2}, -t, 1 \right\rangle$$

$$|\vec{r}\,' \times \vec{r}\,''| = \sqrt{\left(\frac{t^2}{2}\right)^2 + t^2 + 1} = \sqrt{\left(\frac{t^2}{2}+1\right)^2} = \frac{t^2}{2}+1$$

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} =$$

$$= \frac{\frac{t^2}{2}+1}{\left(\frac{t^2}{2}+1\right)^3} = \left(\frac{t^2}{2}+1\right)^{-2}.$$

$$\kappa(6) = 19^{-2}.$$

Exercise 45 Find the curvature of the parabola $y = x^2$ at $(2,4)$.

Solution. The parabola consists of points (x, x^2) and hence

$$\vec{r}(t) = \langle t, t^2 \rangle$$

is its parametrization. We have

$$\vec{r}\,'(t) = \langle 1, 2t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1+4t^2}$$

$$\vec{r}\,''(t) = \langle 0, 2 \rangle$$

In order to find the cross product we need to interpret $\vec{r}\,'$ and $\vec{r}\,''$

as vectors in space

$$\vec{r}\,'(t) = \langle 1, 2t, 0 \rangle$$
$$\vec{r}\,''(t) = \langle 0, 2, 0 \rangle.$$

Now

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$|\vec{r}\,' \times \vec{r}\,''| = 2$$

$$\kappa(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{2}{(1+4t^2)^{3/2}}$$

The point $(2,4)$ corresponds to $t=2$ and hence the curvature at that point is

$$\kappa(2) = \frac{2}{(1+4\cdot 4)^{3/2}} = \frac{2}{17^{3/2}}.$$

Exercise 46 Prove that the curvature of the graph of $y = f(x)$ at $(x, f(x))$ equals

$$\kappa(x) = \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}.$$

<u>Proof</u> The proof is rey similar to the computation in the previous problem, but now we need to work with a general function instead of x^2.

The graph of $y = f(x)$ has a parametrization

$$\vec{r}(t) = \langle t, f(t) \rangle.$$

We have

$$\vec{r}\,'(t) = \langle 1, f'(t) \rangle$$

$$|\vec{r}\,'| = \sqrt{1 + (f'(t))^2}$$

$$\vec{r}\,'' = \langle 0, f''(t) \rangle.$$

Vectors $\vec{r}\,'$ and $\vec{r}\,''$ can be regarded as vectors in space

$$\vec{r}\,' = \langle 1, f'(t), 0 \rangle$$

$$\vec{r}\,'' = \langle 0, f''(t), 0 \rangle$$

Hence

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f' & 0 \\ 0 & f'' & 0 \end{vmatrix} = \langle 0, 0, f''(t) \rangle$$

$$|\vec{r}\,' \times \vec{r}\,''| = |f''(t)|$$

$$\kappa(t) = \frac{|f''(t)|}{\left(\sqrt{1+f'(t)^2}\right)^3} = \frac{|f''(t)|}{\left(1+f'(t)^2\right)^{3/2}}.$$

Observe that $t = x$ gives the point $(x, f(x))$, so the corresponding curvature is

$$\kappa(x) = \frac{|f''(x)|}{\left(1+(f'(x))^2\right)^{3/2}}.$$

<u>Exercise 47</u> Find the curvature of $\vec{r}(t) = \langle 3t-7,\ 5t+2,\ -t+6 \rangle$

<u>Solution</u> $\vec{r}\,''(t) = 0$. Hence

$$\kappa(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{0}{|\vec{r}\,'|^3} = 0.$$

<u>Remark</u> No surprise. $\vec{r}$ is a straight line, so it is not curved and hence its curvature should be zero.

Exercise 48 Reparametrize the curve

$$\vec{r}(t) = \left(\frac{2}{t^2+1} - 1\right)\vec{i} + \frac{2t}{t^2+1}\vec{j}$$

with respect to arc-length measured from the point $(1,0)$ in the direction of increasing t. Express the reparametrization in its simplest form. What can you conclude about the curve?

Solution First we find the arc-length parameter

$$s(t) = \int_0^t |\vec{r}\,'(u)|\, du$$

$$\vec{r}\,'(t) = \left\langle \frac{-4t}{(t^2+1)^2}, \frac{2(t^2+1) - 2t \cdot 2t}{(t^2+1)^2} \right\rangle$$

$$\vec{r}\,'(u) = \frac{2}{(u^2+1)^2}\langle -2u, 1-u^2 \rangle$$

$$|\vec{r}\,'(u)| = \frac{2}{(u^2+1)^2}\sqrt{4u^2 + (1-u^2)^2} =$$

$$= \frac{2}{(u^2+1)^2}\sqrt{4u^2+1-2u^2+u^4} =$$

$$= \frac{2}{(u^2+1)^2}\sqrt{u^4+2u^2+1} = \frac{2}{(u^2+1)^2}\sqrt{(u^2+1)^2}$$

$$= \frac{2}{u^2+1}.$$

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \frac{2\,du}{u^2+1} =$$

$$= 2\arctan u \Big|_0^t = 2\arctan t.$$

Now we want to express t as a function of s (i.e. we find the inverse function)

$$s = 2\arctan t$$

$$t = \tan\left(\frac{s}{2}\right).$$

Hence the arc-length parametrization of the curve is

$$\vec{r}_1(s) = \vec{r}(t(s)) = \vec{r}\left(\tan\left(\tfrac{s}{2}\right)\right)$$

$$= \left\langle \frac{2}{(\tan^2(\frac{s}{2}))+1} - 1, \frac{2\tan(\frac{s}{2})}{(\tan^2(\frac{s}{2}))+1} \right\rangle.$$

This expression can be simplified.
Let $\alpha = s/2$.

$$\tan^2\alpha + 1 = \frac{\sin^2\alpha}{\cos^2\alpha} + 1 = \frac{\sin^2\alpha + \cos^2\alpha}{\cos^2\alpha}$$

$$= \frac{1}{\cos^2\alpha}$$

$$\frac{2}{(\tan^2\alpha)+1} - 1 = 2\cos^2\alpha - 1 = \cos 2\alpha = \cos s$$

$$\frac{2\tan\alpha}{\tan^2\alpha + 1} = 2\tan\alpha\cos^2\alpha. = 2\frac{\sin\alpha}{\cos\alpha}\cos^2\alpha$$

$$= 2\sin\alpha\cos\alpha = \sin 2\alpha = \sin s.$$

Thus

$$\vec{r}_1(s) = \langle \cos s, \sin s \rangle.$$

This is the unit circle! So simple!
$s = 0$ gives $\vec{r}_1(0) = \langle 1, 0 \rangle$.

Hence the arc-length parametrization from $(1, 0)$ in the direction of increasing t is

$$\vec{r}_1(s) = \langle \cos s, \sin s \rangle, \qquad s \geq 0.$$

<u>Remark</u> Since the answer is so simple, is it possible to find a simpler solution? Yes, first we check directly that the curve $\vec{r}(t)$ satisfies the equation of the unit circle

$$x^2 + y^2 = 1.$$

We have

$$\left(\frac{2}{t^2+1} - 1\right)^2 + \left(\frac{2t}{t^2+1}\right)^2 =$$

$$\frac{(2-(t^2+1))^2 + (2t)^2}{(t^2+1)^2} = \frac{(1-t^2)^2 + (2t)^2}{(t^2+1)^2} =$$

$$\frac{1 - 2t^2 + t^4 + 4t^2}{(t^2+1)^2} = \frac{t^4 + 2t^2 + 1}{(t^2+1)^2} = \frac{(t^2+1)^2}{(t^2+1)^2} = 1.$$

Thus $\vec{r}(t)$ describes the unit circle $x^2+y^2=1$, but the unit circle has the arc-length parametrization

$$\vec{r}_1(s) = \langle \cos s, \sin s \rangle .$$

This parametrization is arc-length, because the speed is 1

$$|\vec{r}_1{}'(s)| = 1.$$

<u>Exercise 49</u> For the curve $\vec{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$ find the vectors $\vec{T}, \vec{N}, \vec{B}$ at the point $(1,0,0)$.

<u>Solution</u> Observe that the curve is defined only for t such that $\cos t > 0$. The point $(1,0,0)$ corresponds to $t=0$. We have

$$\vec{r}\,'(t) = \left\langle -\sin t, \cos t, \frac{-\sin t}{\cos t} \right\rangle$$

$$= \langle -\sin t, \cos t, -\tan t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\tan t)^2}$$

$$= \sqrt{1+\tan^2 t} = \sqrt{1 + \frac{\sin^2 t}{\cos^2 t}} =$$

$$\sqrt{\frac{\cos^2 t + \sin^2 t}{\cos^2 t}} = \frac{1}{\cos t}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left\langle -\frac{\sin t}{\cos t}, 1, -\frac{\tan t}{\cos t} \right\rangle$$

$$= \left\langle -\tan t, 1, -\frac{\tan t}{\cos t} \right\rangle$$

$$\vec{T}\,'(t) = \left\langle -\frac{1}{\cos^2 t}, 0, -\frac{\frac{1}{\cos^2 t}\cos t - \tan t\,(-\sin t)}{\cos^2 t} \right\rangle$$

$$= \left\langle -\frac{1}{\cos^2 t}, 0, -\frac{\frac{1}{\cos t} + \frac{\sin^2 t}{\cos t}}{\cos^2 t} \right\rangle =$$

$$= \left\langle -\frac{1}{\cos^2 t}, 0, -\frac{1+\sin^2 t}{\cos^3 t} \right\rangle$$

$$\vec{T}'(0) = \langle -1, 0, -1 \rangle$$

$$\vec{N}(0) = \frac{\vec{T}'(0)}{|\vec{T}'(0)|} = \frac{\langle -1, 0, -1 \rangle}{\sqrt{2}}$$

$$\vec{T}(0) = \langle 0, 1, 0 \rangle$$

$$\vec{B}(0) = \vec{T}(0) \times \vec{N}(0) =$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix} = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

Thus the vectors $\vec{T}, \vec{N}, \vec{B}$ at the point $(1, 0, 0)$ are

$$\langle 0, 1, 0 \rangle, \left\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle, \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle.$$

Exercise 50 Find the normal and binormal vectors to the helix
$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$

Solution $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle$

$|\vec{r}\,'(t)| = \sqrt{2}$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle -\sin t, \cos t, 1\rangle}{\sqrt{2}}$$

$$\vec{T}\,'(t) = \frac{\langle -\cos t, -\sin t, 0\rangle}{\sqrt{2}}$$

$$|\vec{T}\,'(t)| = \frac{1}{\sqrt{2}}$$

$$\vec{N}(t) = \vec{T}\,'(t)/|\vec{T}\,'(t)| = \langle -\cos t, -\sin t, 0\rangle$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) =$$

$$\frac{1}{\sqrt{2}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t, & -\sin t & 0 \end{vmatrix} = \frac{\langle \sin t, -\cos t, 1\rangle}{\sqrt{2}}$$

Answer

$\vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle$, $\vec{B}(t) = \dfrac{\langle \sin t, -\cos t, 1\rangle}{\sqrt{2}}$

Exercise 51 Prove that
$$\frac{d\vec{T}}{ds} = \kappa\,\vec{N}$$

Proof. Recall that
$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = |\vec{T}'(s)|.$$
Hence
$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \frac{\vec{T}'(s)}{\kappa}$$
$$\vec{T}'(s) = \kappa\,\vec{N}(s)$$
$$\frac{d\vec{T}}{ds} = \kappa\,\vec{N}.$$

Velocity and acceleration

Recall that for a curve
$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$$
velocity, speed and acceleration are defined as follows
$$\vec{v}(t) = \vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$
$$|\vec{v}(t)| = |\vec{r}'(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$$

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \langle f''(t), g''(t), h''(t) \rangle$.

Here is a typical example

Example An object starts at an initial position $\vec{r}(0) = \langle 1, 0, 1 \rangle$. Its acceleration is $\vec{a}(t) = \langle t, 1, t^2 \rangle$ and $\vec{v}(1) = \langle \frac{3}{2}, 2, \frac{4}{3} \rangle$. Find $\vec{r}(2)$.

Solution $\vec{a}(t) = \vec{v}'(t) = \langle t, 1, t^2 \rangle$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle \frac{t^2}{2} + c_1, t + c_2, \frac{t^3}{3} + c_3 \rangle$$

$$\langle \frac{3}{2}, 2, \frac{4}{3} \rangle = \vec{v}(1) = \langle \frac{1}{2} + c_1, 1 + c_2, \frac{1}{3} + c_3 \rangle$$

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 1$$

$$\vec{v}(t) = \langle \frac{t^2}{2} + 1, t + 1, \frac{t^3}{3} + 1 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \frac{t^3}{6} + t + d_1, \frac{t^2}{2} + t + d_2, \frac{t^4}{12} + t + d_3 \rangle$$

$$\langle 1, 0, 1 \rangle = \vec{r}(0) = \langle d_1, d_2, d_3 \rangle$$

$$d_1 = 1, \quad d_2 = 0, \quad d_3 = 1$$

$$\vec{r}(t) = \langle \frac{t^3}{6} + t + 1, \frac{t^2}{2} + t, \frac{t^4}{12} + t + 1 \rangle$$

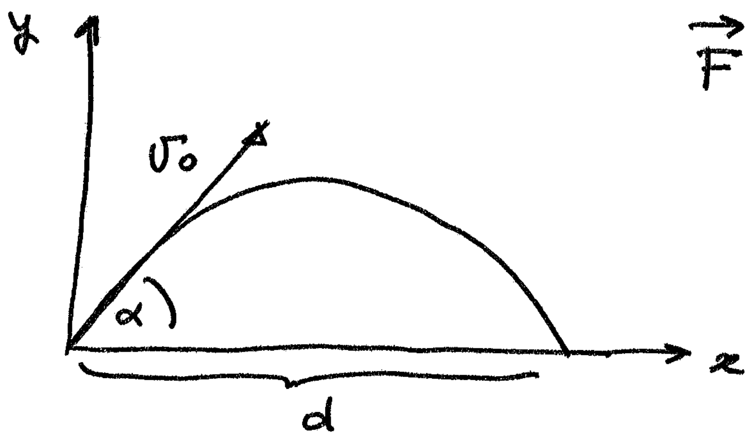
$$\vec{r}(2) = \langle \frac{8}{6} + 3, 4, \frac{16}{12} + 3 \rangle.$$

Many problems require some background
in physics. Recall that Newton's Second Law of Motion states that if a force $\vec{F}(t)$ acts on an object of mass m, then it moves with an acceleration such that

$$\vec{F}(t) = m\vec{a}(t).$$

<u>Example</u> A projectile is fired with angle of elevation α and initial velocity $\vec{v}_0$. Find the position $\vec{r}(t)$ of the projectile. What value of α maximizes the range?

<u>Solution</u> We assume that the air resistance is negligible and that the only force acting on the object is due to gravity



$$\vec{F} = m\vec{a} = -mg\vec{j}.$$

Hence $\vec{a} = -g\vec{j}$. We have

$$\vec{v}(t) = \int \vec{a}\,dt = -gt\,\vec{j} + \vec{c}$$

where $\vec{v_0} = \vec{v}(0) = -g \cdot 0\,\vec{j} + \vec{c}$,

$$\vec{c} = \vec{v_0}$$

Thus

$$\vec{v}(t) = -gt\,\vec{j} + \vec{v_0}$$

Integrating again yields

$$\vec{r}(t) = -\frac{1}{2}gt^2\,\vec{j} + t\vec{v_0} + \vec{D}$$

Now $\vec{0} = \vec{r}(0) = \vec{D}$

$$\vec{D} = \vec{0}$$

$$\vec{r}(t) = -\frac{1}{2}gt^2\,\vec{j} + t\,\vec{v_0}$$

Let $|\vec{v_0}| = v_0$. Then

$$\vec{v_0} = v_0 \cos\alpha\,\vec{i} + v_0 \sin\alpha\,\vec{j}$$

so

$$\vec{r}(t) = (v_0\cos\alpha)t\vec{i} + \left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right]\vec{j}$$

or

$$x(t) = (v_0\cos\alpha)t, \quad y(t) = (v_0\sin\alpha)t - \frac{1}{2}gt^2.$$

The projectile hits the surface when $y=0$

$$(v_0 \sin\alpha)t - \frac{1}{2}gt^2 = 0$$

$$t\left((v_0 \sin\alpha) - \frac{1}{2}gt\right) = 0.$$

We can assume that $t>0$ so

$$v_0 \sin\alpha - \frac{1}{2}gt = 0$$

$$t = \frac{2v_0 \sin\alpha}{g}$$

and

$$d = x(t) = (v_0 \cos\alpha)t = (v_0 \cos\alpha)\frac{2v_0 \sin\alpha}{g}$$

$$= \frac{v_0^2 \, 2\sin\alpha\cos\alpha}{g}$$

$$= \frac{v_0^2 \sin 2\alpha}{g}$$

The maximal value is when $\sin 2\alpha = 1$, i.e. $2\alpha = \pi/2$, $\alpha = \pi/4$.

<u>Remark</u> $g \approx 9.8\ m/s^2$.

Exercise 52 A projectile is fired with an initial speed of 200 m/s and angle of elevation 60°. Find
(a) the range
(b) the maximum height
(c) the speed at impact

Solution It is convenient to remember that

$$\vec{r}(t) = -\frac{1}{2} g t^2 \vec{j} + t\vec{v_0}$$

i.e.

$$x(t) = t v_0 \cos\alpha, \quad y(t) = t v_0 \sin\alpha - \frac{1}{2} g t^2.$$

Maximum height is when the y component of velocity equals zero

$$y' = v_0 \sin\alpha - gt = 0$$

$$t = \frac{v_0 \sin\alpha}{g}$$

$$y_{max} = \frac{v_0 \sin\alpha}{g} v_0 \sin\alpha - \frac{1}{2} g \left(\frac{v_0 \sin\alpha}{g}\right)^2 =$$

$$= \frac{v_0^2 \sin^2\alpha}{g} - \frac{v_0^2 \sin^2\alpha}{2g} = \frac{v_0^2 \sin^2\alpha}{2g}$$

Thus

(b) maximum height is $y_{max} = \dfrac{v_0^2 \sin^2\alpha}{2g}$.

The range is attained when $y = 0$

$$t v_0 \sin\alpha - \frac{1}{2} g t^2 = 0 \qquad (t \neq 0)$$

$$v_0 \sin\alpha = \frac{1}{2} g t$$

$$t = \frac{2 v_0 \sin\alpha}{g}$$

The range is

(a) $d = x(t) = \dfrac{2 v_0 \sin\alpha}{g} v_0 \cos\alpha = \dfrac{v_0^2 \sin 2\alpha}{g}$

To find the speed at impact we need to find the speed at

$$t = \frac{2 v_0 \sin\alpha}{g}$$

which is the time of impact.

$$\vec{v}(t) = -gt\,\vec{j} + \vec{v_0}$$

$$= \langle v_0 \cos\alpha,\ v_0 \sin\alpha - gt \rangle$$

$$|\vec{v}(t)| = \sqrt{(v_0 \cos\alpha)^2 + (v_0 \sin\alpha - gt)^2}.$$

Taking

$$t = \frac{2 v_0 \sin\alpha}{g}$$

we obtain the speed at impact

(c)

$$|\vec{v}(t)| = \sqrt{(v_0 \cos\alpha)^2 + \left(v_0 \sin\alpha - g\,\frac{2v_0 \sin\alpha}{g}\right)^2}$$

$$= v_0 \sqrt{\cos^2\alpha + (\sin\alpha - 2\sin\alpha)^2}$$

$$= v_0 \sqrt{\cos^2\alpha + \sin^2\alpha} = v_0$$

Finding numerical values is left to the reader.

Remark One could immediately obtain the answer v_0 at (c) using the conservation of energy principle.

<u>Exercise 53</u> A force with magnitude F acts directly upward from the xy plane on an object with mass m. The object starts from the origin with initial velocity $\vec{v}(0) = \vec{i} - \vec{j}$. Find its position function and its speed at time t.

<u>Solution</u> $\vec{F} = F\vec{k}$, $m\vec{a} = F\vec{k}$

$$\vec{a} = \frac{F}{m}\vec{k}$$

$$\vec{v} = \int \vec{a} = \frac{F}{m}t\vec{k} + \vec{c} \quad \text{(const. vector)}$$

$$\vec{i} - \vec{j} = \vec{v}(0) = \vec{c}$$

$$\boxed{\vec{v} = \vec{i} - \vec{j} + \frac{Ft}{m}\vec{k}}$$

$$\vec{r} = \int \vec{v} = t\vec{i} - t\vec{j} + \frac{Ft^2}{2m}\vec{k} + \vec{D} \quad \text{(const. vector)}$$

$$\vec{0} = \vec{r}(0) = \vec{D}$$

$$\boxed{\vec{r}(t) = t\vec{i} - t\vec{j} + \frac{Ft^2}{2m}\vec{k}}$$

Functions of several variables

A function of two variables

$$z = f(x,y)$$

assigns a value of z to a point (x,y) in the plane.

Example $f(x,y) = 3x^2y$

The domain of f is a subset of the plane. If the domain is not specified, we consider all points (x,y) for which the function $f(x,y)$ is well defined.

Example The domain of

$$f(x,y) = \sqrt{1-x^2-y^2}$$

consists of all points (x,y) such that

$$1 - x^2 - y^2 \geq 0$$

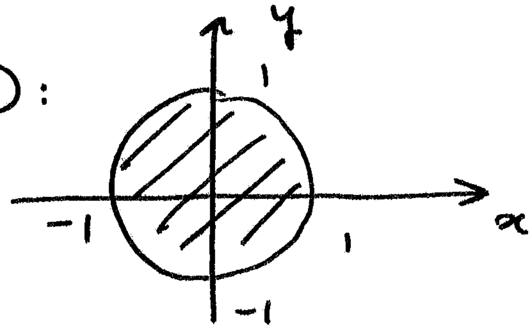
$$x^2+y^2 \leq 1,$$

i.e. the domain is the unit disc.
We are often asked to skech the domain. This is for a better visualization of the situation (read between the lines: to bother students with

annoying problems on the exams).

In this particular example the task was easy

Domain D:



We can also describe the domain D of the function f as follows

$$D = \{ (x,y) \mid x^2 + y^2 \leq 1 \}$$

the set of all (x,y) | such that | this condition is satisfied

Example Find the domain of

$$f(x,y) = \frac{\ln(x^2+y^2-1)}{x+y}$$

Solution

The domain D consists of all points (x, y) such that

$$x^2+y^2-1>0 \quad \text{and} \quad x+y\neq 0,$$

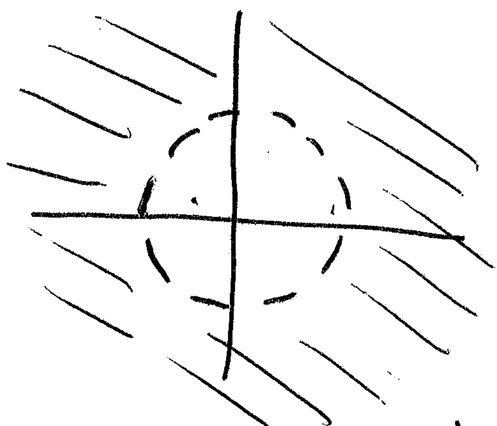
i.e.

$$x^2+y^2>1 \quad \text{and} \quad y\neq -x$$

i.e.

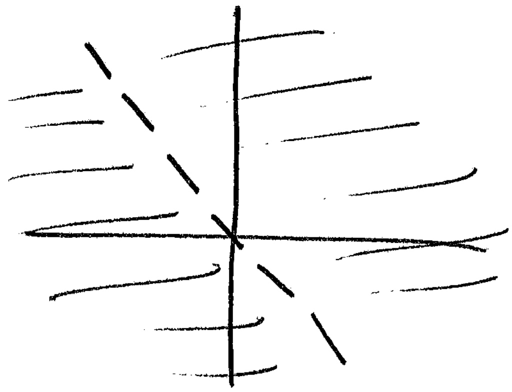
$$D=\{(x,y) \mid x^2+y^2>1 \text{ and } y\neq -x\}$$

The inequality $x^2+y^2>1$ describes the region outside the unit disc



The circle is denoted by a dotted line. That means the circle is not included in the domain.

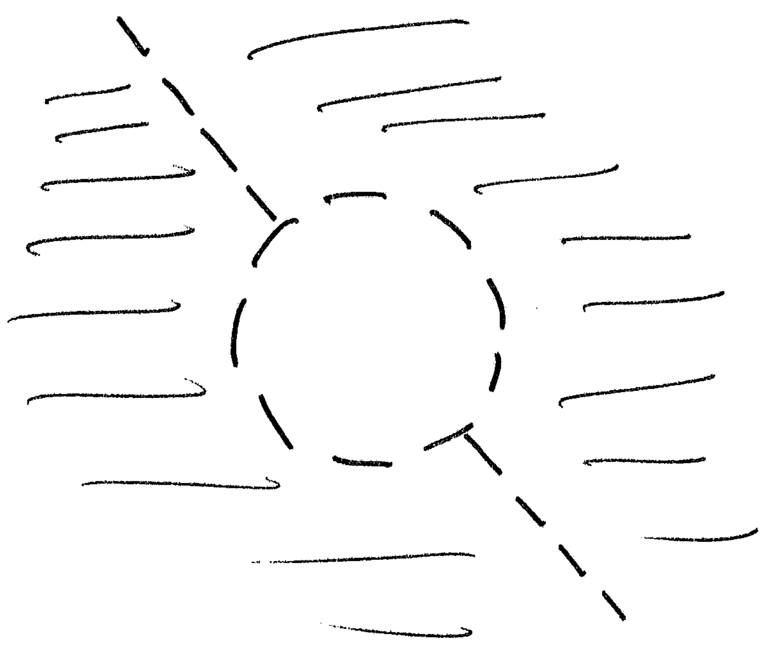
The condition $y \neq -x$ describes the set of points not on the line $y = x$



However, both conditions

$$x^2+y^2>1 \quad \text{and} \quad y \neq -x$$

must be satisfied at the same time. Hence we have to take the common part (called intersection) of both of the sets, i.e. the domain looks as follows



Similarly we define functions of three variables

$$w = f(x, y, z)$$

or even more variables

$$y = f(x_1, x_2, x_3, x_4, x_5, x_6, x_7).$$

The domain of $w = f(x, y, z)$ consists of points (x, y, z) in the three dimensional space $\mathbb{R}^3$, and of course it is much more difficult to sketch such domains.

<u>Example</u> Find the domain of

$$f(x, y, z) = \sqrt{1 - \max(|x|, |y|, |z|)}$$

<u>Solution</u> The condition is that

$$1 - \max(|x|, |y|, |z|) \geq 0$$

$$\max(|x|, |y|, |z|) \leq 1.$$

That means

$$|x| \leq 1, \quad |y| \leq 1, \quad |z| \leq 1$$

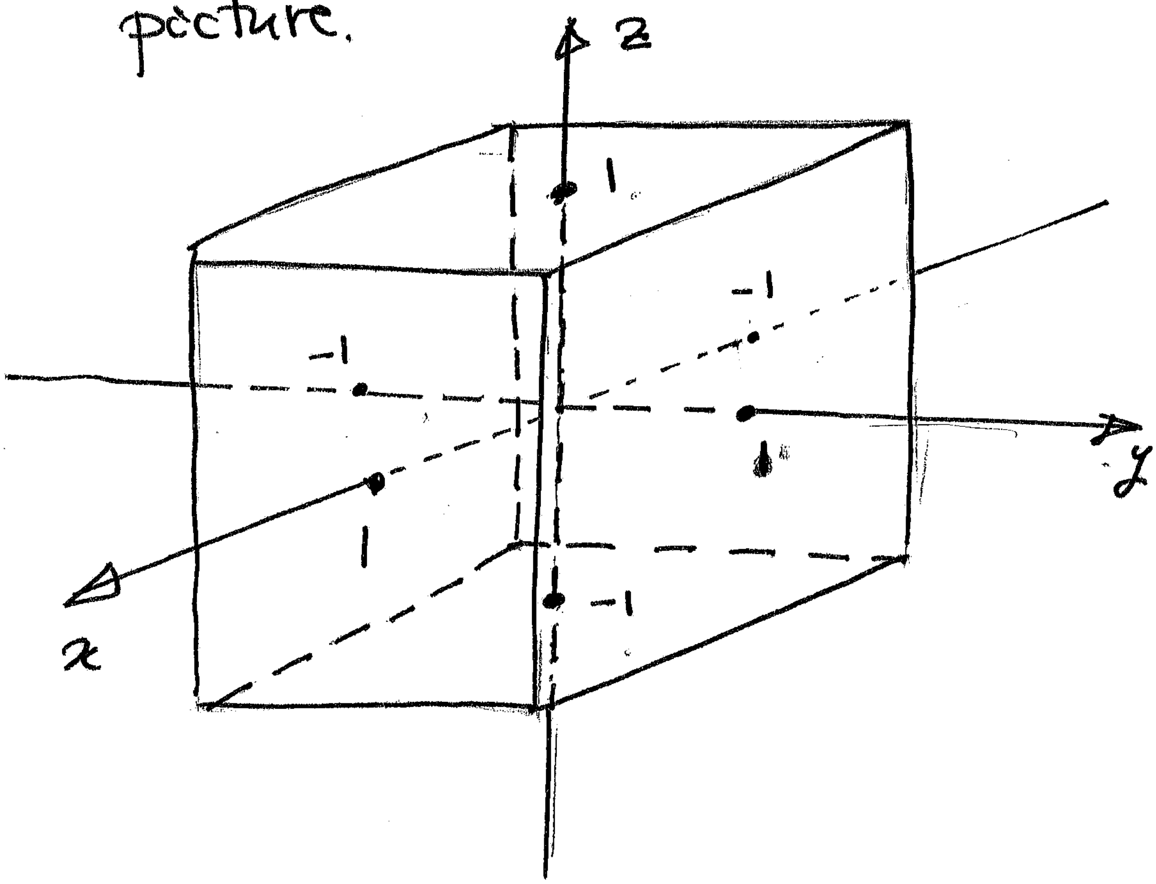
or

$$-1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad -1 \leq z \leq 1$$

The domain is the solid cube

$$D = \{(x,y,z) \mid -1 \le x \le 1,\ -1 \le y \le 1,\ -1 \le z \le 1\}$$

I will try to sketch it, but do not blame me for a low quality ~~of~~ the picture.



Not so bad after all, but I do not expect you to draw nice pictures on the exam.

The domain is the solid cube with the boundary (i.e. surface) and interior included.

The graph of
$$z = f(x,y)$$
consists of all points
$$(x,y,z) = (x,y,f(x,y)), \quad (x,y) \in D.$$
We will explain it in examples.

Example Sketch the graph of
$$f(x,y) = x+y-1$$

Solution This is a linear function.* The graph is the plane given by the equation
$$z = x+y-1.$$

To sketch a plane it is convenient to find points where the plane intersects with the, x, y, and z axis,

---

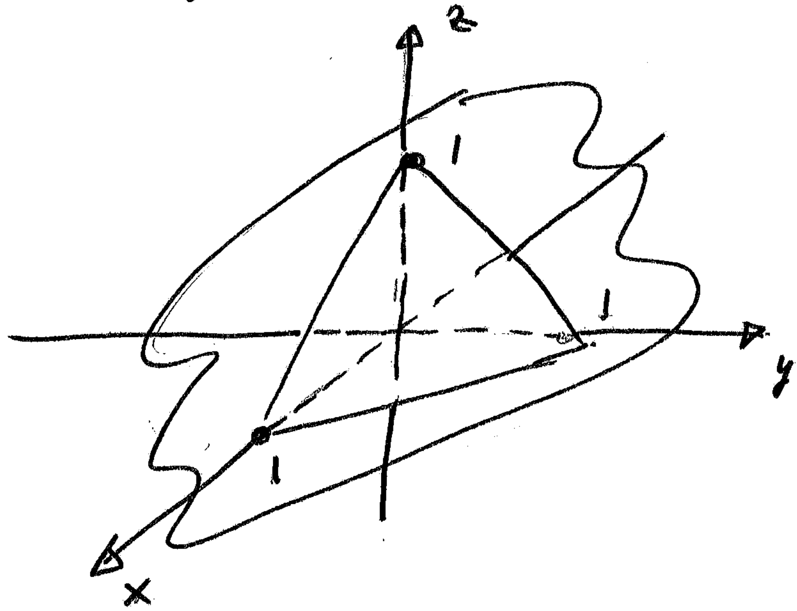
* A general linear function is
$$f(x,y) = ax+by+c$$

So called, x, y and z intercepts.

$$y = z = 0 \implies x = 1$$
$$x = z = 0 \implies y = 1$$
$$x = y = 0 \implies z = 1$$

Thus the plane intersects the coordinate system at points

$(1,0,0)$, $(0,1,0)$, $(0,0,1)$

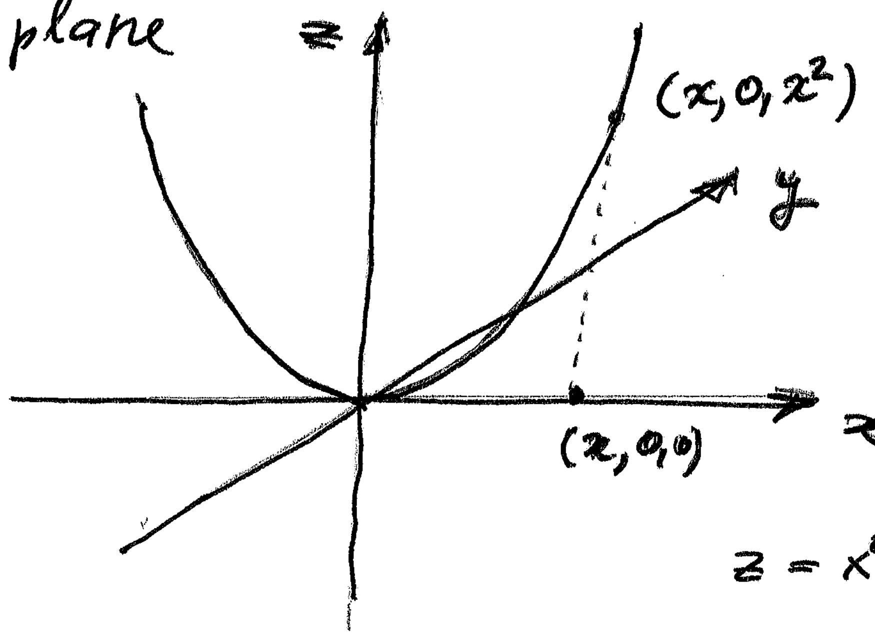


The graph is the plane that contains the triangle shown on the picture.

<u>Example</u> Sketch the graph of

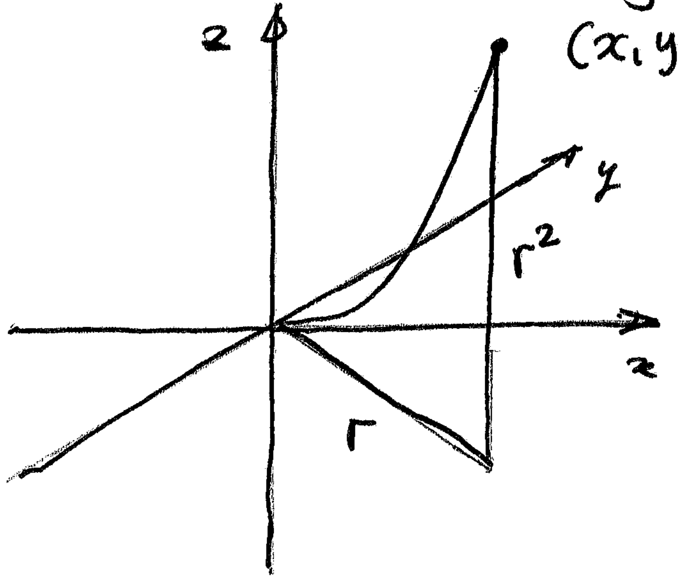
$$z = x^2 + y^2$$

<u>Solution</u> Let us check first the trace along the xz plane, i.e. the intersection of the graph with the xz plane



$$z = x^2 + 0^2 = x^2$$

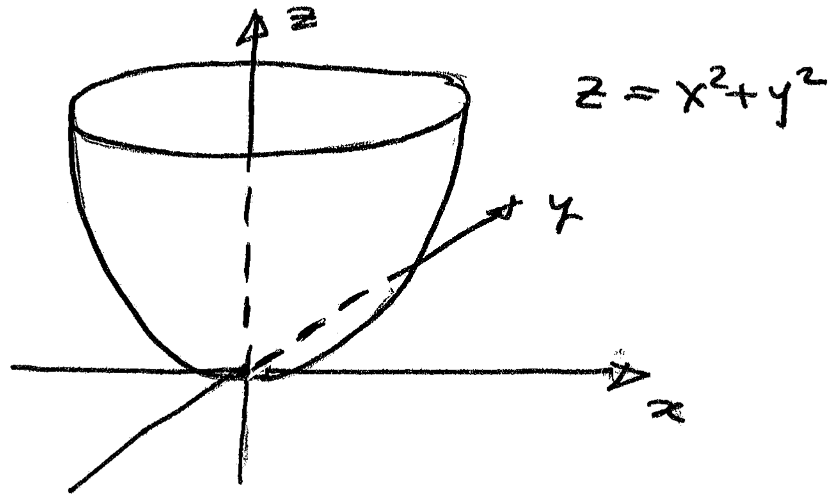
The trace is the parabola $z = x^2$.
Now let us take any direction in the xy plane



$$z = x^2 + y^2 = r^2$$

Thus in any direction we obtain the

same parabola $r \to r^2$. Thus the graph of f is the paraboloid obtained from the parabola $z = x^2$ by rotating it about the z-axis



Example Sketch the graph of

$$z = \sqrt{1-x^2-y^2}$$

Solution The domain is the unit disc

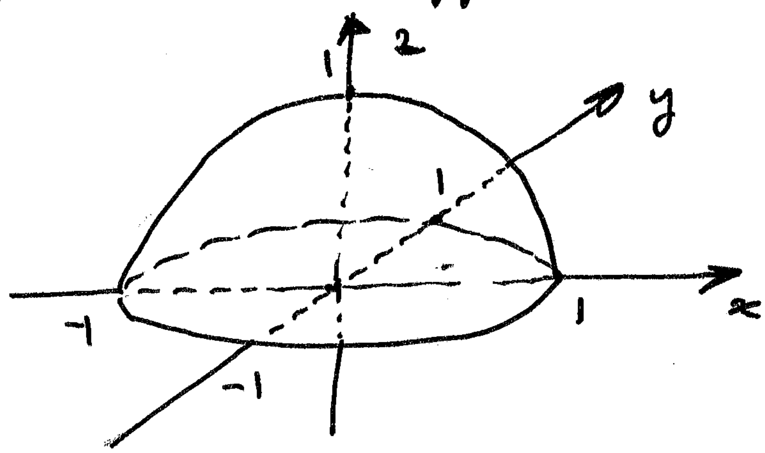
$$D = \{ (x,y) \mid x^2+y^2 \le 1 \}$$

Since

$$x^2+y^2+z^2 = x^2+y^2+(1-x^2-y^2) = 1$$

the points of the graph are on the unit sphere. Since

$z \geq 0$, it is the upper hemisphere



Example Sketch the graph of

$$z = -\sqrt{1-x^2-y^2}$$

Solution The domain is still the unit disc and

$$x^2+y^2+z^2 = x^2+y^2+(1-x^2-y^2) = 1$$

So again the points of the graph are on the unit sphere, but now $z \leq 0$, so it is the lower hemisphere. I will not sketch the picture. I am tired. Sorry.

One can think in an abstract way of graphs of functions of three variables

$$\omega = f(x, y, z),$$

but it would consist of points in the four dimensional space

$$(x, y, z, \omega) = (x, y, z, f(x, y, z))$$

and sketching such a graph is beyond our abilities.

A good way to visualize behaviour of a function $z = f(x, y)$ is to sketch the level curves which are described by the equation

$$(*) \qquad f(x, y) = k \qquad (k - \text{constant})$$

The level curve consists of all points (x, y) that satisfy the equation $(*)$

$$\{ (x, y) \mid f(x, y) = k \}.$$

Different values of k give different curves.

You should be familiar with the notion of level curves from topographic maps of mountains. Lines on such maps shows constant elevation.

By looking at such curves we can imagine the shape of the mountains.

Mountain are like graphs and sketching the level curves of a graph is like sketching a topographic map of a function.

<u>Example</u> Find the level curves of the function

$$z = x^2 + y^2.$$

<u>Solution</u> The level curves

$$x^2 + y^2 = k$$

are circles of radius $\sqrt{k}$ and they are defined only if $k \geq 0$.

We have seen on p. 138 that the graph of $z = x^2+y^2$ is a paraboloid, and indeed, if we intersect that paraboloid with the plane

$$z = k$$

we will obtain a circle (of radius $\sqrt{k}$).

The fact that there are no level curves for $k<0$ simply means that the graph rises above the xy plane.

I will not show any pictures here, because it is difficult to draw them correctly and the book contains plenty of good examples.

In the case of functions of three variables, a counterpart of level curves will be <u>level surfaces</u>

$$f(x, y, z) = k, \quad k - \text{constant}$$

<u>Example</u> Level surfaces of the function

$$w = x^2+y^2+z^2$$

are spheres

$$x^2+y^2+z^2=k$$

of radius $\sqrt{k}$. They are defined for $k \geq 0$. When $k=0$ the sphere is reduced to a single point $(0,0,0)$.

Problems

I will not include problems for this section, but it does not mean that you can skip it. The book contains many exercises and you need to study them. Do not neglect the problems 41–46.

Limits

Similarly as in the case of functions of one variable we can define the notion of limit for functions of several variables. We say that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \in \mathbb{R}$$

if the values of $f(x,y)$ approach to L when (x,y) approaches to (x_0, y_0), but $(x,y) \neq (x_0,y_0)$. We also assume that (x,y) is taken from the domain of f, otherwise the expression $f(x,y)$ would make no sense.

(x,y) approaches to (x_0, y_0)

$$(x,y) \to (x_0, y_0)$$

means that the distance between (x,y) and (x_0,y_0) decreases to zero, i.e.

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} \longrightarrow 0$$

or equivalently

$$|x-x_0| \to 0 \text{ and } |y-y_0| \to 0 \text{ simultaneously}$$

or equivalently

$$x \to x_0 \text{ and } y \to y_0 \text{ simultaneously.}$$

Remember that when we study the limit we always assume that $(x,y) \neq (x_0, y_0)$.

Then $f(x,y)$ approaches to $L \in \mathbb{R}$ in a sense that the distance between $f(x,y)$ and L decreases to zero.

$$|f(x,y) - L| \to 0.$$

Thus

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

means that

$$\left.\begin{array}{l} x \to x_0 \,\&\, y \to y_0 \text{ simultaneously} \\ \text{and } D \ni (x,y) \neq (x_0,y_0) \end{array}\right| \Longrightarrow |f(x,y) - L| \to 0$$

We also consider limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = +\infty$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = -\infty,$$

but I leave it to the reader to figure out what it means.

Finally, just like in the case of functions of one variable it may happen that the limit does not exist.

<u>Example</u> Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

does not exist.

<u>Proof</u>. Take $(x,y)=(t,0)$, $t\to 0$, $t\neq 0$. The point $(t,0)$ is in the domain of the function, $(t,0)\neq(0,0)$ and

$$f(t,0) = \frac{t^2-0^2}{t^2+0^2} = 1 \longrightarrow 1$$

That shows that the limit, if

it exists, must be equal 1.

On the other hand, if

$(x,y) = (0,t)$, $t \to 0$, $t \neq 0$,

then

$$f(0,t) = \frac{0^2 - t^2}{0^2 + t^2} = -1 \longrightarrow -1$$

so if the limit exist must be equal -1. Thus the limit, if it exists, must be 1 and -1 at the same time which is impossible. That simply means the limit does not exist.

In order to show that the limit does not exist, it suffices to find two different ways (paths) to approach (x_0, y_0) so that each time we get a different limit, just like in the example explained above.

Here is a more surprising example.

Example Does the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

exist?

Solution. Let $(x,y)=(t,0)$, $t\to 0$, $t\neq 0$.

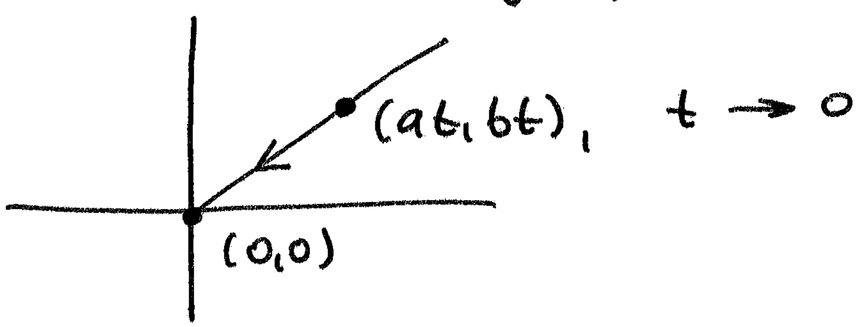
$$f(t,0) = \frac{0}{t^2} = 0 \longrightarrow 0.$$

Let now $(x,y) = (0,t)$, $t\neq 0$, $t\to 0$,

Then

$$f(0,t) = \frac{0}{t^4} = 0 \longrightarrow 0.$$

There are, however, other directions from which we may approach $(0,0)$.



Let $(x,y) = (at, bt)$, $t\to 0$, $t\neq 0$,
$a, b\neq 0$. Then

$$f(at, bt) = \frac{at\,(bt)^2}{(at)^2+(bt)^4} =$$

$$= \frac{ab^2t^3}{a^2t^2+b^4t^4} \underset{\text{div. by } t^2}{=} \frac{ab^2t}{a^2+b^4t^2} \to \frac{0}{a^2} = 0.$$

$ab^2t \to 0$, $b^4t^2 \to 0$

Thus if we approach $(0,0)$ from any direction we get the limit equal 0. Does it mean that

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = 0 \quad ?$$

Not at all. In the above computations we always approached $(0,0)$ along a straight line, but it is not the only possibility. Take

$$(x,y) = (t^2, t), \quad t \to 0, \quad t \neq 0.$$

Now we approach to $(0,0)$ along a parabola and

$$f(t^2, t) = \frac{t^2 \cdot t^2}{t^4+t^4} = \frac{1}{2} \longrightarrow \frac{1}{2}.$$

Since $\frac{1}{2} \neq 0$, the limit does not exist. That is really surprising and it is hard to imagine how it can be possible. You can find the graph of the function in the book, but it does not really help to understand what is happening there.

Now we need some positive examples where the limit does exist.

We need the following observation which is a version of the squeeze theorem

If

$$|f(x,y)-L| \leq g(x,y) \xrightarrow[\substack{(x,y)\to(x_0,y_0)\\(x,y)\neq(x_0,y_0)}]{} 0$$

then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

It may happen that the formula for f is very complicated, but we are able to find an estimate

$$|f(x,y)-L| \le g(x,y)$$

by a much simpler function g for which it is easy to show that

$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = 0.$$

Then we conclude that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Example Find

$$\lim_{(x,y)\to(x_0,y_0)} \frac{3x^2y}{x^2+y^2}$$

if it exists.

Solution. Take $(x,y)=(t,0)$, $t\to 0$, $t\neq 0$. We see that

$$f(t,0) = 0 \longrightarrow 0$$

so if the limit exist, it must be equal 0. Now we will prove that actually the limit exists and it is equal to 0.

We have

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| = 3\,\frac{x^2}{x^2+y^2}\,|y| \leq 3|y| \xrightarrow[(x,y)\to(0,0)]{} 0.$$

Hence

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

Limits have similar properties as limits of functions of one variable.

If $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, $\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$, $L, M \in \mathbb{R}$,

then

(1) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) \pm g(x,y) = L \pm M$

(2) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)\, g(x,y) = LM$

(3) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, if $M \neq 0$.

Sometimes (2) and (3) are called product and quotient rules for limits.

Here is a very powerful result

<u>Theorem</u> If a function $g(t)$ of one variable is continuous at $t = L$ and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

then

$$\lim_{(x,y)\to(x_0,y_0)} g(f(x,y)) = g(L).$$

In the next example we will explain how to use the above two results.

<u>Example</u> Prove that

$$\lim_{(x,y)\to(0,0)} \cos(xy)\, e^{\frac{3x^2y}{x^2+y^2}} = 1.$$

<u>Proof</u> The expression is very complicated and we need to break it into pieces.

It suffices to prove that

$$(*) \quad \lim_{(x,y)\to(0,0)} \cos(xy) = 1 \text{ and } \lim_{(x,y)\to(0,0)} e^{\frac{3x^2y}{x^2+y^2}} = 1,$$

because then the result will follow from the product rule (2) on p. 152.

Clearly $x \to 0$ as $(x,y) \to (0,0)$ and $y \to 0$ as $(x,y) \to (0,0)$, so the product rule (2) on p. 152 yields

$$\lim_{(x,y)\to(0,0)} xy = 0.$$

Since $\cos t$ is a continuous function the theorem from p. 153 implies

$$\lim_{(x,y)\to(0,0)} \cos(xy) = \cos 0 = 1.$$

Now if we take $g(t) = e^t$, we can write

$$e^{\frac{3x^2y}{x^2+y^2}} = g\left(\frac{3x^2y}{x^2+y^2}\right).$$

As we know (Example p. 151)

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

Since $g(t) = e^t$ is continuous, we get

$$\lim_{(x,y)\to(0,0)} e^{\frac{3x^2y}{x^2+y^2}} = \lim_{(x,y)\to(0,0)} g\left(\frac{3x^2y}{x^2+y^2}\right)$$

$$= g(0) = e^0 = 1.$$

Thus we proved $(*)$ p. 154 and hence

$$\lim_{(x,y)\to(0,0)} \cos(xy)\, e^{\frac{3x^2y}{x^2+y^2}} = 1.$$

<u>Continuity</u>

We say that a function $f(x,y)$ is <u>continuous</u> at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

and we say that f is continuous, if it is <u>continuous</u> at every point of its domain D.

If functions $f(x,y)$ and $g(x,y)$ are continuous at (x_0, y_0), then the functions

$$f(x,y) \pm g(x,y)$$

$$f(x,y)\, g(x,y)$$

$$\frac{f(x,y)}{g(x,y)} \quad \text{provided } g(x,y) = 0 \text{ for } (x,y) \text{ near } (x_0, y_0)$$

are continuous at (x_0, y_0). As an example we will verify continuity of $f(x,y)\, g(x,y)$. We have

$$\lim_{(x,y) \to (x_0, y_0)} f(x,y) = f(x_0, y_0)$$

$$\lim_{(x,y) \to (x_0, y_0)} g(x,y) = g(x_0, y_0)$$

and hence the product rule for the limits gives

$$\lim_{(x,y) \to (x_0, y_0)} f(x,y)\, g(x,y) = f(x_0, y_0)\, g(x_0, y_0)$$

which means that the function

$$z = f(x,y)\, g(x,y)$$

is continuous at (x_0, y_0).

Clearly $\lim_{(x,y)\to(x_0,y_0)} x = x_0$, hence the function $f(x,y) = x$ is continuous. (Yes, this is a function of two variables despite the fact that y is not present in the formula).

Similarly $g(x,y) = y$ is continuous. Adding and multiplying these functions several times we obtain that the polynomial

$$h(x,y) = 3x^5y^4 - 12x^3y + y^7 - 12xy^2 + 4$$

is continuous.

More generally <u>polynomials of two variables</u> are sums of terms of the form cx^ny^m where c is a constant and $n, m \geq 0$ are integers.

Polynomials are continuous.

A quotient of two polynomials is called a <u>rational function</u>. Rational functions are continuous at all points where the denominator is non zero.

Example

$$f(x,y) = \frac{3x^2y + 5xy + 7}{x^2 + y^2 + 1}$$

is a rational function. It is continuous on the whole plane, because the denominator is never equal zero.

Example

$$f(x,y) = \frac{x^4 + 5x^2y^2 + 7}{x - y} + \frac{1}{x^2+y^2-1}$$

is also a rational function. (To see this think of a common denominator). It is continuous at all points (x,y) such that

$$x - y \neq 0 \quad \text{and} \quad x^2 + y^2 - 1 \neq 0$$

at the same time, i.e. (x,y) cannot belong to the line $x = y$ and it cannot belong to the circle $x^2+y^2=1$.

Thus $f(x,y)$ is continuous in all points of the set

This set is the domain of f.
This is true in general: rational functions are continuous at all points in the domain.

Example For what values of a is the function

$$f(x,y) = \begin{cases} \dfrac{3x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ a & \text{if } (x,y) = (0,0) \end{cases}$$

continuous?

Solution The function f is continuous at all points $(x,y) \neq (0,0)$ as a rational function with non-zero denominator. The only question is what happens at $(0,0)$.

Since

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0,$$

$(x,y) \neq (0,0)$, so
$f(x,y) = \dfrac{3x^2y}{x^2+y^2}$

the function will be continuous if and only if

$$f(0,0) = 0$$

so the function is continuous if and only if $a=0$.

<u>Theorem</u> If $f(x,y)$ is continuous at (x_0, y_0) and the function $g(t)$ of one variable is continuous at $t_0 = f(x_0, y_0)$, then

$$h(x,y) = g(f(x,y))$$

is continuous at (x_0, y_0)

<u>Example</u> $f(x,y) = \cos(x^2y + 5xy - 7)$ is continuous.

<u>Remark</u> When we investigate continuity of $f(x,y)$ at (x_0, y_0), the point (x_0, y_0) is in the domain of f. However, when we study just the limit

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

it is not required that (x_0, y_0) is in the domain. For example

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

despite the fact that the function is not defined at $(0,0)$.

All the results about limits and continuity easily extend to functions of three or more variables.

Problems

<u>Exercise 54</u> Suppose that $\lim_{(x,y)\to(1,2)} f(x,y) = 5$. What can you say about the value of $f(1,2)$?

Solution Nothing. It can even happen that f is not defined at $(1,2)$. However, if f is continuous at $(1,2)$, then $f(1,2)=6$.

Exercise 55 Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$$

does not exist.

Solution $f(t,0) = \frac{t^4}{t^2} = t^2 \to 0$ as $t\to 0$

but

$$f(0,t) = -\frac{4t^2}{2t^2} = -2 \longrightarrow -2 \text{ as } t\to 0.$$

Since $0\neq -2$, the limit does not exist.

Exercise 56 Find the limit of it exists

$$\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2}$$

Solution $f(t,0) = 0 \to 0$ as $t\to 0$, So if the limit exists it must be equal zero. We will prove that the limit equals zero.

$$\left| \frac{x^2 \sin^2 y}{x^2+2y^2} - 0 \right| = \left| \frac{x^2}{x^2+2y^2} \right| \left| \sin^2 y \right| \leq$$

$$\leq \left| \sin^2 y \right| \longrightarrow 0 \quad \text{as} \quad (x,y) \to 0.$$

Indeed, of $(x,y) \to 0$, then $y \to 0$ and hence

$$\sin^2 y \longrightarrow 0$$

because $\sin^2 y$ is a continuous function of the variable y.

Exercise 57 Find the limit

$$\lim_{(x,y) \to (0,0)} \frac{\sin(xy)}{xy}$$

Solution If $(x,y) \to 0$, then $t = xy \to 0$ and hence

$$\frac{\sin(xy)}{xy} = \frac{\sin t}{t} \xrightarrow{t \to 0} 1$$

Thus the limit exists and equals 1.

Exercise 58 Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$$

Solution $t = x^2+y^2 \to 0$ if $(x,y)\to(0,0)$

and

$$\frac{e^{-x^2-y^2}-1}{x^2+y^2} = \frac{e^{-t}-1}{t}$$

Now

$$\lim_{t\to 0} \frac{e^{-t}-1}{t} \overset{\text{l'Hospital}}{=\!=} \lim_{t\to 0} -e^{-t} = -1.$$

Thus

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = -1.$$

Exercise 59 For what values of a is the function

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2+x^2\sin(xy)}{x^2+y^2}, & \text{if } (x,y)\neq(0,0) \\ a & \text{if } (x,y)=(0,0) \end{cases}$$

continuous?

Solution Clearly f is continuous at all $(x,y) \neq (0,0)$ and we only need to investigate its limit at $(0,0)$.

Observe that

$$f(t,0) = \frac{t^2+0+t^2\sin 0}{t^2+0} = 1 \to 1 \text{ as } t\to 0$$

so the limit if it excists equals 1.

We have

$$\frac{x^2+y^2+x^2\sin(xy)}{x^2+y^2} = 1 + \frac{x^2\sin(xy)}{x^2+y^2}.$$

Hence

$$\left|\frac{x^2+y^2+x^2\sin(xy)}{x^2+y^2} - 1\right| = \frac{x^2}{x^2+y^2}|\sin(xy)|$$

$$\leq |\sin(xy)| \longrightarrow 0 \text{ as } (x,y)\to(0,0)$$

Indeed, $xy \to 0$ and hence $\sin(xy)\to 0$ by continuity of the function $\sin t$.

Thus

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+x^2\sin(xy)}{x^2+y^2} = 1$$

and hence the function is continuous at $(0,0)$ if and only if $a=0$.

<u>Exercise 60</u> Prove that the function

$$f(x,y) = \begin{cases} e^{\left(\frac{\sin(xy)}{xy}\right)} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

is discontinuous at $(0,0)$.

<u>Solution</u> We already proved in Exercise 57 that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = 1$$

and hence continuity of the function e^t yields

$$\lim_{(x,y)\to(0,0)} e^{\left(\frac{\sin(xy)}{xy}\right)} = e^{0} = 1 \neq f(0,0).$$

Partial derivatives

The partial derivative $\frac{\partial f}{\partial x}(a,b)$ of a function $f(x,y)$ with respect to x at the point (a,b) is defined as follows.

Fix $y=b$. Then $g(x) = f(x,b)$ is a function of one variable and the partial derivative $\partial f/\partial x\,(a,b)$ is defined as derivative of that function at $x=a$, i.e.

$$\frac{\partial f}{\partial x}(a,b) = g'(a) = \frac{d}{dx}\Big|_{x=a} f(x,b).$$

Similarly we define the partial derivative with respect to y. Let $h(y) = f(a,y)$. Then

$$\frac{\partial f}{\partial y}(a,b) = h'(b) = \frac{d}{dy}\Big|_{y=b} f(a,y).$$

Recall that

$$g'(a) = \lim_{t\to 0} \frac{g(a+t)-g(a)}{t}.$$

Hence

$$\frac{\partial f}{\partial x}(a,b) = \lim_{t \to 0} \frac{f(a+t,b) - f(a,b)}{t}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{t \to 0} \frac{f(a,b+t) - f(a,b)}{t}.$$

If the function f is sufficiently nice (regular), partial derivatives can be computed at any point of the domain so they become functions of two variables

$$\frac{\partial}{\partial x} f(x,y) \quad , \quad \frac{\partial}{\partial y} f(x,y).$$

Example Find partial derivatives of

$$f(x,y) = e^{\sin(x^2 y)}$$

Solution

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x} e^{\sin(x^2 y)} = e^{\sin(x^2 y)} \frac{\partial}{\partial x} \sin(x^2 y)$$

$$= e^{\sin(x^2 y)} \cos(x^2 y)\, 2xy.$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y} e^{\sin(x^2y)} = e^{\sin(x^2y)} \frac{\partial}{\partial y} \sin(x^2y)$$

$$= e^{\sin(x^2y)} \cos(x^2y)\, x^2 .$$

There are other ways to denote partial derivatives

$$\frac{\partial f}{\partial x} = D_x f = D_1 f = f_x = f_1 .$$

Here $D_1 f$ and f_1 mean that we take the partial derivative with respect to the first variable. Similarly

$$\frac{\partial f}{\partial y} = D_y f = D_2 f = f_y = f_2 .$$

Partial derivatives have a nice geometric interpretation. The graph of

$$g(x) = f(x, b)$$

is obtained from the graph of

$$z = f(x, y)$$

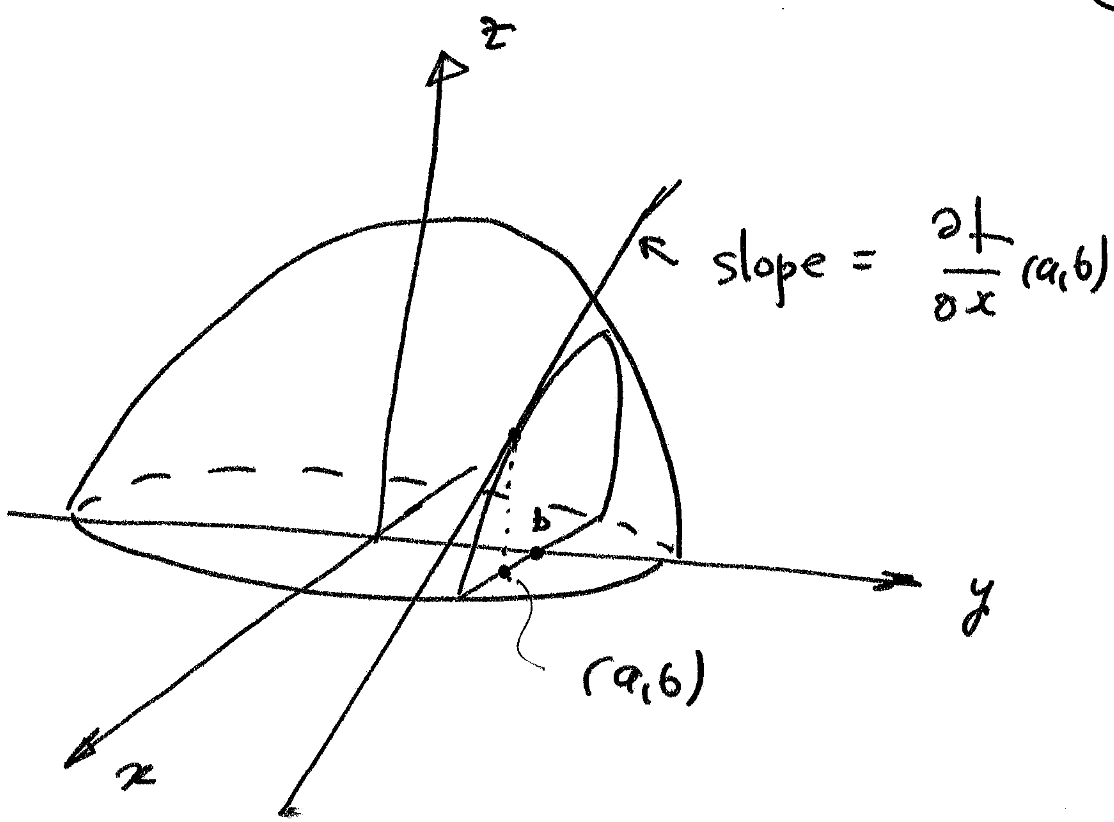
by intersecting it with the plane $y = b$, and $\frac{\partial f}{\partial x}(a, b)$ is the slope of the tangent line at $x = a$ to that graph. Indeed,

$$\frac{\partial f}{\partial x}(a, b) = g'(a) = \text{the slope}.$$

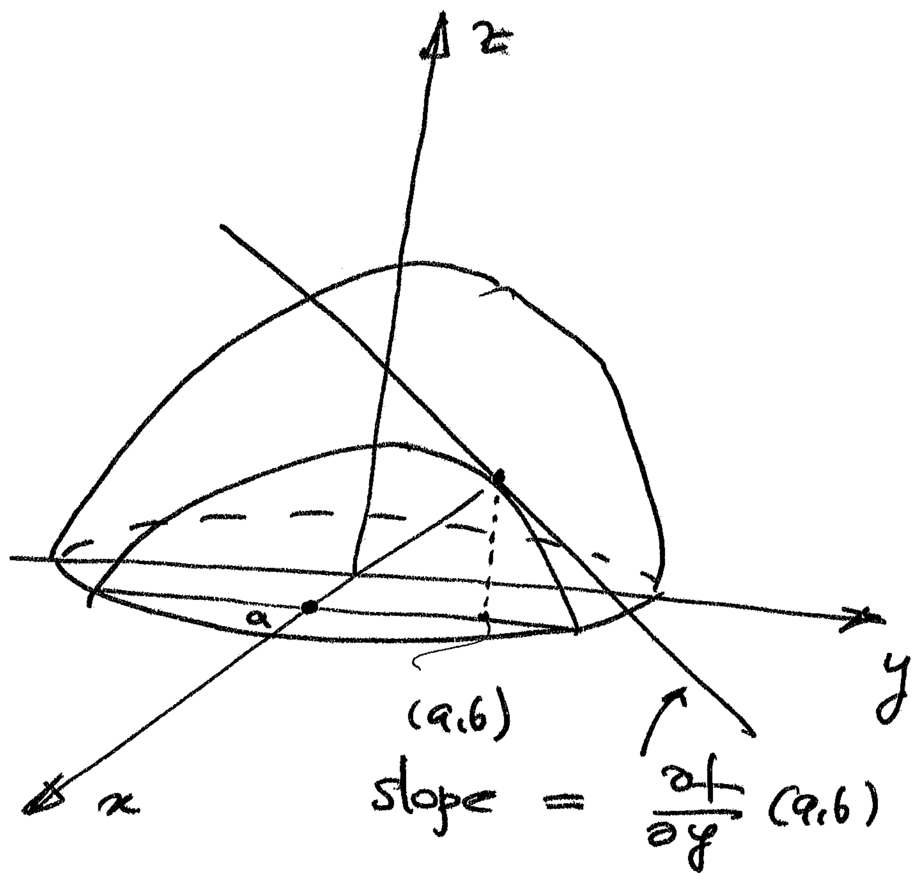
In other words, $\frac{\partial f}{\partial x}(a, b)$ is the slope of the tangent line to the graph of

$$z = f(x, y)$$

at $(x, y) = (a, b)$ in the direction of the x-axis.



Similarly $\frac{\partial f}{\partial y}(a,b)$ is the slope of the tangent line to the graph of $z = f(x,y)$ at $(x,y) = (a,b)$ in the direction of the y-axis.



Let us recall the technique of implicit differentiation from Calculus I.

Sometimes it is convenient to define a function $y = y(x)$ implicitly through the equation

$$F(x, y) = 0.$$

For example

$$x^2 + y^2 = 1$$

(or $F(x,y) = x^2 + y^2 - 1 = 0$)

defines

$$y = \sqrt{1-x^2}$$

if we are in the upper semicircle

or

$$y = -\sqrt{1-x^2}$$

if we are in the lower semicircle.

The technique of implicit differentiation allows us to differentiate

y with respect to x, even without solving the equation for y.

Assuming that y is a function of x we have

$$x^2 + y(x)^2 = 1$$

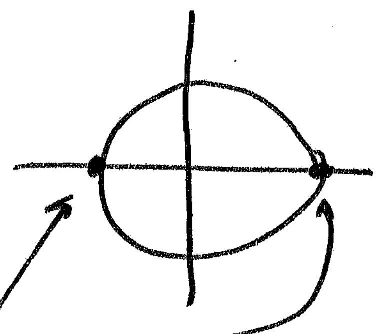
Taking the derivative with respect to x yields

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y} \qquad (*)$$

and it is well defined if $y \neq 0$.

Indeed



$y=0$ at these points and y is not a function of x near these points (vertical line test).

In our example

$$y = \sqrt{1-x^2}$$

so

$$y' = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}} = -\frac{x}{y}$$

or

$$y = -\sqrt{1-x^2}$$

so

$$y' = -\frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{-\sqrt{1-x^2}} = -\frac{x}{y}$$

which is consistent with (*) p. 173.

What is important is that it was not necessary to solve the equation for y in order to find the derivative y'.

Similar technique applies to functions of two variables.

Example Find $\frac{\partial z}{\partial x}$ if z is a function of variables x, y defined through the equation

$$yz - \ln z = x + y$$

Solution We have

$$y\, z(x,y) - \ln z(x,y) = x + y.$$

Hence

$$\frac{\partial}{\partial x}\left(y z(x,y) - \ln z(x,y)\right) = \frac{\partial}{\partial x}(x+y)$$

$$y \frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}} = \frac{z}{yz - 1}$$

Observe that the derivative does not exist at the points when $yz = 1$.

We can also define partial derivatives of functions of three and more variables.

Example Find $\partial f / \partial x_3$ where

$$f(x_1, x_2, x_3, x_4) = x_1^3 x_2^2 x_3^5 \sin x_4 .$$

Solution

$$\frac{\partial f}{\partial x_3} = 5 x_1^3 x_2^2 x_3^4 \sin x_4 .$$

Higher order partial derivatives are defined as partial derivatives taken several times. The notation is as follows

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \, \partial x} = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \, \partial y} = (f_y)_x = f_{yx}$$

and

$$\frac{\partial^5 f}{\partial x\, \partial y\, \partial z^3} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)\right)\right)\right).$$

<u>Example</u> Let $f(x,y) = x^3 + x^2y^3 - 2y^2$.
Find $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$.

<u>Solution</u>

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^3, \quad \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 4y, \quad \frac{\partial^2 f}{\partial x \partial y} = 6xy^2.$$

There is something interesting about this example. Although the first order partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ look quite differently, after taking the second order derivative we obtained that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

This equality turns out to be true in a very general situation.

<u>Theorem (Clairaut)</u> If the second order partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}$$

are continuous in a region that contains (a,b), then

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

Thus if the partial derivatives are continuous (that is usually the case if a function is given by a single formula), then it does not matter in what order we take derivatives.

That applies to derivatives of order higher than 2 as well and to functions of more than two variables:

$$\frac{\partial^5 f}{\partial x \partial y \partial x \partial z \partial x} = \frac{\partial^5 f}{\partial x^3 \partial y \partial z} = \frac{\partial^5 f}{\partial z \partial y \partial x^3} = \cdots$$

Partial differential equations

Partial differential equations appear in numerous applications of mathematics to physics and engineering.

We say that a function $u(x,y)$ is <u>harmonic</u> if it solves the <u>Laplace equation</u>

$$u_{xx} + u_{yy} = 0.$$

Example Show that the function
$$u(x,y) = e^x \sin y$$
is harmonic.

Solution $u_x = e^x \sin y$, $u_{xx} = e^x \sin y$

$u_y = e^x \cos y$, $u_{yy} = -e^x \sin y$

$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0.$

A function of three variables $u(x,y,z)$ is called harmonic if
$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Example Electric potential at (x,y,z) caused by electric charge located at the origin equals
$$U(x,y,z) = \frac{k}{r}$$
where k is some constant and r is the distance to the origin.

We will prove that U is a harmonic function. This fact plays a fundamental role in physics. Since $r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$ we can write

$$U(x,y,z) = k\,(x^2+y^2+z^2)^{-1/2}.$$

Thus

$$U_x = \left(-\frac{1}{2}\right) k\,(x^2+y^2+z^2)^{-3/2}\,(2x)$$

$$= -k\,(x^2+y^2+z^2)^{-3/2}\,x$$

$$U_{xx} = \left(-\frac{3}{2}\right)(-k)\,(x^2+y^2+z^2)^{-5/2}\,(2x)x$$

$$+ (-k)(x^2+y^2+z^2)^{-3/2} =$$

$$= 3k\,(x^2+y^2+z^2)^{-5/2}\,x^2 - k(x^2+y^2+z^2)^{-3/2}.$$

Since variables x, y, z play the same role in the formula, by symmetry we get

$$U_{yy} = 3k\,(x^2+y^2+z^2)^{-5/2}\,y^2 - k\,(x^2+y^2+z^2)^{-3/2}$$

$$U_{zz} = 3k\,(x^2+y^2+z^2)^{-5/2}\,z^2 - k\,(x^2+y^2+z^2)^{-3/2}.$$

Hence

$$U_{xx} + U_{yy} + U_{zz} =$$

$$3k\,(x^2+y^2+z^2)^{-5/2}\underbrace{(x^2+y^2+z^2)}_{(x^2+y^2+z^2)^{2/2}} - 3k\,(x^2+y^2+z^2)^{-3/2} =$$

$$3k\,(x^2+y^2+z^2)^{-3/2} - 3k\,(x^2+y^2+z^2)^{-3/2} = 0.$$

The Laplace operator (Laplacian) is defined by

$$\Delta u = u_{xx} + u_{yy} \quad \text{in 2 variables}$$

and

$$\Delta u = u_{xx} + u_{yy} + u_{zz} \quad \text{in 3 variables.}$$

Thus u is harmonic if

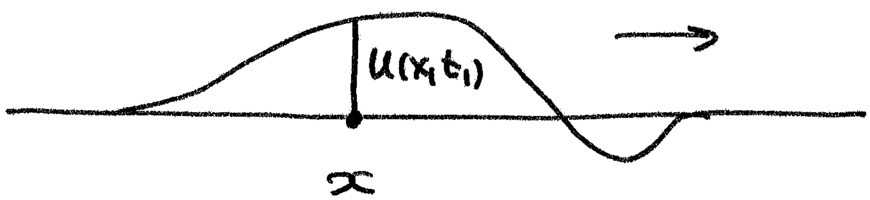
$$\Delta u = 0.$$

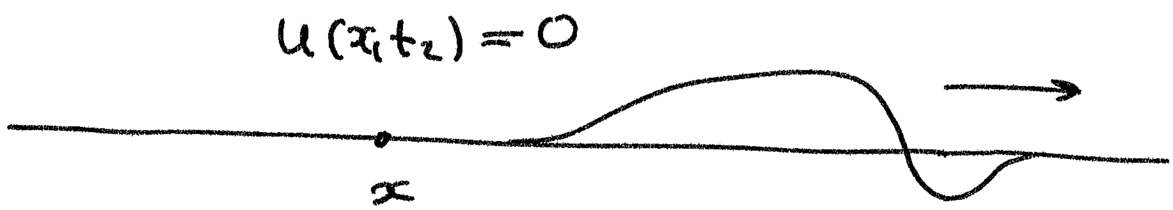
We say that a function $u(x,t)$ solves the wave equation if

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

This equation describes motion of a wave along the x-axis and $u(x,t)$ is the height of the wave at x at time t

Example





The wave moves to the right so the height of the wave at x changes in time.

<u>Example</u> Let $f(x)$ be twice differentiable. Then

$$u(x,t) = f(x-at)$$

solves the wave equation. Indeed,

$$u_t = f'(x-at)(-a)$$

$$u_{tt} = f''(x-at)(-a)^2 = a^2 f''(x-at)$$

$$u_x = f'(x-at)$$

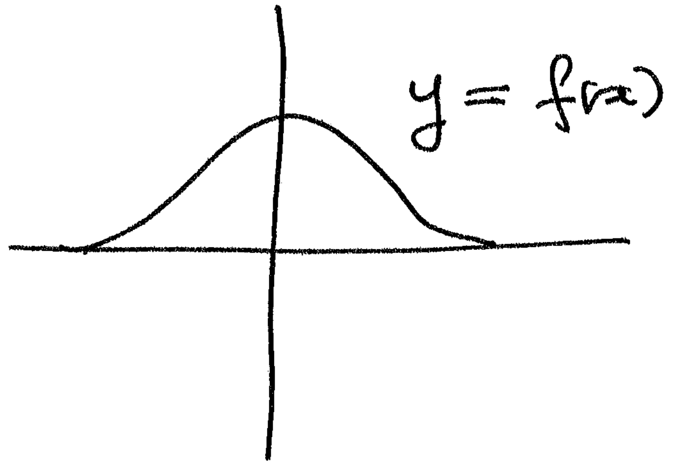
$$u_{xx} = f''(x-at)$$

$$u_{tt} - a^2 u_{xx} = a^2 f''(x-at) - a^2 f''(x-at) = 0$$

$$u_{tt} = a^2 u_{xx}.$$

The solution described above has a nice geometric interpretation.

Suppose that the graph of $f(x)$ looks as follows

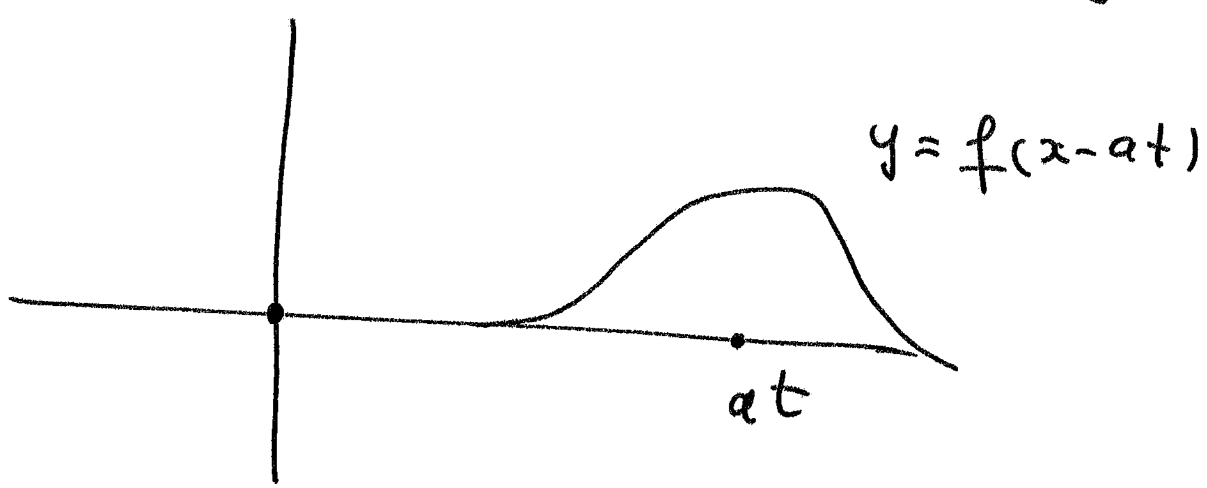


Note that $u(x,0) = f(x)$, so the shape of the wave at $t = 0$ coincides with the shape of f.

Now at time t the shape of the wave is

$$u(x,t) = f(x - at)$$

which is the translation of the graph of f to the right by at.



Thus the graph of f moved to the right by at units when the elapsed time was t, so it moves with the speed

$$\begin{array}{l} \text{distance} \longrightarrow \\ \text{time} \longrightarrow \end{array} \frac{at}{t} = a.$$

That means the coefficient a in the wave equation is the speed of waves.

<u>Example</u> Vibrating string is described by the wave equation.

There are also analogs of the wave equation in dimensions 2 and 3 and the equation describes motion of sound, electromagnetic and water waves.

Problems

Exercise 61 Find $f_x(3,4)$, where

$$f(x,y) = \ln\left(x + \sqrt{x^2+y^2}\right)$$

Solution

$$f_x = \frac{1 + \frac{2x}{2\sqrt{x^2+y^2}}}{x + \sqrt{x^2+y^2}} = \frac{1 + \frac{x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}$$

$$f_x(3,4) = \frac{1 + \frac{3}{\sqrt{3^2+4^2}}}{3 + \sqrt{3^2+4^2}} =$$

$$= \frac{1 + \frac{3}{5}}{3+5} \underset{\cdot 5}{=} \frac{8}{8 \cdot 5} = \boxed{\frac{1}{5}}$$

Exercise 62 Find $F_x(x,y)$ where

$$F(x,y) = \int_x^y \cos(e^t)\,dt$$

Solution We have

$$F(x,y) = -\int_y^x \cos(e^t)\,dt$$

and hence the Fundamental Theorem of Calculus yields

$$F_x(x,y) = -\frac{d}{dx}\int_y^x \cos(e^t)\,dt$$

$$= -\cos(e^x).$$

Exercise 63 Suppose that a function $F(x,y)$ has the property

$$F(x,y) = -F(y,x).$$

Prove that

$$F_x(x,x) + F_y(x,x) = 0.$$

Proof We want to prove that

$$F_1(x,x) + F_2(x,x) = 0$$

(This notation will be more convenient as less confusing). We have

$$F(x,y) + F(y,x) = 0$$

Hence

$$0 = \frac{\partial}{\partial x}\left(F(x,y) + F(y,x)\right) =$$

$F_1(x,y) + F_2(y,x)$

Taking $y = x$ yields

$$F_1(x,x) + F_2(x,x) = 0$$

Exercise 64 Find g_{rst}, where $g(r,s,t) = e^r \sin(st)$.

Solution $g_r = e^r \sin(st)$,

$$g_{rs} = e^r \cos(st)\, t$$

$$g_{rst} = e^r \cos(st) + e^r(-\sin(st)\, s)\, t$$

$$= e^r \cos(st) - e^r \sin(st)\, st$$

Exercise 65 If $u = e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}$, where $a_1^2 + a_2^2 + \dots + a_n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = u$$

Proof

$$u_{x_i} = e^{a_1x_1+\dots+a_nx_n}\, a_i$$

$$u_{x_ix_i} = e^{a_1x_1+\dots+a_nx_n}\, a_i^2 = u\, a_i^2$$

Hence

$$u_{x_1x_1}+\dots+u_{x_nx_n} = u\,\underbrace{(a_1^2+\dots+a_n^2)}_{1} = u$$

Exercise 66 If a, b, c are sides of a triangle, then the opposite angles A, B, C can be regarded as functions of a, b and c.
Prove that

$$(\sin A)(\sin B)(\sin C)\,\frac{\partial A}{\partial a}\,\frac{\partial B}{\partial b}\,\frac{\partial C}{\partial c} = \frac{1}{abc}.$$

Solution Sides and angles are connected through the Law of Cosines

$$a^2 = b^2+c^2-2bc\,(\cos A)$$

Taking the derivative with respect to a (implicit differentiation) yields

$$2a = -2bc(-\sin A)\frac{\partial A}{\partial a}$$

$$bc(\sin A)\frac{\partial A}{\partial a} = a$$

similarly

$$ab(\sin C)\frac{\partial C}{\partial c} = c$$

$$ac(\sin B)\frac{\partial B}{\partial b} = b$$

$$a^2b^2c^2(\sin A)(\sin B)(\sin C)\frac{\partial A}{\partial a}\frac{\partial B}{\partial b}\frac{\partial C}{\partial c} = abc$$

$$(\sin A)(\sin B)(\sin C)\frac{\partial A}{\partial a}\frac{\partial B}{\partial b}\frac{\partial C}{\partial c} = \frac{1}{abc}$$

Exercise 67 Under notation from Exercise 66, find $\frac{\partial A}{\partial b}$.

Solution

$$a^2 = b^2 + c^2 - 2bc(\cos A)$$

Taking derivative with respect to b yields

$$0 = 2b - 2c(\cos A) - 2bc(-\sin A)\frac{\partial A}{\partial b}$$

$$\frac{\partial A}{\partial b} = \frac{c(\cos A) - b}{bc(\sin A)}.$$

<u>Exercise 68</u> Is there a function $f(x,y)$ such that

$$f_x = x + 4y, \quad f_y = 3x - y \;?$$

<u>Solution</u> Such a function does not exist, because

$$f_{xy} = 4, \quad f_{yx} = 3$$

and hence

$$f_{xy} \neq f_{yx}.$$

<u>Exercise 69</u> Find $f_x(1,0)$, where

$$f(x,y) = x(x^2+y^2)^{-3/2} e^{\sin(x^2 y)}$$

<u>Solution</u> The function seems difficult

to differentiate, but the problem is actually very simple. Recall that

$$f_x(a,b) = g'(a) \quad \text{where } g(x) = f(x,b).$$

In our situation $(a,b) = (1,0)$ and

$$g(x) = f(x,0) = x\,(x^2)^{-3/2}\,e^0 = x^{-2}$$

$$g'(x) = -2x^{-3}, \quad \text{so}$$

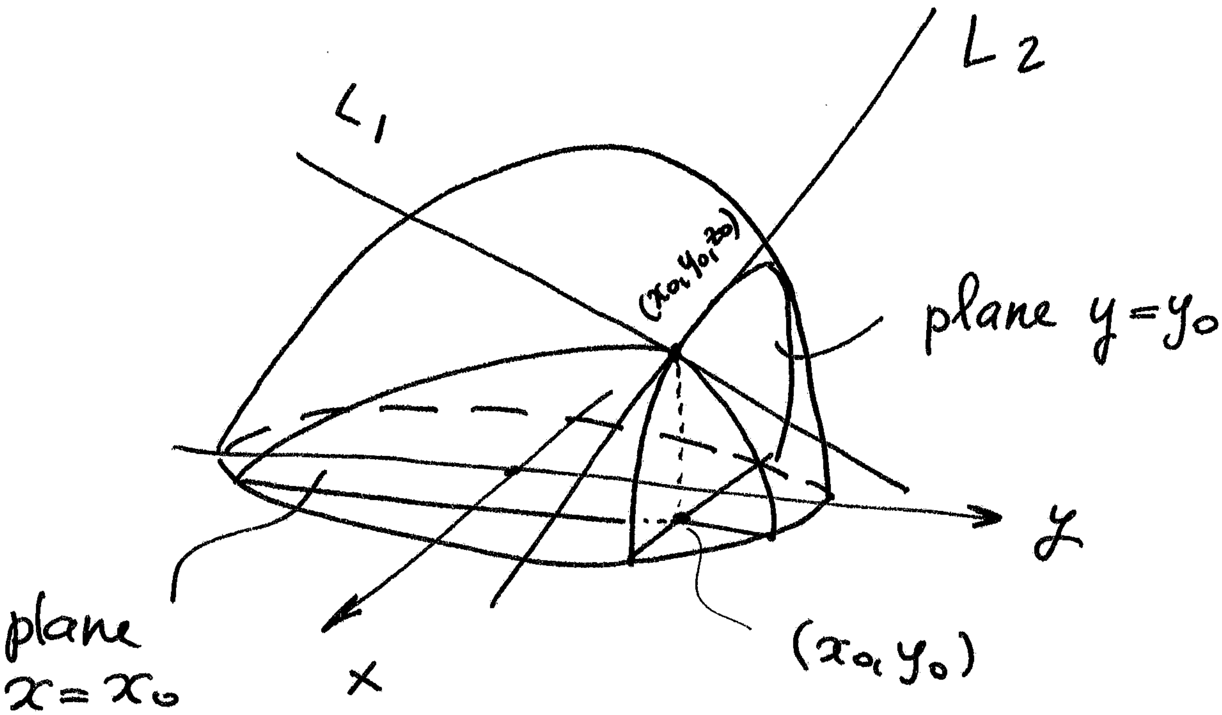
$$f_x(1,0) = g'(1) = -2.$$

Tangent planes

Suppose that S is the surface of the graph of $z = f(x,y)$, where the function f has continuous partial derivatives. In this situation the surface S is smooth and it has the tangent plane at every point of S. What is a tangent plane to S at (x_0, y_0, z_0), $z_0 = f(x_0, y_0)$?

Intersect the surface S with the plane $x = x_0$. In the intersection we obtain a curve. The line L_1 tangent to that curve at (x_0, y_0, z_0) is also tangent to S. Now we intersect the surface S with the plane $y = y_0$. In the intersection we obtain another curve. The line L_2 tangent to that curve at (x_0, y_0, z_0) is also tangent to S. Since both lines L_1 and L_2 are tangent to S at (x_0, y_0, z_0) it is natural to define the tangent plane to S at (x_0, y_0, z_0) as the plane that contains both lines L_1 and L_2.

The next picture explains geometrically the situation.



Observe that this picture combines both pictures from p. 171. The only difference is that we have now the point (x_0, y_0) instead of (a, b). But this is just a notation.

As we observed on p. 171 the slope of L_1 is $\frac{\partial f}{\partial y}(x_0, y_0)$ and the slope of L_2 is $\frac{\partial f}{\partial x}(x_0, y_0)$. Hence the equation of the tangent plane at (x_0, y_0, z_0), i.e. the plane that contains both lines L_1 and L_2 should involve partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$.

The curve obtained in the intersection of the surface S with the plane $y = y_0$ has a parametrization

$$\vec{r_2}(x) = \langle x, y_0, f(x, y_0) \rangle$$

and

$$\vec{r_2}'(x_0) = \langle 1, 0, f_x(x_0, y_0) \rangle$$

is tangent to that curve at (x_0, y_0, z_0). Hence this vector is parallel to L_2.

Similarly the curve obtained in the intersection of the surface S with the plane $x = x_0$ has a parametrization

$$\vec{r_1}(y) = \langle x_0, y, f(x_0, y) \rangle$$

with the tangent vector

$$\vec{r_1}(y_0) = \langle 0, 1, f_y(x_0, y_0) \rangle$$

at (x_0, y_0, z_0). This vector is parallel to L_1.

Now a normal vector to the plane is

$$\vec{r_2}(x_0) \times \vec{r_1}(y_0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} =$$

$$\langle -f_x, -f_y, 1 \rangle$$

Hence the equation of the tangent plane is given by

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

i.e.

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

The equation of the tangent plane should approximate very well values of $f(x, y)$ when (x, y) is near (x_0, y_0)

We can write

$$(*) \quad z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

for the equation of the tangent plane.

The value of z given by this equation is not necessarily equal $f(x,y)$, but the difference between $f(x,y)$ and z given by $(*)$ should be a small error $E(x,y)$, so we can write

$$f(x,y) = f(x_0, y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + E(x,y)$$

One can prove that if the function f has continuous partial derivatives f_x and f_y at (x_0,y_0), then

$$(**) \qquad \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \xrightarrow[(x,y)\to(x_0,y_0)]{} 0$$

i.e. the error $E(x,y)$ is much smaller than the distance between (x,y) and (x_0,y_0).

We say that a function f is differentiable at (x_0,y_0) if partial derivatives $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and $(**)$ is true

Thus we have

Theorem If partial derivatives f_x, f_y are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0)

We should understand the definition of differentiability as follows. Imagine that (x, y) is very close to (x_0, y_0), so close that

$$\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \approx 0.01$$

We draw both the graph of the function and the tangent plane with a large magnification so large that on the picture the distance between (x, y) and (x_0, y_0) is 10 cm

Since $\sqrt{(x-x_0)^2+(y-y_0)^2}$ is 10 cm on the picture,

$$E(x,y) \approx 0.01 \cdot 10\,\text{cm} = 0.1\,\text{cm}$$

on the same picture. That means

the distance between the graph of $f(x,y)$ and the graph of the tangent plane is about 1 mm on that picture, and we are unable to distinguish the graph of f from the tangent plane. In other words in a small neighborhood of (x_0, y_0) the graph of f looks pretty much like the tangent plane.

That means the equation of the tangent plane approximates $f(x,y)$ very well if (x,y) is close to (x_0, y_0).

The function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

i.e. the function which defines the tangent plane is called linearization of f at (x_0,y_0) or linear approximation of f at (x_0,y_0).

If f is differentiable at (x_0, y_0)
$L(x, y)$ is close to $f(x, y)$ near (x_0, y_0)
and we can write

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

when (x, y) is close to (x_0, y_0).

<u>Example</u> Find an approximate value of

$$\sqrt{0.03 + \cos^2 0.045}$$

without using calculator.

<u>Solution</u> Let $f(x, y) = \sqrt{x + \cos^2 y}$

Then

$$f_x = \frac{1}{2\sqrt{x + \cos^2 y}}$$

$$f_y = \frac{2 \cos y (-\sin y)}{2\sqrt{x + \cos^2 y}} = \frac{\sin y \cos y}{\sqrt{x + \cos^2 y}}$$

The functions are continuous at $(0,0)$ and hence f is differentiable at $(0,0)$. Hence

$$f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$
$$= 1 + \frac{1}{2}x$$

when (x,y) is close to $(0,0)$. Since $(0.03, 0.045)$ is close to $(0,0)$ we obtain approximation

$$\sqrt{0.03 + \cos^2 0.045} = f(0.03, 0.045) \approx$$
$$1 + \frac{0.03}{2} = 1.015. \qquad (*)$$

You can check on the calculator that the actual value is

$$\sqrt{0.03 + \cos^2 0.045} = 1.013891694\ldots$$

which shows that $(*)$ is a pretty good approximation.

Theorem If $f(x,y)$ is differentiable at (x_0,y_0), then f is continuous at (x_0,y_0)

Indeed,

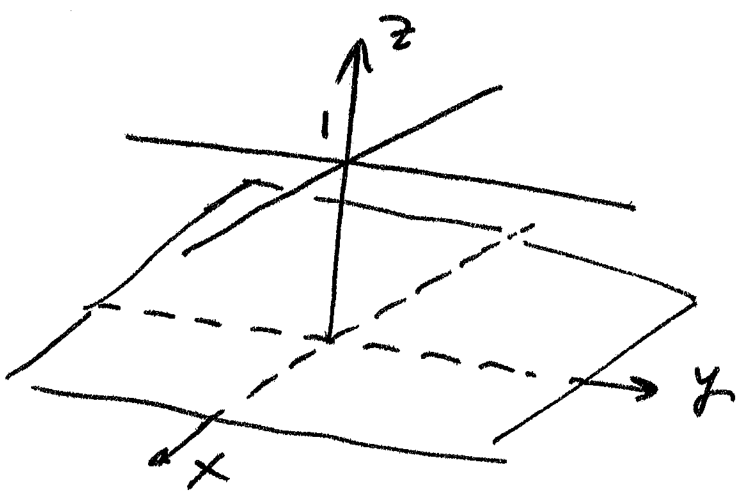
$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\underbrace{(x-x_0)}_{\to 0} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{\to 0} + \underbrace{E(x,y)}_{\to 0}$$

$$\longrightarrow f(x_0,y_0) \quad \text{as } (x,y) \longrightarrow (x_0,y_0).$$

Existence of partial derivatives at (x_0,y_0) does not guarantee differentiability, not even continuity

Example Consider the function

$$f(x,y) = \begin{cases} 1 & \text{if } x=0 \text{ or } y=0 \\ 0 & \text{if } x\neq 0 \text{ and } y \neq 0 \end{cases}$$



Thus the graph is the plane $z=0$ with the x and y axes cut out and elevated to $z=1$. Since

$$g(x) = f(x,0) = 1$$

$$\frac{\partial f}{\partial x}(0,0) = g'(0) = 0$$

and $h(y) = f(0,y) = 1$

yields

$$\frac{\partial f}{\partial y}(0,0) = h'(0) = 0.$$

Thus the equation of the tangent space at $(0,0)$ should be

$$z = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$z = 1.$$

This is the plane spanned by the elevated x and y axis to $z=1$.

However this is not a tangent plane because it does not approximate f well near $(0,0)$ as most of the values of f are equal 0.

The function is not differentiable

at $(0,0)$. It is not even continuous because

$$0 = f(t,t) \to 0 \neq f(0,0).$$

However if the partial derivatives are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0) and the graph of f has a tangent plane at that point.

All the results extend to three and more variables. For example if $w = f(x,y,z)$ has continuous partial derivatives at (x_0, y_0, z_0), then f is differentiable and

$$f(x,y,z) \approx f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

if (x,y,z) is close enough to (x_0, y_0, z_0).

The following fact is often useful.

If a curve $\vec{r}(t) = \langle a(t), b(t), c(t) \rangle$ lies on the graph of

$$z = f(x, y),$$

i.e. $c(t) = f(a(t), b(t))$, then the tangent vector to that curve

$$\vec{r}\,'(t_0) = \langle a'(t_0), b'(t_0), c'(t_0) \rangle$$

is also tangent to the surface at $(x_0, y_0, z_0) = (a(t_0), b(t_0), c(t_0))$ and hence it is in the tangent space at (x_0, y_0, z_0):

Under the above notation $\langle a'(t_0), b'(t_0), c'(t_0) \rangle$ is parallel to the tangent plane to

$$z = f(x, y)$$

at $(x_0, y_0, z_0) = (a(t_0), b(t_0), c(t_0))$

Problems

Exercise 70 Find the equation of the tangent plane to

$$z = \sqrt{xy} \quad \text{at } (1,1,1)$$

Solution

$$\frac{\partial z}{\partial x} = \frac{y}{2\sqrt{xy}} \qquad \frac{\partial z}{\partial x}(1,1) = \frac{1}{2}$$

$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{xy}} \qquad \frac{\partial z}{\partial y}(1,1) = \frac{1}{2}$$

Thus the tangent plane at $(1,1,1)$ is

$$z-1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

i.e.

$$2z = x+y$$

Exercise 71 Explain why is the function

$$f(x,y) = 1 + x \ln(xy-5)$$

differentiable at $(2,3)$. Then find the linearization of f at $(2,3)$.

Solution Partial derivatives

$$\frac{\partial f}{\partial x} = \ln(xy-5) + x\,\frac{y}{xy-5}$$

$$\frac{\partial f}{\partial y} = x\,\frac{x}{xy-5}$$

are continuous at $(2,3)$, so the function is differentiable at $(2,3)$.

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= 1 + 6(x-2) + 4(y-3) = 6x + 4y - 23.$$

Exercise 72 Find the approximate value of

$$1 + 2.01\ln(1.0501).$$

Use the previous exercise.

Solution If $x = 2.01$, $xy - 5 = 1.0501$, then $y = 3.01$ and

$$1 + 2.01\ln(1.0501) = f(2.01, 3.01),$$

where $f(x,y) = 1 + x\ln(xy-5)$.

Since

$$f(x,y) \approx 6x + 4y - 23 \quad \text{near } (2,3)$$

$$f(2.01, 3.01) \approx 12.06 + 12.04 - 23 = 1.1$$

Remark Checking on the calculator we get

$$1 + 2.01 \ln(1.0501) \approx 1.098259\ldots \approx 1.1.$$

Exercise 73 Find the linear approximation of $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ at $(3,2,6)$ and use it to approximate

$$\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$$

Solution $f_x = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$

Similarly

$$f_y = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad f_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$f_x(3,2,6) = \frac{3}{7}, \quad f_y(3,2,6) = \frac{2}{7}, \quad f_z(3,2,6) = \frac{6}{7}$$

$$f(3,2,6) = 7.$$

$L(x, y, z) =$

$f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$

$= 7 + \frac{3}{7}(x-3) + \frac{2}{7}(x-2) + \frac{6}{7}(x-6)$

$= \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z.$

Thus

$\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99)$

$\approx L(3.02, 1.97, 5.99) = \frac{3}{7} \cdot 3.02 + \frac{2}{7} \cdot 1.97 + \frac{6}{7} \cdot 5.99$

$= 6.9914\ldots$

We could use this approximation, because $(3.02, 1.97, 5.99)$ is close to $(3, 2, 6)$.

<u>Exercise 74</u> Suppose you need to know an equation of the tangent plane to a surface S at the point $(2, 1, 3)$. You don't have an equation for S but you know that the curves

$\vec{r}_1(t) = \langle 2+3t, 1-t^2, 3-4t+t^2 \rangle$ (211)

$\vec{r}_2(t) = \langle 1+u^2, 2u^3-1, 2u+1 \rangle$

both lie on S. Find an equation of the tangent plane at (2,1,3)

<u>Solution</u> Tangent vectors to $\vec{r}_1$ and $\vec{r}_2$ are tangent to the surface

$$\vec{r}_1(0) = (2,3,1)$$

$$\vec{r}_2(1) = (2,3,1)$$

Thus the vectors

$$\vec{r}_1{}'(0) \text{ and } \vec{r}_2{}'(1)$$

are tangent to S at (2,3,1) and hence they define the tangent plane.

We have

$\vec{r}_1{}' = \langle 3, -2t, -4+2t \rangle$, $\vec{r}_1{}'(0) = \langle 3, 0, -4 \rangle$

$\vec{r}_2{}' = \langle 2u, 6u^2, 2 \rangle$, $\vec{r}_2{}'(1) = \langle 2, 6, 2 \rangle$.

Thus the tangent plane at (2,1,3)

is parallel to vectors

$$\vec{r}_1{}'(0) = \langle 3, 0, -4 \rangle, \quad \vec{r}_2{}'(1) = \langle 2, 6, 2 \rangle$$

and it passes through $(2, 3, 1)$.

The normal vector is

$$\vec{r}_1{}'(0) \times \vec{r}_2{}'(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} = \langle 24, -14, 18 \rangle$$

and hence the equation is

$$24(x-2) - 14(y-3) + 18(z-1) = 0.$$

<u>Exercise 75</u> A function $z(x, y)$ is given implicitly by the equation

$$yz - \ln z = x + y.$$

Find equation of the tangent plane to the graph of z at the point $(-1, 0, e)$.

<u>Solution</u> We have already seen

this function in an example on page 175, but we will repeat all computations. To find partial derivatives we will use implicit differentiation

$$yz - \ln z = x + y$$

$$\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x+y)$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}} = \frac{z}{yz-1}$$

$$\frac{\partial z}{\partial x}(-1,0) = \frac{e}{0 \cdot e - 1} = -e$$

$$\frac{\partial}{\partial y}(yz - \ln z) = \frac{\partial}{\partial y}(x+y)$$

$$z + y\frac{\partial z}{\partial y} - \frac{1}{z}\frac{\partial z}{\partial y} = 1$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial y} = 1 - z$$

$$\frac{\partial z}{\partial y} = \frac{1-z}{y - \frac{1}{z}} = \frac{z - z^2}{yz - 1}$$

$$\frac{\partial z}{\partial y}(-1,0) = \frac{e-e^2}{0\cdot e-1} = e^2-e.$$

The tangent plane is

$$z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$$z-e = -e(x+1)+(e^2-e)(y-0)$$

$$ex+(e-e^2)y+z = 0.$$

Remark Notation is quite confusing. In the last equation z is not the same as the original function z.

The chain rule

The chain rule tells us how to differentiate a composition of functions. For functions of two variables we have two most important cases

Case I Suppose that $z = f(x,y)$ and that $x(t)$ and $y(t)$ are functions of one variable. Then

$$z = f(x(t), y(t))$$

is also a function of one variable and

$$\frac{dz}{dt} = \frac{d}{dt}\left(f(x(t), y(t))\right) =$$

$$\frac{\partial f}{\partial x}(x(t), y(t))\frac{dx(t)}{dt} + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy(t)}{dt}.$$

Shortly:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example If $z = x^2y + 3xy^4$, $x = \sin 2t$, $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$

Solution 1 According to the chain rule we have

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} =$$

$$(2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6$$

Solution II. In this case we could also use the methods of Calculus I. Namely

$$z = x^2y + 3xy^4 = (\sin 2t)^2\cos t + 3(\sin 2t)(\cos t)^4$$

$$\frac{dz}{dt} = 2(\sin 2t)(2\cos 2t)\cos t +$$
$$(\sin 2t)^2(-\sin t) +$$
$$3(2\cos 2t)(\cos t)^4 +$$
$$3(\sin 2t)\,4(\cos t)^3(-\sin t) =$$

At $t=0$

$$\left.\frac{dz}{dt}\right|_{t=0} = 6.$$

Observe that taking out the common factors $2\cos 2t$ and $-\sin t$

we obtain

$$\frac{dz}{dt} = \left(2(\sin 2t)\cos t + 3(\cos t)^4\right)(2\cos 2t) +$$

$$+\left((\sin 2t)^2 + 12(\sin 2t)(\cos t)^3\right)(-\sin t)$$

$$= \left(2xy + 3y^4\right)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

In that case we could verify the chain rule from Calculus 3 from the chain rule from Calculus 1, but in general it will not be possible.

Now we will use the chain rule to verify the result from p. 206.

Theorem If $\vec{r}(t) = \langle a(t), b(t), c(t)\rangle$ lies on the graph of $z = f(x,y)$, then $\vec{r}\,'(t_0) = \langle a'(t_0), b'(t_0), c'(t_0)\rangle$ is parallel to the tangent plane to the graph of f at $(x_0, y_0, z_0) = (a(t_0), b(t_0), c(t_0))$.

Proof The fact that $\vec{r}(t)$ lies on the graph of f means that

$$c(t) = f(a(t), b(t)).$$

Hence

$$\vec{r}\,'(t) = \langle a'(t), b'(t), c'(t) \rangle =$$

$$\langle a'(t), b'(t), \frac{d}{dt} f(a(t), b(t)) \rangle.$$

$$\frac{d}{dt} f(\underbrace{a(t)}_{x}, \underbrace{b(t)}_{y}) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= f_x\, a'(t) + f_y\, b'(t)$$

$$\left.\frac{d}{dt}\right|_{t=t_0} f(a(t), b(t)) =$$

$$= f_x(x_0, y_0)\, a'(t_0) + f_y(x_0, y_0)\, b'(t_0)$$

because $x_0 = a(t_0)$, $y_0 = b(t_0)$.

Hence

$$\vec{r}\,'(t_0) = \langle a'(t_0), b'(t_0), f_x(x_0, y_0)a'(t_0) + f_y(x_0, y_0)\, b'(t_0) \rangle$$

The equation of the tangent plane at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Hence

$$\vec{n} = \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \rangle$$

is normal to the tangent plane.
In order to show that $\vec{r}\,'(t_0)$ is tangent it suffices to show that

$$\vec{n} \cdot \vec{r}\,'(t_0) = 0.$$

We have

$$\langle -f_x, -f_y, 1 \rangle \cdot \langle a', b', f_x a' + f_y b' \rangle =$$

$$= -f_x a' - f_y b' + f_x a' + f_y b' = 0. \quad \square$$

<u>Case II</u> If $z = f(x, y)$ and $x = g(s, t)$, $y = h(s, t)$ depend on two variables, then

$$z = f(g(s,t), h(s,t))$$

is also a function of two variables and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Indeed, in order to find for example $\frac{\partial z}{\partial t}$ we fix variable s, so s is regarded as a constant and hence

$$z = f\,(g(s,t),\, h(s,t))$$

is regarded as a function of one variable t. Thus

$$\frac{\partial z}{\partial t} = \frac{dz}{dt} = \frac{d}{dt} f\,(\underbrace{g(s,t)}_{x(t)},\, \underbrace{h(s,t)}_{y(t)}) \underset{(\text{case I})}{=} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} =$$

$$= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

<u>Example</u> If $g(s,t) = f\,(s^2 - t^2,\, t^2 - s^2)$, where f is differentiable, then g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

Indeed, $x = s^2 - t^2$, $y = t^2 - s^2$ and hence

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Hence

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(\cancel{\frac{\partial f}{\partial x}\cdot 2ts} - \cancel{\frac{\partial f}{\partial y}2ts}\right) +$$

$$+ \left(-\cancel{\frac{\partial f}{\partial x}\cdot 2st} + \cancel{\frac{\partial f}{\partial y}2st}\right) = 0.$$

The chain rule generalizes to more than two variables. For example

$$\frac{d}{dt} f(x(t), y(t), z(t)) =$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\frac{\partial}{\partial s} f(x(s,t), y(s,t), z(s,t)) =$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial}{\partial t} f(x(s,t), y(s,t), z(s,t)) =$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Example This is a difficult example, but if you understand it, it means you understand a lot.

The usual way to compute the derivative of $f(x) = x^x$ is to write the function as $f(x) = e^{x \ln x}$ and use the chain rule from Calculus 1. In this example we will find a different argument based on a chain rule from Calculus 3. Let

$$g(x,y) = x^y, \quad x(t) = t, \quad y(t) = t.$$

Thus

$$t^t = g(x(t), y(t)).$$

We have

$$(t^t)' = \frac{d}{dt}\, g(x(t), y(t)) = \frac{\partial g}{\partial x}\underbrace{\frac{dx}{dt}}_{1} + \frac{\partial g}{\partial y}\underbrace{\frac{dy}{dt}}_{1} =$$

$$= \frac{\partial}{\partial x}(x^y) + \frac{\partial}{\partial y}(x^y) = \heartsuit.$$

Recall that

$$(x^a)' = ax^{a-1}, \quad (a^y)' = a^y \ln a$$

$$\heartsuit = y\, x^{y-1} + x^y \ln x =$$

$$= t \cdot t^{t-1} + t^t \ln t =$$

$$= t^t (1 + \ln t).$$

$$(t^t)' = t^t (1 + \ln t)$$

or

$$(x^x)' = x^x (1 + \ln x).$$

Implicit differentiation

We have already seen how to use implicit differentiation for functions of several variables, but the chain rule allows us to show a general approach.

Suppose that a curve is given by an equation

$$F(x,y) = 0.$$

Suppose that y is a function of x, so

$$F(x, y(x)) = 0.$$

Thus

$$0 = \frac{d}{dx} F(x\, y(x)) =$$

$$= \frac{\partial F}{\partial x} \underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y} \frac{dy(x)}{dx}$$

$$\frac{\partial F}{\partial y} \frac{dy}{dx} = - \frac{\partial F}{\partial x}$$

and hence

$$\frac{\partial y}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y},$$

provided $F_y \neq 0$.

Similar method applies to functions of three variables.

Suppose that a surface is described by an equation

$$F(x, y, z) = 0.$$

Suppose also that z is a function of x and y, so

$$F(x, y, z(x,y)) = 0.$$

then

$$0 = \frac{\partial}{\partial x} F(x, y, z(x,y)) = \frac{\partial F}{\partial x} \underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y} \underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{F_x}{F_z}$$

and

$$0 = \frac{\partial}{\partial y} F(x, y, z(x, y)) = \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial y}}_{0} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial y}}_{1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$$

$$= \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{F_y}{F_z}.$$

We obtained

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

This is true provided $F_z \neq 0$.

Problems

Exercise 76 Find $\partial z/\partial y$ where

$$xyz = \cos(x+y+z)$$

Solution I We regard z as a function of two variables x and y. Recall that if

$$F(x,y,z) = 0,$$

then

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

In our problem

$$F(x,y,z) = xyz - \cos(x+y+z) = 0$$

$$F_y = xz + \sin(x+y+z)$$

$$F_z = xy + \sin(x+y+z)$$

$$\frac{\partial z}{\partial y} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}.$$

Solution II We can also find the derivative $\partial z/\partial y$ more directly without memorizing any formula.

$$\frac{\partial}{\partial y}(xyz) = \frac{\partial}{\partial y}\cos(x+y+z)$$

Since x and y are independent variables and z is a function of x and y we get

$$xz + xy\frac{\partial z}{\partial y} = -\sin(x+y+z)\left(1+\frac{\partial z}{\partial y}\right)$$

$$xz + xy\frac{\partial z}{\partial y} = -\sin(x+y+z) - \sin(x+y+z)\frac{\partial z}{\partial y}$$

$$xz + \sin(x+y+z) = -(xy+\sin(x+y+z))\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} = -\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}.$$

Exercise 77 Prove that if $u(x,y)$ is harmonic, then $\omega(s,t) = u(s+t, s-t)$ is harmonic too.

Proof We can write

$$\omega(s,t) = u(x(s,t), y(s,t))$$

where $x(s,t) = s+t$, $y(s,t) = s-t$.

Then the chain rule gives

$$w_s = \frac{\partial w}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= u_x(s+t, s-t)\frac{\partial (s+t)}{\partial s} + u_y(s+t, s-t)\frac{\partial (s-t)}{\partial s}$$

$$= u_x(s+t, s-t) + u_y(s+t, s-t)$$

$$= u_x + u_y.$$

Now we compute a second order derivative using the chain rule

$$w_{ss} = (u_x)_x \frac{\partial(s+t)}{\partial s} + (u_x)_y \frac{\partial(s-t)}{\partial s} +$$

$$+ (u_y)_x \frac{\partial(s+t)}{\partial s} + (u_y)_y \frac{\partial(s-t)}{\partial s}$$

$$= u_{xx} + u_{xy} + u_{yx} + u_{yy}$$

$$= u_{xx} + 2u_{xy} + u_{yy}$$

Similar computations apply to derivatives with respect to t

$$w_t = u_x \frac{\partial(s+t)}{\partial t} + u_y \frac{\partial(s-t)}{\partial t} = u_x - u_y$$

$$w_{tt} = (u_x)_x \frac{\partial(s+t)}{\partial t} + (u_x)_y \frac{\partial(s-t)}{\partial t} -$$

$$-\left((u_y)_x \frac{\partial(s+t)}{\partial t} + (u_y)_y \frac{\partial(s-t)}{\partial t} \right)$$

$$= u_{xx} - u_{xy} - (u_{yx} - u_{yy})$$

$$= u_{xx} - 2u_{xy} + u_{yy}$$

Now

$$w_{ss} + w_{tt} = u_{xx} + 2u_{xy} + u_{yy} + u_{xx} - 2u_{xy} + u_{yy}$$

$$= 2(u_{xx} + u_{yy}) = 0$$

because u is harmonic.

Exercise 78 Suppose that the equation $F(x,y,z) = 0$ implicitly defines each of the three variables x, y and z as a function of the other two $z = f(x,y)$, $y = g(x,z)$, $x = h(y,z)$. Prove that if F is differentiable, $F_x \neq 0$, $F_y \neq 0$ and $F_z \neq 0$, then

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1$$

Recall that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.$$

By analogy we also have

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}, \quad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$

Hence

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = \left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)$$

$$= -\frac{F_x F_y F_z}{F_z F_x F_y} = -1.$$

<u>Exercise 79</u> A function f with continuous second-order partial derivatives satisfies the identity

$$(*) \qquad f(tx, ty) = t^5 f(x,y)$$

for all $(x,y) \in \mathbb{R}^2$ and all $t > 0$. Simplify the expression

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2}.$$

<u>Hint:</u> Differentiate (*) twice with respect to t. Apply the chain rule and then take $t=1$.

<u>Solution</u> Clearly

$$\frac{d^2}{dt^2}\left(f(tx,ty)\right) = \frac{d^2}{dt^2}\left(t^5 f(x,y)\right)$$

because (*) is true for all $t>0$.

$$LHS = \frac{d}{dt}\left(\frac{\partial f}{\partial x}(tx,ty)\,x + \frac{\partial f}{\partial y}(tx,ty)\,y\right)$$

$$= \left(\frac{\partial^2 f}{\partial x^2}(tx,ty)x + \frac{\partial^2 f}{\partial y\,\partial x}(tx,ty)y\right)x + \left(\frac{\partial^2 f}{\partial x\,\partial y}(tx,ty)x + \frac{\partial^2 f}{\partial y^2}(tx,ty)y\right)y$$

$$= \frac{\partial^2 f}{\partial x^2}(tx,ty)\,x^2 + 2\frac{\partial^2 f}{\partial y\,\partial x}(tx,ty)\,xy + \frac{\partial^2 f}{\partial y^2}(tx,ty)\,y^2$$

$$RHS = 20t^3 f(x,y)$$

Taking $t=1$ we obtain

$$LHS = x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2}$$

and

$$RHS = 20\, f(x,y).$$

Since $LHS = RHS$ for all $t>0$ we have equality for $t=1$, i.e.

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 20 f(x,y).$$

Exercise 80 Let

$$f(x) = \int_0^{\arctan x} e^{\tan t}\, dt.$$

Find $f'(x)$.

This and the next problem have nothing to do with the chain rule for functions of several variables, but the two problems are just an introduction to Exercise 82.

<u>Solution</u> If $g(s) = \int_0^s e^{\tan t}\,dt$,

then

$$f(x) = g(\arctan x)$$

Recall that according to the Fundamental Theorem of Calculus

$$g'(s) = \frac{d}{ds} \int_0^s e^{\tan t}\,dt = e^{\tan s}.$$

Hence the chain rule yields

$$f'(x) = \frac{d}{dx}\, g(\arctan x) =$$

$$g'(\arctan x)\,\frac{1}{1+x^2} =$$

$$e^{\tan(\arctan x)}\,\frac{1}{1+x^2} = \frac{e^x}{1+x^2}.$$

<u>Exercise 81</u> Let

$$F(x) = \int_{\sin x}^{\cos x} e^{(t^2+t)}\,dt.$$

Find $F'(0)$.

Solution Let

$$g(s) = \int_0^s e^{t^2+t}\,dt.$$

Then

$$F(x) = \int_{\sin x}^{\cos x} e^{t^2+t}\,dt = \int_0^{\cos x} + \int_{\sin x}^{0}$$

$$= \int_0^{\cos x} e^{t^2+t}\,dt - \int_0^{\sin x} e^{t^2+t}\,dt$$

$$= g(\cos x) - g(\sin x).$$

Thus as in the previous problem

$$F'(x) = g'(\cos x)(-\sin x) - g'(\sin x)\cos x$$

$$= e^{\cos^2 x + \cos x}(-\sin x) - e^{\sin^2 x + \sin x}\cos x$$

$$F'(0) = e^2 \cdot 0 - e^0 \cdot 1 = -1.$$

Exercise 82 Let

$$F(x) = \int_{\sin x}^{\cos x} e^{(t^2+xt)}\,dt.$$

Find $F'(0)$.

Remark This problem is very different from Exercise 81. Indeed, now the function

$$g = \int_0^s e^{t^2+xt}\,dt$$

depend both on s and x. Previously x was only present in the limits of the integral, but now it's also present in the function that we integrate. Thus we can write

$$g(s, x) = \int_0^s e^{t^2+xt}\,dt$$

and

$$(*) \quad F(x) = g(\cos x, x) - g(\sin x, x)$$

<u>Solution</u> We could differentiate F using (*) and the chain rule in two variables, but we will show here a slightly different approach with the chain rule in three variables.

Let

$$G(u,v,w) = \int_v^u e^{t^2+wt}\,dt.$$

Then

$$F(x) = G(\cos x, \sin x, x).$$

Hence

$$F'(x) = \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial G}{\partial w}\frac{\partial w}{\partial x}.$$

$$\frac{\partial G}{\partial u} = e^{u^2+wu} \qquad \text{(Fund. Thm. Calc.)}$$

$$\frac{\partial G}{\partial v} = -e^{v^2+wv} \qquad \text{(Fund. Thm. Calc.)}$$

$$\frac{\partial G}{\partial w} = \int_v^u e^{t^2+wt}\cdot t\,dt.$$

We just differentiate under the

sign of the integral. Thus

$$F'(x) = e^{u^2+\omega u}(-\sin x) - e^{v^2+\omega v}\cos x$$

$$+ \int_v^u t\, e^{t^2+\omega t}\, dt \cdot 1$$

$u(x) = \cos x$, $u(0) = 1$. Hence

$u^2 + \omega u = 1^2 + 0\cdot 1 = 1$ at $x = 0$

$v(x) = \sin x$, $v(0) = 0$. Hence

$v^2 + \omega v = 0^2 + 0\cdot 0 = 0$.

Thus

$$F'(0) = e^1\cdot 0 - e^0\cdot 1 + \int_0^1 t\, e^{t^2+0\cdot t}\, dt$$

$$= -1 + \int_0^1 t e^{t^2} dt = -1 + \frac{1}{2} e^{t^2}\Big|_0^1$$

$$= -1 + \frac{1}{2}(e^1 - e^0) = \frac{e-3}{2}.$$

Directional derivative and the gradient vector

Recall that

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

$$= \lim_{h\to 0} \frac{f((x_0,y_0)+h(1,0))-f(x_0,y_0)}{h}$$

Thus $f_x(x_0,y_0)$ is the rate of change of f in the direction of the vector $\vec{i} = \langle 1,0\rangle$; it is a directional derivative in the direction of $\vec{i}$.

Similarly

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

$$= \lim_{h\to 0} \frac{f((x_0,y_0)+h(0,1))-f(x_0,y_0)}{h}$$

so $f_y(x_0,y_0)$ is the rate of change of f in the direction of the vector

$\vec{j} = \langle 0,1 \rangle$; it is a directional derivative in the direction of $\vec{j}$.

More generally we define the directional derivative of f in the direction of any unit

$$\vec{u} = \langle a, b \rangle$$

by

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f((x_0, y_0) + h(a,b)) - f(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

With this notation

$$f_x = D_{\vec{i}} f \quad \text{and} \quad f_y = D_{\vec{j}} f.$$

We assume that $\vec{u}$ is a unit vector, because we are interested in the direction in which we take the derivative and the magnitude of $\vec{u}$ does not change the direction. Thus we assume that

$$|\vec{u}| = 1.$$

Theorem If a function $f(x,y)$ is differentiable, then it has a directional derivative in any direction $\vec{u} = \langle a, b \rangle$ and

$$D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b.$$

Proof. Let $g(h) = f(x_0+ah, y_0+bh)$. Then

$$g'(0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0, y_0)}{h}$$

$$= D_{\vec{u}} f(x_0, y_0).$$

On the other hand the chain rule gives

$$g'(0) = \frac{d}{dh}\Big|_{h=0} f(x_0+ah, y_0+bh) =$$

$$f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b.$$

Comparing the two formulas for $g'(0)$ yields

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b. \quad \square$$

Observe that

$$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b =$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a,b \rangle =$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}.$$

This observation justifies the following definition.

If f is a function of two variables, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle.$$

Hence

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}.$$

<u>Example</u> Find the directional derivative of $f(x,y) = x^2 + xy$ at the point $(1,2)$ in the direction of the vector $\vec{v} = \vec{i} + \vec{j}$.

Solution $\nabla f(x,y) = \langle 2x+y, x\rangle$.

The unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1,1\rangle}{\sqrt{2}}.$$

Hence

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u} =$$

$$= \langle 2x+y, x\rangle \cdot \frac{\langle 1,1\rangle}{\sqrt{2}} = \frac{2x+y}{\sqrt{2}} + \frac{x}{\sqrt{2}}$$

$$= \frac{3x+y}{\sqrt{2}}$$

$$D_{\vec{u}} f(1,2) = \frac{3\cdot 1+2}{\sqrt{2}} = \frac{5}{\sqrt{2}}.$$

The definition of directional derivative and that of the gradient easily extend to three and more variables.

For a unit vector $\vec{u} = \langle a,b,c\rangle$ we define

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0+ah, y_0+bh, z_0+ch) - f(x_0, y_0, z_0)}{h}$$

If f is differentiable, then

$$D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$$

where

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle.$$

Maximizing the directional derivative.

In general a function of two or more variables changes with different rates in different directions. Indeed, the rate of change of f in a direction $\vec{u}$ is precisely $D_{\vec{u}} f$. The question is: In what direction is the function f increasing most rapidly and what is the maximum rate of change?

In other words we want to find the direction (unit vector) $\vec{u}$ for which the directional derivative $D_{\vec{u}} f$ has the maximum value and we want to find that maximum value.

<u>Theorem</u> Suppose f is a differentiable function of two or more variables. The maximum value of the directional derivative $D_{\vec{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$

Here $\mathbf{x}$ stands for a point in the domain. Depending on the number of variables we have $\mathbf{x} = (x, y)$ or $\mathbf{x} = (x, y, z)$ or ...

<u>Proof</u> $D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos\theta$

$$= |\nabla f| \cos\theta$$

The maximum value is attained when $\cos\theta = 1$, i.e. $\theta = 0$. This maximum value equals

$$|\nabla f| \cos 0 = |\nabla f|$$

and $\theta = 0$ means that the vector $\vec{u}$ has the same direction as ∇f. $\square$

Since f increases most rapidly in the direction of ∇f, it decreases most rapidly in the opposite direction, i.e. in the direction of $-\nabla f$ and the rate of change is $-|\nabla f|$.

<u>**Example**</u> Find the directions in which $f(x,y) = \frac{1}{2}(x^2+y^2)$

(a) Increases most rapidly at the point $(1,1)$

(b) Decreases most rapidly at the point $(1,1)$

(c) Has zero rate of change at the point $(1,1)$.

<u>Solution</u> $\nabla f(x,y) = \langle x, y \rangle$

$\nabla f(1,1) = \langle 1,1 \rangle$

(a) f increases most rapidly in the direction

$$\vec{u} = \frac{\langle 1,1 \rangle}{\sqrt{2}}.$$

It increases with the rate

$$|\nabla f(1,1)| = \sqrt{2}$$

(b) f decreases most rapidly in the direction

$$-\vec{u} = \frac{-\langle 1,1 \rangle}{\sqrt{2}}$$

It decreases with the rate

$$-|\nabla f(1,1)| = -\sqrt{2}$$

(c) We are looking for unit vectors $\vec{v}$ such that

$$D_{\vec{v}} f(1,1) = \nabla f(1,1) \cdot \vec{v} = 0$$

If $\vec{v} = \langle a, b \rangle$ then the equation becomes

$$\langle 1,1\rangle \cdot \langle a,b\rangle = 0$$
$$a+b=0$$
$$b=-a \qquad (*)$$

Since $\vec{v}$ is unit vector,

$$\sqrt{a^2+b^2} = 1$$

Hence $(*)$ gives

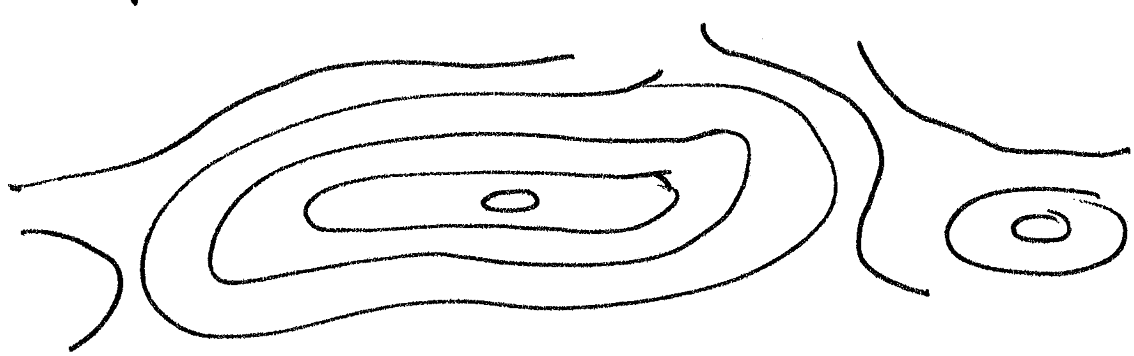
$$\sqrt{2a^2} = 1$$
$$|a| = \frac{1}{\sqrt{2}}$$

and the answer is

$$\vec{v}_1 = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle, \quad \vec{v}_2 = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

Level curves and level surfaces

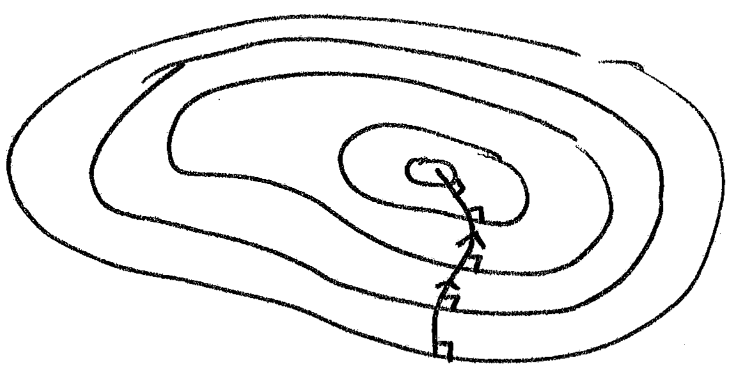
Think of a topographic map that represent mountains.



In mathematical terms such a topographic map represents level curves

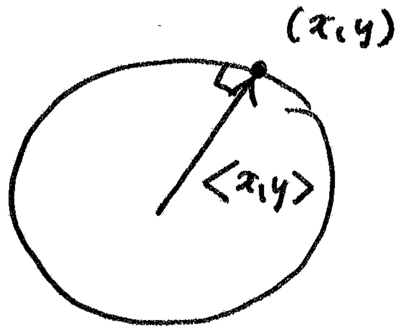
$$f(x,y) = k\ , \qquad k = \text{const}.$$

Now it is intuitively obvious that in order to claimb a mountain as fast as possible we should go in the direction orthogonal to level curves



Claimbing a mountain "as fast as possible" simply means going in the direction in which the function $f(x,y)$ that represents height of the mountain increases most rapidly, i.e. going in the direction of $\nabla f(x,y)$. Thus the intuition suggest that the gradient vector should be orthogonal to level curves.

<u>Example</u> If $f(x,y) = \frac{1}{2}(x^2+y^2)$, then the level curves are circles centered at the origin. $\nabla f(x,y) = \langle x, y \rangle$ is clearly orthogonal to these circles



Similarly one can expect that the gradient $\nabla f(x,y,z)$ should be orthogonal to the level surfaces

$$f(x,y,z) = k, \quad k\text{-constant.}$$

Now we will focus on a rigorous proof of this fact.

Implicit differentiation again

Consider a surface described by the equation

$$F(x, y, z) = k. \quad k - \text{const}$$

If $F_z(x_0, y_0, z_0) \neq 0$, then in a neighborhood of (x_0, y_0, z_0), z can be represented as a function of x, y and

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

If $F_y(x_0, y_0, z_0) \neq 0$, then in a neighborhood of (x_0, y_0, z_0), y can be represented as a function of x, z and

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}, \quad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$

If $F_x(x_0, y_0, z_0) \neq 0$, then in a neighborhood of (x_0, y_0, z_0), x can be represented as a function of y, z

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}, \quad \frac{\partial x}{\partial z} = -\frac{F_z}{F_x}.$$

Note that $\nabla F = \langle F_x, F_y, F_z \rangle$.

If $\nabla F(x_0, y_0, z_0) \neq \vec{0}$ then one of the partial derivatives F_x, F_y, F_z is $\neq 0$ at (x_0, y_0, z_0) and hence the corresponding variable can be represented as a function of the remaining two variables. For example if $F_y(x_0, y_0, z_0) \neq 0$, then $y = y(x, z)$,

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}, \quad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}.$$

Thus if $\nabla F \neq \vec{0}$ at every point of the surface

$$F(x, y, z) = k$$

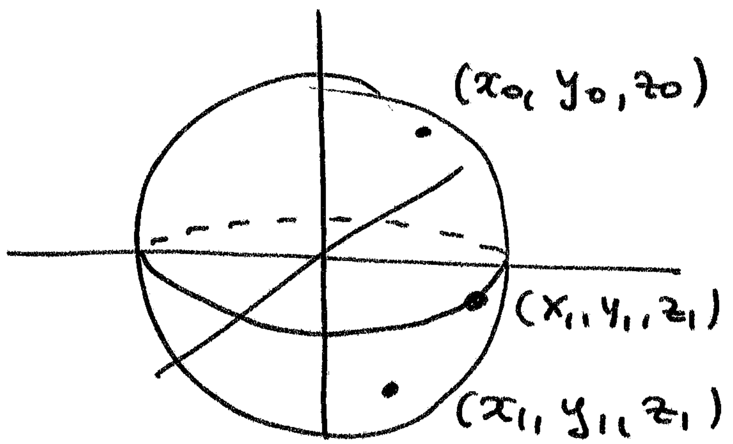
then locally the surface is a graph of a function of two variables. Depending where we are on the surface it may be a graph of $z = f(x, y)$, $y = g(x, z)$ or $x = h(x, y)$.

Let us explain it on a simple example.

<u>Example</u> $F(x,y,z) = x^2+y^2+z^2=1$ represents the unit sphere.

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x, 2y, 2z \rangle \neq \vec{0}$$

because $x \neq 0$, $y \neq 0$ or $z \neq 0$ on the sphere.



In the upper hemisphere, above the equator, $z_0 \neq 0$ and in a neighborhood of (x_0, y_0, z_0) the surface is the graph of

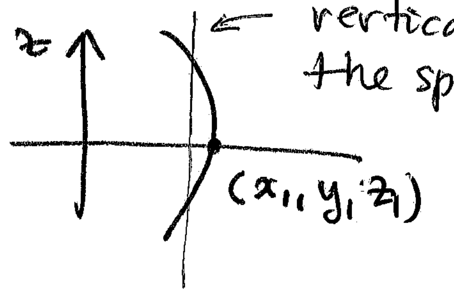
$$z = \sqrt{1-x^2-y^2}.$$

Below the equator $z_1 \neq 0$ and still in a neighborhood of (x_1, y_1, z_1) the surface of the sphere is the graph of

$$z = -\sqrt{1 - x^2 - y^2}$$

although we have a different function now.

On the equator $z_1 = 0$ and in a neighborhood of (x_1, y_1, z_1) z cannot be represented as a function of x, y because the vertical line test fails



vertical line intersects the sphere in two points no matter how close to (x_1, y_1, z_1) we are.

But then $x_1 \neq 0$ or $y_1 \neq 0$. If $x_1 \neq 0$ then x is a function of y, z and of $y_1 \neq 0$ then $y = y(x, z)$.

If $x_0 \neq 0$, $y_0 \neq 0$, $z_0 \neq 0$, say $x_0 > 0$, $y_0 < 0$, $z_0 > 0$ then we can represent the sphere near the point (x_0, y_0, z_0) in three different ways

$$x = \sqrt{1-x^2-y^2} \quad \text{or} \quad y = -\sqrt{1-x^2-y^2}$$

$$\text{or} \quad z = \sqrt{1-x^2-y^2}.$$

Thus no matter at what point on the sphere we are, a part of the sphere near that point can be represented as a graph of

$$x = f(y,z), \quad y = g(x,z) \quad \text{or} \quad z = h(x,y).$$

The same is true for any surface

$$F(x,y,z) = k$$

if $\nabla F \neq 0$ on that surface.

Let us repeat again:

Consider a surface

$$F(x,y,z) = k, \qquad k = \text{const}.$$

If $\nabla F(x_0,y_0,z_0) \neq \vec{0}$, then a part of the surface near the point (x_0,y_0,z_0) can be represented as one of the graphs

$$x = f(y,z) \quad \text{or} \quad y = g(x,z) \quad \text{or} \quad z = h(x,y).$$

The surface is smooth near this point and it has a tangent plane at (x_0,y_0,z_0). We will find the equation of the tangent plane.

If $F_z(x_0,y_0,z_0) \neq 0$, then

$$z = f(x,y)$$

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

The tangent plane to the surface

$$F(x, y, z) = k$$

at (x_0, y_0, z_0) is the tangent plane to $z = f(x, y)$ at (x_0, y_0, z_0) and hence it has the equation

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$z - z_0 = -\frac{F_x}{F_z}(x - x_0) - \frac{F_y}{F_z}(y - y_0)$$

$$F_z(z - z_0) = -F_x(x - x_0) - F_y(y - y_0)$$

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

If we assume that e.g. $F_x(x_0, y_0, z_0) \neq 0$, $x = g(y, z)$, and repeat the above calculations, we will get the same equation.

From the equation of the tangent plane we see that

$$\nabla F = \langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0) \rangle$$

is a normal rector to the tangent plane, i.e. ∇F is orthogonal to lerer surfaces.

The easiest way to remember the equation of the tangent plane to

$$F(x, y, z) = k$$

is to remember that $\nabla F(x_0, y_0, z_0)$ is orthogonal to level surfaces. Thus if $\nabla F(x_0, y_0, z_0) \neq \vec{0}$, the normal vector to the tangent plane at (x_0, y_0, z_0) is

$$\nabla F(x_0, y_0, z_0) = \langle F_x, F_y, F_z \rangle$$

from which we immediately obtain the equation

$$\boxed{F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0}$$

For level curves

$$F(x,y)=k\ ,\quad k-\text{constant}$$

we have a similar situation. ∇F is orthogonal to level curves. If $\nabla F(x_0,y_0)\neq\langle 0,0\rangle$, the equation of the tangent line to the level curve at (x_0,y_0) is

$$\boxed{F_x(x_0,y_0)(x-x_0)+F_y(x_0,y_0)(y-y_0)=0}$$

Problems

Exercise 83 Find the equation of the tangent plane to

$$yz-\ln z=x+y$$

at the point $(-1,0,e)$.

Remark We have already solved this problem (Exercise 75) but now we have tools to present a very short solution.

Solution

$$F(x,y,z) = yz - \ln z - x - y = 0$$

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle -1, z-1, y - \tfrac{1}{z} \rangle$$

$$\nabla F(-1,0,e) = \langle -1, e-1, -\tfrac{1}{e} \rangle$$

The equation is

$$-1(x+1) + (e-1)(y-0) - \tfrac{1}{e}(z-e) = 0$$

Multiply by e

$$-ex - \cancel{e} + (e^2 - e)y - z + \cancel{e} = 0$$

$$ex + (e - e^2)y + z = 0$$

which is the same answer as on page 213.

Exercise 84 Find an equation of the tangent line to
$$e^{xy} = e^2$$
at the point $(1,2)$.

Solution The equation is $F(x,y) = e^2$ where $F(x,y) = e^{xy}$.

$$\nabla F(x,y) = \langle y e^{xy}, \; x e^{xy} \rangle$$

$$\nabla F(1,2) = \langle 2e^2, \; e^2 \rangle.$$

Thus the equation of the tangent line is

$$(*) \quad F_x(1,2)(x-1) + F_y(1,2)(y-2) = 0$$

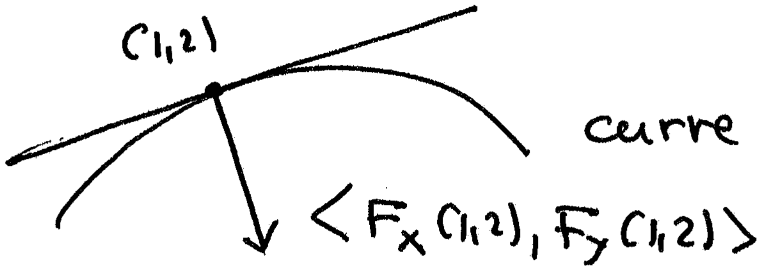
$$2e^2(x-1) + e^2(y-2) = 0 \qquad | \div e^2$$

$$2(x-1) + (y-2) = 0$$

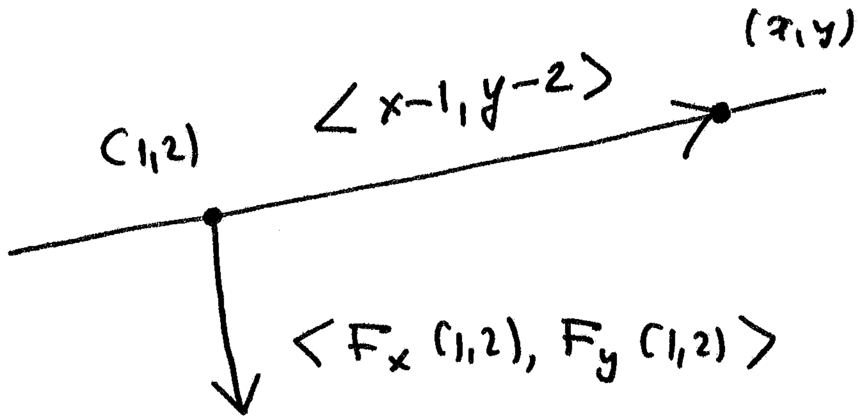
$$2x + y - 4 = 0$$

For better understanding of the problem, let us recall where does the equation $(*)$ come from.

$\nabla F(1,2)$ is orthogonal to the curve $F(x,y) = e^2$ at the point $(1,2)$



so it is orthogonal to the tangent line at the point $(1,2)$



$$\langle F_x(1,2), F_y(1,2) \rangle \cdot \langle x-1, y-2 \rangle = 0$$

$$F_x(1,2)(x-1) + F_y(1,2)(y-2) = 0.$$

Exercise 85 Show that the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point (x_0, y_0, z_0) can be written as

$$\frac{x x_0}{a^2} + \frac{y y_0}{b^2} + \frac{z z_0}{c^2} = 1.$$

Proof $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\nabla F = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

The equation of the tangent line is

$$\frac{2x_0}{a^2}(x-x_0) + \frac{2y_0}{b^2}(y-y_0) + \frac{2z_0}{c^2}(z-z_0) = 0$$

$$\frac{x_0 x}{a^2} - \frac{x_0^2}{a^2} + \frac{y_0 y}{b^2} - \frac{y_0^2}{b^2} + \frac{z_0 z}{c^2} - \frac{z_0^2}{c^2} = 1$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = \underbrace{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}}_{1}$$

Exercise 86 Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1,1,2)$

Solution Denote the tangent line by L. It is tangent both to the paraboloid and the ellipsoid.

Let

$F(x,y,z) = z - x^2 - y^2$, $G(x,y,z) = 4x^2 + y^2 + z^2$.

$\nabla F(-1,1,2)$ is orthogonal to the paraboloid at $(-1,1,2)$ and hence orthogonal to L.

$\nabla G(-1,1,2)$ is orthogonal to the ellipsoid at $(-1,1,2)$ and hence orthogonal to L.

We have two vectors orthogonal to L, so

$$\vec{v} = \nabla F(-1,1,2) \times \nabla G(-1,1,2)$$

is parallel to L.

$\nabla F = \langle -2x, -2y, 1 \rangle$, $\nabla F(-1,1,2) = \langle 2, -2, 1 \rangle$

$\nabla G = \langle 8x, 2y, 2z \rangle$, $\nabla G(-1,1,2) = \langle -8, 2, 4 \rangle$

$\vec{v} = \langle 2, -2, 1 \rangle \times \langle -8, 2, 4 \rangle = \langle -10, -16, -12 \rangle$

Thus L is through $(-1,1,2)$ and parallel to $\langle -10, -16, -12 \rangle$, hence parametric equations are

$$x = -1 - 10t, \quad y = 1 - 16t, \quad z = 2 - 12t.$$

Exercise 87 Show that the ellipsoid $3x^2+2y^2+z^2=9$ and the sphere $x^2+y^2+z^2-8x-6y-8z+24=0$ are tangent to each other at the point $(1,1,2)$

Solution Two surfaces are tangent at a point if they have a common tangent plane at the point. Thus it suffices to show that both surfaces have parallel normal vectors at the point $(1,1,2)$.

Let $F(x,y,z) = 3x^2+2y^2+z^2$.

Then

$$\nabla F = \langle 6x, 4y, 2z \rangle$$

$$\nabla F(1,1,2) = \langle 6, 4, 4 \rangle$$

is normal to the ellipsoid at the point $(1,1,2)$.

Let

$G(x,y,z) = x^2+y^2+z^2-8x-6y-8z+24.$

Then

$$\nabla G = \langle 2x-8,\ 2y-6,\ 2z-8 \rangle$$

$$\nabla G(1,1,2) = \langle -6, -4, -4 \rangle.$$

is normal to the sphere at the point $(1,1,2)$.

The vectors $\langle 6,4,4 \rangle$ and $\langle -6,-4,-4 \rangle$ are parallel and hence the surfaces are tangent at the point $(1,1,2)$.

Exercise 88 Show that the sum of the x, y and z intercepts of any tangent plane to the surface

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$$

is a constant.

Proof. Let $F = \sqrt{x} + \sqrt{y} + \sqrt{z}$.

$$\nabla F(x_0, y_0, z_0) = \left\langle \frac{1}{2\sqrt{x_0}}, \frac{1}{2\sqrt{y_0}}, \frac{1}{2\sqrt{z_0}} \right\rangle$$

The tangent plane at (x_0, y_0, z_0) is

$$\frac{1}{2\sqrt{x_0}}(x-x_0)+\frac{1}{2\sqrt{y_0}}(y-y_0)+\frac{1}{2\sqrt{z_0}}(z-z_0)=0$$

$$\frac{x}{\sqrt{x_0}}-\underbrace{\frac{x_0}{\sqrt{x_0}}}_{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}-\underbrace{\frac{y_0}{\sqrt{y_0}}}_{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}-\underbrace{\frac{z_0}{\sqrt{z_0}}}_{\sqrt{z_0}}=0$$

$$\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{c}$$

To find the x- intercept we set $y=z=0$

$$\frac{x}{\sqrt{x_0}}=\sqrt{c}, \quad x=\sqrt{c x_0}$$

Similarly y and z intercepts are

$$y=\sqrt{c y_0}, \quad z=\sqrt{c z_0}.$$

Hence

$$x+y+z=\sqrt{c}\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)=c$$

is a constant.

Exercise 89 Suppose that partial derivatives f_x, f_y of a function $f(x,y)$ are continuous at (x_0, y_0). Show that there is a direction $\vec{u}$ such that

$$D_{\vec{u}} f(x_0, y_0) = 0.$$

Proof From the continuity of partial derivatives we conclude that f is differentiable at (x_0, y_0). Hence

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u} =$$

$$= |\nabla f(x_0, y_0)| |\vec{u}| \cos \alpha.$$

Taking $\alpha = \pi/2$, $\cos \alpha = 0$ and hence $D_{\vec{u}} f(x_0, y_0) = 0$.

In other words $D_{\vec{u}} f(x_0, y_0) = 0$ if $\vec{u}$ is orthogonal to $\nabla f(x_0, y_0)$.

Exercise 90 Let $f(x,y)$ be a function with continuous partial derivatives f_x, f_y. Let

$$\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle, \quad \vec{v} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle.$$

Given that

$$D_{\vec{u}} f(3,-7) = 8, \quad D_{\vec{v}} f(3,-7) = -1$$

find $\nabla f(3,-7)$.

Solution Let

$$\nabla f(3,-7) = \langle f_x(3,-7), f_y(3,-7) \rangle = \langle \alpha, \beta \rangle.$$

Since for $\vec{\omega} = \langle a, b \rangle$

$$D_{\vec{\omega}} f(3,-7) = \nabla f(3,-7) \cdot \vec{\omega} = \alpha a + \beta b$$

we have

$$8 = D_{\vec{u}} f(3,-7) = \alpha \frac{\sqrt{2}}{2} + \beta \frac{\sqrt{2}}{2}$$

$$-1 = D_{\vec{v}} f(3,-7) = \alpha \left(-\frac{\sqrt{2}}{2}\right) + \beta \frac{\sqrt{2}}{2}$$

$$\begin{cases} \alpha \frac{\sqrt{2}}{2} + \beta \frac{\sqrt{2}}{2} = 8 \\ \alpha \left(-\frac{\sqrt{2}}{2}\right) + \beta \frac{\sqrt{2}}{2} = -1 \end{cases}$$

$$\begin{cases} \alpha+\beta = 8\sqrt{2} \\ -\alpha+\beta = -\sqrt{2} \end{cases}$$

$2\beta = 7\sqrt{2}$, $\beta = \frac{7\sqrt{2}}{2}$

$\alpha = 8\sqrt{2} - \beta = \frac{9\sqrt{2}}{2}$.

$\nabla f(3,-7) = \langle \alpha, \beta \rangle = \langle \frac{9\sqrt{2}}{2}, \frac{7\sqrt{2}}{2} \rangle$

Exercise 91 The temperature at a point (x,y,z) is given by

$$T(x,y,z) = 200\, e^{-x^2-3y^2-9z^2}$$

where T is measured in °C and x, y, z in meters

(a) Find the rate of change of temperature at the point $P(2,-1,2)$ in the direction toward the point $(3,-3,3)$

(b) In which direction does the temperature increase fastest at P?

(c) Find the maximum rate of increase at P.

Solution

$$\nabla T = -400\, e^{-x^2-3y^2-9z^2} \langle x, 3y, 9z \rangle.$$

(a) The direction toward $(3,-3,3)$ is given by

$$\vec{u} = \frac{\langle 3-2, -3-(-1), 3-2 \rangle}{|\langle 3-2, -3-(-1), 3-2 \rangle|} = \frac{\langle 1,-2,1 \rangle}{\sqrt{6}}$$

$$\nabla T(2,-1,2) = -400\, e^{-43} \langle 2,-3,18 \rangle$$

Hence the rate of change of T in the direction $\vec{u}$ is

$$D_{\vec{u}} T(2,-1,2) = -400\, e^{-43} \langle 2,-3,18 \rangle \cdot \frac{\langle 1,-2,1 \rangle}{\sqrt{6}}$$

$$= \left(-\frac{400\, e^{-43}}{\sqrt{6}}\right) \cdot 26 = -\frac{400 \cdot 26 \sqrt{6}}{6 \cdot e^{43}}$$

$$= \frac{-5200\sqrt{6}}{3 e^{43}} \ {}^{\circ}C/m$$

(b) In the direction of

$$\nabla T(2,-1,2) = 400\, e^{-43} \langle -2, 3, -18 \rangle$$

or equivalently in the direction of $\langle -2, 3, -18 \rangle$.

(c) The maximum rate of increase at P is

$$|\nabla T(2,-1,2)| = 400e^{-43}\sqrt{2^2+3^2+18^2}$$

$$= 400e^{-43}\sqrt{337} \quad {}^\circ C/m.$$

Maximum and minimum values

In this section we will describe methods of finding maximum and minimum values of a function of two or more variables.

There are two kinds of maxima and minima.

We say that $f(x,y)$ has a <u>local maximum</u> at (x_0, y_0) if $f(x_0,y_0) \geq f(x,y)$ for all (x,y) that are near (x_0, y_0).
If $f(x_0,y_0) \geq f(x,y)$ for <u>all</u> (x,y) in the domain of f, we say that f has <u>absolute maximum</u> (or <u>global maximum</u>) at (x_0, y_0).

Similarly we define local and absolute (global) minimum.

<u>Strange example</u> $f(x,y) = 1$ for all $(x,y) \in \mathbb{R}^2$ has local and global maximum and also local and global minimum at every point $(x,y) \in \mathbb{R}^2$.

This is because in our definition we use inequality "$\geqslant$" and we allow equality.

The domain D on which a function f is defined has interior and boundary (If $D = \mathbb{R}^2$, there is no bounday).
When searching for maxima and minima we have to deal with the interior and bounday separately.

First we will deal with the interior, i.e. with points that are not on the boundary.

<u>Theorem</u> If $f(x,y)$ has a local maximum or local minimum at a point (x_0, y_0) in the interior of D and partial derivatives $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist, then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

<u>Proof</u> The function $g(x) = f(x, y_0)$ attains local maximum or local minimum at $x = x_0$, so $g'(x_0) = 0$ but

$$0 = g'(x_0) = \frac{\partial f}{\partial x}(x_0, y_0).$$

Similarly $h(y) = f(x_0, y)$ has local maximum or minimum at $y = y_0$ and thus

$$\frac{\partial f}{\partial y}(x_0, y_0) = h'(y_0) = 0.$$

Thus in order to find local maxima and minima in the interior of D we have to find all points (x_0, y_0) such that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

We also have to check points where partial derivatives do not exist, because to such points the theorem does not apply, but perhaps a local maximum or minimum is there.

This justifies the following definition.

A point (x_0, y_0) is called critical if

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

or if one of these partial derivatives does not exist.

Example Consider the function

$$f(x, y) = \sqrt{x^2 + y^2}$$

We have

$$f_x(x,y) = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y(x,y) = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

There is no point (x_0, y_0) such that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

Indeed, if $(x_0, y_0) = (0,0)$ then the partial derivatives do not exist (zero in the denominator), but if $(x_0, y_0) \neq (0,0)$, then $x_0 \neq 0$ or $y_0 \neq 0$ and hence at least one of the partial derivatives $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ is not zero.

However, the function f has one critical point $(0,0)$. It is critical because partial derivatives do not exist at that point.

If fact f has a local and global minimum at the critical point $(0,0)$ because

$$f(x,y) > 0 = f(0,0)$$

if $(x,y) \neq (0,0)$. $\square$

If at some point (x_0, y_0) we have

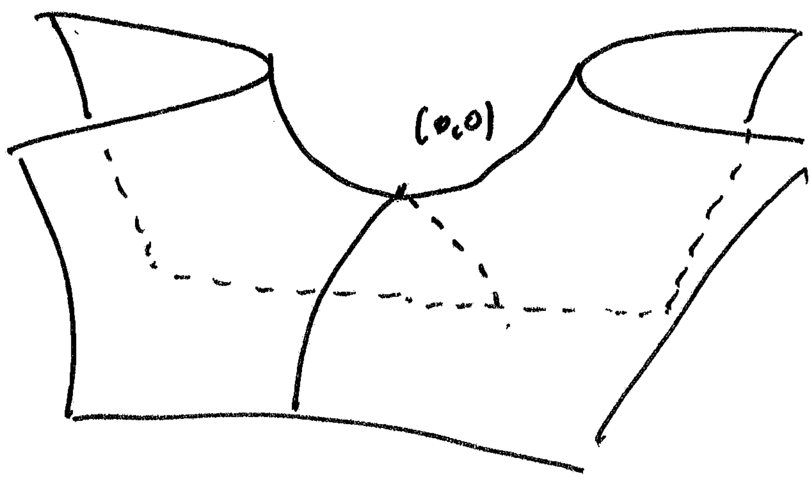
$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

then this point is a good candidate to be a local maximum or minimum, but we do not know yet whether it is maximum or minimum. Moreover it may happen that it is neither maximum or minimum. You should not be surprised. You know a simil situation from Calculus I. $f'(x) = 0$ does not guarantee maximum or minimum. It may be an inflection point.

<u>Example</u> The graph of

$$f(x,y) = x^2 - y^2$$

looks as follows



Along the x axis we have a parabola "up" $z = x^2$ and along the y axis we have a parabola "down" $z = -y^2$.

Since $f_x(0,0) = f_y(0,0) = 0$ the point $(0,0)$ is critical but we have neither maximum nor minimum at $(0,0)$ because along the parabola $z = x^2$ it is minimum but along the parabola

$z = -y^2$ it is maximum.

The graph of f near $(0,0)$ looks like a saddle and for this reason $(0,0)$ is called a saddle point.

For functions of one variable a method of of checking whether a point x_0 where $f'(x_0)=0$ is a local maximum or minimum is called second derivative test:
If $f''(x_0) > 0$, it is a local minimum
If $f''(x_0) < 0$, it is a local maximum
If $f''(x_0) = 0$ we cannot say anything. More information is needed. It still may be a local minimum or maximum but also it may happen that it is not a local maximum or minimum.
Similar test is available for functions of two variables.

Theorem (Second Derivatives Test)

Suppose that the second order partial derivatives of $f(x,y)$ are continuous in a neighborhood of (x_0, y_0), i.e. in some disc centered at (x_0, y_0). Suppose also that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

Let

$$D = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}$$

$$= f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

(a) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum

(c) If $D < 0$, then $f(x_0, y_0)$ is not a local maximum or minimum.

(d) If $D = 0$ we cannot say anything. More information is needed.

If $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, but $f(x_0, y_0)$ is not a local maximum or minimum, we say that (x_0, y_0) is a <u>saddle</u> point. Note that in that case the graph of f crosses the tangent plane near (x_0, y_0).

The next examples shows in a very simple situation how the test works.

<u>Example</u>

$f(x,y) = x^2+y^2$ has local (and absolute) minimum at $(0,0)$. Note that

$$f_x(0,0) = f_y(0,0) = 0$$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$D(0,0) = 4 > 0$$
$$f_{xx}(0,0) = 2 > 0.$$

<u>Example</u>

$f(x,y) = -(x^2+y^2)$ has local (and absolute) maximum at $(0,0)$. Note that

$$f_x(0,0) = f_y(0,0) = 0$$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$$

$$D(0,0) = 4 > 0$$

$$f_{xx}(0,0) = -2 < 0.$$

Example

$f(x,y) = x^2 - y^2$ has no maximum or minimum at $(0,0)$. It is a saddle point. Note that

$$f_x(0,0) = f_y(0,0) = 0$$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4$$

$$D(0,0) = -4 < 0.$$

The next example has two parts. The first part is easy, but the second one is tricky.

<u>Example</u>

For what values of k does $f(x,y) = x^2 + kxy + 4y^2$ have a critical point at $(0,0)$?

<u>Solution</u> $f_x = 2x + ky$, $f_y = kx + 8y$

$$f_x(0,0) = 0, \quad f_y(0,0) = 0.$$

Thus $(0,0)$ is a critical point for <u>any</u> value of k.

For what values of k does $f(x,y) = x^2 + kxy + 4y^2$ have a local minimum at $(0,0)$?

<u>Solution</u> We already know that $(0,0)$ is a critical point for any value of k and we need to look at second order derivatives.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & k \\ k & 8 \end{vmatrix} = 16 - k^2.$$

Recall that

$D = 16 - k^2 > 0$, $f_{xx} = 2 > 0$ local minimum

$D = 16 - k^2 < 0$ saddle point

$D = 16 - k^2 = 0$ not enough information.

We have

$D = 16 - k^2 > 0$, $k^2 < 16$, $-4 < k < 4$.

Thus for $k \in (-4, 4)$ we have local minimum (because $f_{xx} = 2 > 0$).

$D = 16 - k^2 < 0$, $k^2 > 16$,

$k \in (-\infty, -4) \cup (4, \infty)$.

Thus for $k \in (-\infty, -4) \cup (4, \infty)$ we have a saddle point.

We are left with the case

$D = 16 - k^2 = 0$, $k = \pm 4$.

In that case the second derivative test does not apply and we need to find another argument.

If $k = 4$

$$f(x,y) = x^2 + 4xy + 4y^2 = (x+2y)^2$$

and hence we have local minimum at $(0,0)$.

If $k = -4$

$$f(x,y) = x^2 - 4xy + 4y^2 = (x-2y)^2$$

and still we have local minimum at $(0,0)$.

Putting all the cases together we obtain the answer.

$f(x,y) = x^2 + kxy + 4y^2$ has a local minimum at $(0,0)$ if and only if

$$-4 \le k \le 4.$$

Example Find the distance between the lines L_1 and L_2 given by

$$\vec{r}_1(t) = \langle t, t, t \rangle, \quad \vec{r}_2(t) = \langle -1+t, 2, 1-t \rangle$$

Solution 1 The first method is based on the technique of lines and planes. It is quite complicated, but then we will see that the method of finding critical points gives us a much simpler solution.

Vectors

$$\vec{v}_1 = \langle 1, 1, 1 \rangle \text{ and } \vec{v}_2 = \langle 1, 0, -1 \rangle$$

are parallel to the lines, so

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \langle -1, 2, -1 \rangle$$

is orthogonal to both lines. Hence the planes of the form

$$-1(x-x_0)+2(y-y_0)-1(z-z_0)=0$$

are parallel to both lines L_1 and L_2.

The point $(0,0,0)$ is on the line L_1 (we take $\vec{r}_1(0)$). Hence the plane P_1 with the equation

$$-1\cdot x+2y-1\cdot z=0$$

contains L_1, because L_1 is parallel to P_1 and it has a common point $(0,0,0)$ with P_1.

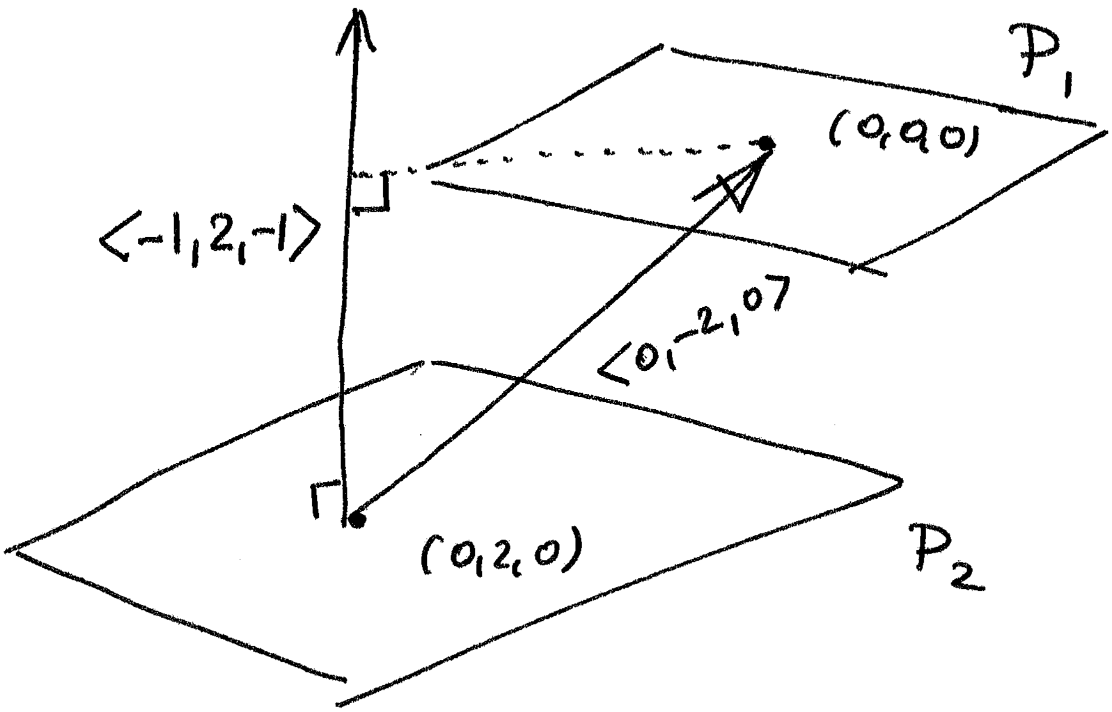
The point $(0,2,0)$ is on the line L_2 (we take $\vec{r}_2(1)$). Hence the plane P_2 with the equation

$$-1\cdot x+2(y-2)-1\cdot z=0$$

contains L_2, because L_2 is parallel to P_2 and it has a common point $(0,2,0)$ with P_2.

The planes P_1 and P_2 are parallel, P_1 contains L_1, P_2 contains L_2, so the distance between the lines equals to the distance between the planes P_1 and P_2. You can find in exercises in the textbook a formula for the distance between the planes, but instead of memorizing a complicated formula I prefer to use logic.

In order to find the distance between planes we take the point $(0,0,0)$ which belongs to P_1 and we find its distance to P_2.



The distance between the planes equals the length of the projection of $\langle 0,-2,0\rangle$ onto $\langle -1,2,-1\rangle$.

$$\left| \operatorname{comp}_{\langle -1,2,-1\rangle} \langle 0,-2,0\rangle \right| = \frac{|\langle 0,-2,0\rangle \cdot \langle -1,2,-1\rangle|}{|\langle -1,2,-1\rangle|}$$

$$= \frac{4}{\sqrt{6}}.$$

Well, that was long and complicated.

Solution 2 Take two arbitrary points on the lines (290)

(t,t,t) on L_1 and $(-1+s, 2, 1-s)$ on L_2.

The distance between the two points is

$$d(s,t) = \sqrt{(t+1-s)^2 + (t-2)^2 + (t-1+s)^2}$$

Now the distance between the lines is the minimum value of the function $d(s,t)$. Note that $d(s,t)$ attains minimum at the same point as the function

$$d^2(s,t) = (t+1-s)^2 + (t-2)^2 + (t-1+s)^2$$

has minimum. It is easier to find minimum of $d^2(s,t)$, because we do not have to differentiate "$\sqrt{\ }$".

From the geometric situation we know that the function $d(s,t)$ (and hence $d^2(s,t)$)

attains minimum at some point (s,t) and this point is a critical point of $d^2(s,t)$. Thus if the function $d^2(s,t)$ has exactly one critical point it will be the point where $d^2(s,t)$ attains minimum and we do not need the Second Derivative Test to check whether it is minimum or maximum*.

To find critical points of $d^2(s,t)$ we have to solve equations:

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* Moreover the Second Derivative Test tells us <u>only</u> whether it is a local minimum or local maximum and we are looking for <u>absolute</u> minimum.

$$\frac{\partial}{\partial t} d^2(s,t) = 2(t+1-s)+2(t-2)+2(t-1+s)$$ 292

$$= 2t+2-2s+2t-4+2t-2+2s$$

$$= 6t-4 = 0$$

$$\frac{\partial}{\partial s} d^2(s,t) = -2(t+1-s)+2(t-1+s)$$

$$= -2t-2+2s+2t-2+2s$$

$$= 4s-4 = 0$$

$$\begin{cases} 6t-4=0 \\ 4s-4=0 \end{cases}$$

$$t = \frac{4}{6} = \frac{2}{3}, \quad s = 1.$$

Thus the point $(s,t) = (1, \frac{2}{3})$ is the <u>only</u> critical point ~~of~~ $d^2(s,t)$, so $d^2(s,t)$ (and hence $d(s,t)$) attains minimum at this point. We have

$$d(1,\tfrac{2}{3}) = \sqrt{(\tfrac{2}{3}+1-1)^2 + (\tfrac{2}{3}-2)^2 + (\tfrac{2}{3}-1+1)^2}$$

$$= \sqrt{\frac{4}{9}+\frac{16}{9}+\frac{4}{9}} = \frac{\sqrt{24}}{3} = \frac{4}{\sqrt{6}}.$$

<u>Remark</u> We will check (although it is not necessary) that $d^2(s,t)$ has a local minimum at $(1, \frac{2}{3})$. We already know that this is a critical point but we still need to check second derivatives

$$D = \begin{vmatrix} d^2_{ss} & d^2_{st} \\ d^2_{ts} & d^2_{tt} \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & 6 \end{vmatrix} = 24 > 0$$

$$d^2_{ss}(1, \tfrac{2}{3}) = 4 > 0$$

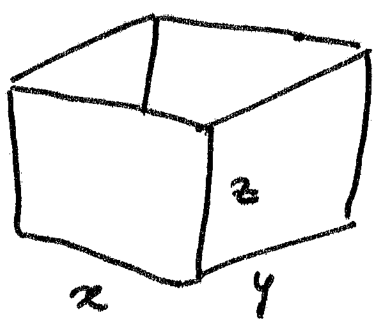
and hence we have <u>local</u> minimum at $(1, \frac{2}{3})$. Note that the second derivatives test does not guarantee that this is an <u>absolute</u> minimum.

<u>Example</u> A rectangular box without a lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such a box

<u>Solution</u> Denote the length, width and height of the box by x, y, z. Then the volume is

$$V = xyz.$$

We have, however a constrain that the surface area is 12. Since the box has no lid the surface area is



$$2xz + 2yz + xy = 12.$$

Solving this equation for z yields

$$z = \frac{12 - xy}{2(x+y)}$$

and hence

$$V = xy\,\frac{12 - xy}{2(x+y)} = \frac{12xy - x^2y^2}{2(x+y)}.$$

In the original formula V depended on three variables, but we could remove one of the variables and now V is a function of (x, y).

Next we find critical points of V.

$$\frac{\partial V}{\partial x} = \frac{(12y - 2xy^2)\cancel{2}(x+y) - (12xy - x^2y^2)\cdot\cancel{2}}{\cancel{4}^{\,2}(x+y)^2}$$

$$= \frac{\cancel{12yx} + 12y^2 - 2x^2y^2 - 2xy^3 - \cancel{12xy} + x^2y^2}{2(x+y)^2}$$

$$= \frac{y^2(12 - x^2 - 2xy)}{2(x+y)^2}$$

Similarly we find

$$\frac{\partial V}{\partial y} = \frac{x^2(12 - y^2 - 2xy)}{2(x+y)^2}.$$

If volume is maximum, then

$$\frac{\partial V}{\partial x} = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = 0$$

(note that $x > 0, y > 0$ because $V = 0$

is not maximum and hence derivatives are well defined).

We have

$$\begin{cases} y^2(12-x^2-2xy)=0 \\ x^2(12-y^2-2xy)=0 \end{cases}$$

$$(*)\quad \begin{cases} 12-x^2-2xy=0 \\ 12-y^2-2xy=0 \end{cases} \qquad (\text{because } x,y>0)$$

Hence $x^2=y^2$, $x=y$.

Now the equations (*) give

$$12-3x^2=0$$

$$x=2,\ y=2,\ z=\frac{12-2\cdot 2}{2(2+2)}=1$$

Since by the physical nature of the problem, the maximum value of V must be attained, it has to be attained at the critical point which is $x=2$, $y=2$, $z=1$ and the second derivatives test is not needed here.

Absolute maximum and minimum values

If f is a continuous function defined on a bounded and closed interval $[a, b]$, then f attains its maximum and minimum values. In order to find the maximum and minimum values of f we check values of f at

(1) all critical points in (a, b)
(2) at the endpoints a and b.

Then absolute maximum of f is the largest of these values and the absolute minimum is the smallest number among these values.

We cannot neglect endpoints. For example $f(x) = x$, $x \in [0, 1]$ has no critical values but it attains maximum at the endpoint $x = 1$ and minimum at the endpoint $x = 0$.

Similar procedure applies to functions of two variables.

We say that a set $D \subset \mathbb{R}^2$ is closed if it contains all its bounday points. (x_0, y_0) is a bounday point of D if in any neighborhood of (x_0, y_0) we can find points that belong to D and also points that do not belong to D.

<u>Example</u> The bounday of the disc

$$D = \{(x,y) \mid x^2+y^2 \leq 1\}$$

is the circle $x^2+y^2=1$. But also the bounday of the open disc

$$\tilde{D} = \{(x,y) \mid x^2+y^2 < 1\}$$

is the same circle $x^2+y^2=1$

The disc D is closed because the circle $x^2+y^2=1$ is a part of D, but $\tilde{D}$ is not closed.

<u>Example</u> $\mathbb{R}^2$ has no boundary and hence it is closed. Every boundary point belongs to $\mathbb{R}^2$ because there are no such points. This looks like a strange kind of logic but it is true: in order to show that $\mathbb{R}^2$ is not closed we would have to find a boundary point which does not belong to $\mathbb{R}^2$ and that cannot happen.

<u>Example</u> The punctured disc

$$D = \{ (x,y) \mid 0 < x^2+y^2 \leq 1 \}$$

is not closed. The boundary consists of the circle $x^2+y^2=1$ and the center $(0,0)$. Indeed, $(0,0)$ is the boundary point because near $(0,0)$ we can find points of D and also points that are not in D, namely $(0,0)$ itself.

<u>Theorem</u> If is a continuous function on a bounded and closed set D in $\mathbb{R}^2$, then f attains an absolute maximum and an absolute minimum value at some points in D.

The method of finding these values is analogous to the case of functions of one variable

1. Find values of f at all critical points inside D i.e. not on the bounday
2. Find the extreme values of f on the boundary

Then the largest value in 1 & 2 is the absolute maximum and the smallest value in 1 & 2 is the absolute minimum.

Example Let $f(x,y) = \frac{1}{x^2+y^2}$
be a function with the domain
$$D = \{ (x,y) \mid 0 < x^2+y^2 \leq 1 \}.$$
The function f does not attain maximum, because
$$f(x,y) \to \infty \quad \text{as} \quad (x,y) \to (0,0).$$

Although D is bounded and the bounday circle is included in D there is no contradiction with the theorem, because D is still not closed as we discussed on p. 299.

To find critical points inside D we solve the equations
$$f_x = 0, \quad f_y = 0$$
Note that in the procedure described above we do not check the points with the Second Derivatives

Test. We take into account all critical points.

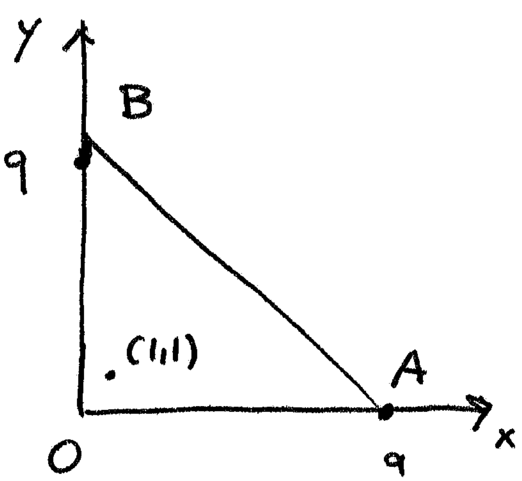
To find max. and min. on the boundary we have to use different methods depending on the shape of the boundary.

<u>Example</u> Find the absolute maximum and minimum values of

$$f(x,y) = 2+2x+2y-x^2-y^2$$

on the triangular region in the first quadrant bounded by the lines $x=0$, $y=0$ and $y=9-x$

<u>Solution</u>



<u>Interior of the triangle</u>

$$\begin{cases} f_x = 2-2x=0 \\ f_y = 2-2y=0 \end{cases}$$

The only solution is $(x,y)=(1,1)$

and the point $(1,1)$ is in the interior of the triangle.

$$f(1,1) = 4.$$

Boundary of the triangle

The boundary consists of three segments and we have to investigate each of the segments separately

On the segment OA, $y = 0$ and hence

$$f(x,y) = f(x,0) = 2+2x-x^2, \quad 0 \le x \le 9.$$

Now we are looking for extreme values of a function of one variable defined on the segment $[0,9]$.

$$g(x) = f(x,0) = 2+2x-x^2, \quad 0 \le x \le 9.$$

$$g'(x) = 2-2x = 0, \quad x = 1$$

So $x = 1$ is a critical point of g in the interior of $[0,9]$

$$g(1) = f(1,0) = 3$$

We still need the values of g at the endpoints

$$f(0,0) = g(0) = 2$$
$$f(9,0) = g(9) = -61$$

On the segment OB , $x=0$
and hence
$$f(x,y) = f(0,y) = 2+2y-y^2, \quad 0 \le y \le 9.$$

This is the same function as on the segment OA, only the variable x is replaced by y. Note also that the domain $[0,9]$ is the same. Thus it has one critical point and two endpoints
$$f(0,1)=3, \quad f(0,0)=2, \quad f(0,9)=-61$$

On the segment AB

We already checked values at the endpoints in other two cases and we only need to look for critical points of a function of one variable in the interior.

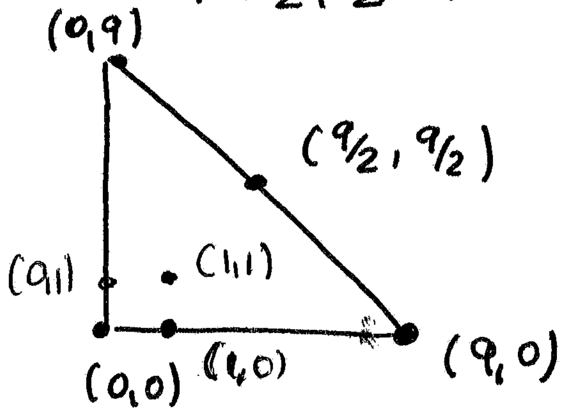
We have $y = 9-x, \ 0 \le x \le 9$ on AB

so
$$f(x,y) = f(x,9-x) = 2+2x+2(9-x)-x^2-(9-x)^2$$
$$= -61+18x-2x^2$$

$$g(x) = -61+18x-2x^2, \quad 0 \le x \le 9$$

$$g'(x) = 18-4x=0$$
$x = 9/2$ satisfies $0 < 9/2 < 9$.

$$\underbrace{f\left(\frac{9}{2},\, 9-\frac{9}{2}\right)}_{f\left(\frac{9}{2},\,\frac{9}{2}\right)} = g\left(\frac{9}{2}\right) = -\frac{41}{2}$$



We have

$f(0,0) = 2$, $f(1,0) = f(0,1) = 3$, $f(0,9) = f(9,0) = -61$

$f(1,1) = 4$

Thus f has the absolute maximum at $(1,1)$, $f(1,1) = 4$ and the absolute minimum at two points $(0,9)$ and $(9,0)$

$f(0,9) = f(9,0) = -61$.

<u>Example</u> Find the absolute maximum and minimum values of

$$f(x,y) = x^2 + y^2 - 2y + 1$$

on the disc

$$D = \{ (x,y) \mid x^2 + y^2 \leq 9 \}$$

<u>Solution.</u>

Critical points inside the circle

$$f_x = 2x = 0 \quad , \quad f_y = 2y - 2 = 0$$

$$(x, y) = (0, 1)$$

and this point is in the interior of the disc.

$$f(0,1) = -1$$

The boundary is the circle $x^2 + y^2 = 9$. We can parametrize the circle by

$$\langle 3\cos t, 3\sin t \rangle \ , \quad t \in \mathbb{R}.$$

Although in this parametrization we go infinitely many times around the circle, it does not matter. With this parametrization, the function f on the circle is

$$f(3\cos t, 3\sin t) = 9\cos^2 t + 9\sin^2 t - 6\sin t + 1$$

$$= 10 - 6\sin t.$$

The maximum is attained when $\sin t = -1$, and hence $\cos t = 0$

$$f(0, -3) = 10 - 6(-1) = 16$$

The minimum is attained when $\sin t = 1$ and hence $\cos t = 0$

$$f(0, 3) = 10 - 6 \cdot 1 = 4$$

We found values at three points

$f(0,1) = -1, \; f(0,-3) = 16, \; f(0,3) = 4.$

The absolute maximum is $f(0,-3) = 16$
and the absolute minimum is $f(0,1) = -1$.

Problems

Exercise 92 Find local maxima, minima and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1.$$

Solution

$$f_x = 4x^3 - 4y \quad f_y = 4y^3 - 4x$$

We are looking for points where

$$f_x = 0 \text{ and } f_y = 0$$

i.e. for solutions of

$$\begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases}$$

i.e.

$$y = x^3 \text{ and } x = y^3$$

Replacing y by $y = x^3$ in the second equation we get $x = x^9$.

$$x^9 - x = 0$$

$$x(x^8-1) = 0$$

$$x(x^4-1)(x^4+1) = 0 \qquad (x^4+1>0)$$

$$x(x^4-1) = 0$$

$$x(x^2-1)(x^2+1) = 0 \qquad (x^2+1>0)$$

$$x(x^2-1) = 0$$

$$x(x-1)(x+1) = 0$$

$x = 0$, $x = 1$ or $x = -1$.

If $x=0$, then $y = 0$, $(0,0)$

If $x=1$, then $y = 1$, $(1,1)$

If $x=-1$, then $y = -1$, $(-1,-1)$

Thus the function f has three critical points $(0,0)$, $(1,1)$ and $(-1,-1)$.
Now we need to use the Second Derivatives test

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix}$$

$$= 144x^2y^2 - 16.$$

$D(0,0) = -16$, $\boxed{(0,0) - \text{saddle point}}$

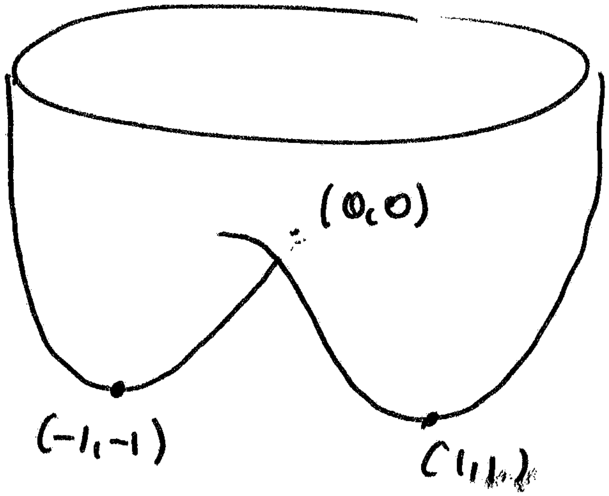
$D(1,1) = 144 - 16 > 0$, $f_{xx}(1,1) = 12 > 0$

$\boxed{(1,1) - \text{local minimum}}$

$D(-1,-1) = 144 - 16 > 0$, $f_{xx}(-1,-1) = 12 > 0$

$\boxed{(-1,-1) - \text{local minimum}}$

The graph of f looks more or less as follows



Exercise 93 The Second Derivatives Test does not take into account the case

$$D > 0 \text{ and } f_{xx} = 0.$$

What can you say about this case?

Solution This will never be the case, because if $f_{xx}=0$, then

$$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$= -(f_{xy})^2 \le 0.$$

Exercise 94 Suppose that

$f_x(0,0)=f_y(0,0)=0$, $D(0,0)=0$, $f_{xx}(0,0)>0$.

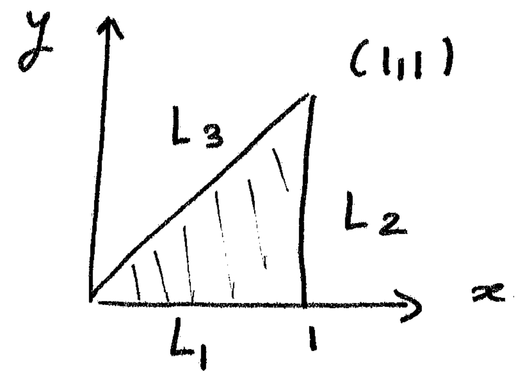
Since $D=0$, the Second Derivatives Test is inconclusive, but nevertheless show that $f(0,0)$ cannot be a local maximum.

Proof Since $f_x(0,0)=0$, $f_{xx}(0,0)>0$, the function $g(x)=f(x,0)$ has local minimum at $x=0$, i.e. the function f has a local minimum at $(0,0)$ when restricted to the x-axis, so $(0,0)$ cannot be a local maximum.

Exercise 95 Find the absolute maximum and minimum values of

$$x^3 - y^3 + 6xy \ , \ 0 \le x \le 1, \ 0 \le y \le x$$

Solution The function $f(x,y) = x^3 - y^3 + 6xy$ is defined on the triangle



Critical points in the interior

$$f_x = 3x^2 + 6y \ , \quad f_y = -3y^2 + 6x$$

$$\begin{cases} 3x^2 + 6y = 0 \\ -3y^2 + 6x = 0 \end{cases}$$

Observe however, that in the interior of the triangle $y > 0$ and hence $3x^2 + 6y > 0$, so the equations have no solutions in the interior of the triangle. Thus the function

f has no critical points in the interior. (Note that e.g. $(0,0)$ is a solution of the system of equations, but it is a vertex of the triangle, so it does not count).

On L_1 $\quad 0 \le x \le 1, \; y = 0$

$$f(x,y) = f(x,0) = x^3, \quad 0 \le x \le 1.$$

This function has min. at $x=0$ and max. at $x=1$

$$f(0,0) = 0, \quad f(1,0) = 1$$

On L_2 $\quad x = 1, \; 0 \le y \le 1$

$$f(x,y) = f(1,y) = 1 - y^3 + 6y$$

Thus we need to investigate the function

$$g(y) = 1 - y^3 + 6y, \quad 0 \le y \le 1.$$

$$g'(y) = -3y^2 + 6 = 3(2 - y^2)$$
$$= 3(\sqrt{2} - y)(\sqrt{2} + y).$$

The roots are $\pm\sqrt{2}$ and they are not in the interval $0 \le y \le 1$.

The function g has no critical points in the interior of $[0,1]$ and we are left with the endpoints.

$$f(1,0) = g(0) = 1$$
$$f(1,1) = g(1) = 6.$$

On L_3 $\quad 0 \leq x \leq 1,\ y = x$, so

$$f(x,y) = f(x,x) = x^3 - x^3 + 6x \cdot x$$
$$f(x,x) = 6x^2, \quad 0 \leq x \leq 1.$$

Since this function is increasing, its extreme values are at the endpoints

$$f(0,0) = 0, \quad f(1,1) = 6.$$

We computed values of f at the following candidates for max/min

$$f(0,0) = 0,\ f(1,0) = 1,\ f(1,1) = 6$$

The absolute maximum is $f(1,1) = 6$ and the absolute minimum is $f(0,0) = 0$.

Exercise 96 Find the absolute maximum and minimum values of

$$f(x,y) = xy^2$$

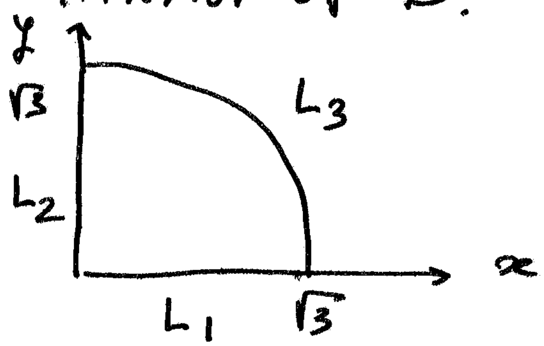
on the set

$$D = \{(x,y) \mid x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 3\}$$

Solution Critical points

$$f_x = y^2, \quad f_y = 2xy$$

$f_x = 0$ if and only if $y=0$ and hence there are no critical points in the interior of D.



The bounday has three parts.

On L_1 $y=0$, $f(x,0)=0$

On L_2 $x=0$, $f(0,y)=0$

On L_3 $y = \sqrt{3-x^2}$, $0 \leq x \leq \sqrt{3}$

and hence

$$f(x,y) = f(x, \sqrt{3-x^2}\,) = x(3-x^2)$$

$$= 3x - x^3 \qquad 0 \leq x \leq \sqrt{3}$$

At the endpoints $x=0$, $x=\sqrt{3}$ we have

$$f(0,\sqrt{3}) = f(\sqrt{3},0) = 0$$

Now we check critical points in $0<x<\sqrt{3}$. Let $g(x) = 3x - x^3$. $g'(x) = 3 - 3x^2$
$= 3(1-x^2) = 3(1-x)(1+x)$.
$g'(x)=0$ when $x = 1$ or $x=-1$.
The root $x=-1$ is outside the interval $(0,\sqrt{3})$, but $1 \in (0,\sqrt{3})$.
$f(1,\sqrt{2}) = g(1) = 2$.

Absolute maximum: $f(1,\sqrt{2}) = 2$
Absolute minimum is 0 and it is attained at any point of L_1 or L_2.

Remark In this problem it was convenient to write $y = \sqrt{3-x^2}$ along the circular part of the boundary while in Example p. 305 we used parametrization

$$\langle 3\cos t, 3\sin t \rangle.$$

Each time it is your decision what argument you choose.

Exercise 97 Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

Solution The distance to the origin is $d = \sqrt{x^2+y^2+z^2}$ where $y^2 = 9 + xz$, so we need to minimize the function

$$d = \sqrt{x^2 + (9+xz) + z^2},$$

but it is easier to minimize the function

$$f(x,z) = d^2 = x^2 + 9 + xz + z^2.$$

Geometrically it is obvious that the minimum is attained and the minimum must be at the critical point of f.

$$f_x = 2x+z \ , \ f_z = x+2z$$

$$\begin{cases} 2x+z=0 \\ x+2z=0 \end{cases}$$

$x=0, \ z=0$

so $(x,z)=(0,0)$ is the only critical point of f and hence $d^2 = f(0,0) = 9$, $d=3$ is the minimum distance. We still have to find y

$$y^2 = 9 + 0 \cdot 0$$

$$y = \pm 3.$$

Thus the minimum distance is attained at two points $(0,3,0)$ and $(0,-3,0)$.

Exercise 98 Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x+2y+3z=6$.

<u>Solution</u> Denoting the sides of the box by x, y, z, the volume is $V = xyz$, where $x + 2y + 3z = 6$.

Note that $x, y, z \geq 0$ (first octant) and the maximum is attained when $x, y, z > 0$ ($V = 0$ is not maximum.) We have

$$z = \frac{6 - x - 2y}{3}$$

and hence

$$V = \frac{xy(6 - x - 2y)}{3} = \frac{6xy - x^2y - 2xy^2}{3}$$

$$V_x = \frac{6y - 2xy - 2y^2}{3} = \frac{1}{3}y(6 - 2x - 2y)$$

$$V_y = \frac{6x - x^2 - 4xy}{3} = \frac{1}{3}x(6 - x - 4y)$$

$$\begin{cases} \frac{1}{3}y(6 - 2x - 2y) = 0 \\ \frac{1}{3}x(6 - x - 4y) = 0 \end{cases}$$

$$\begin{cases} 6 - 2x - 2y = 0 \\ 6 - x - 4y = 0 \end{cases} \quad \text{(because } x > 0 \text{ and } y > 0\text{)}$$

Hence $x = 2, y = 1$.

Geometrically the maximum is attained and (2, 1) is the only critical point, so it must be the point of maximum. The largest volume is

$$V = \frac{2\cdot 1\,(6-2-2\cdot 1)}{3} = \frac{4}{3}.$$

<u>Exercise 99</u> Find the point on the plane $x-y+z=4$ that is closest to the point (1, 2, 3)

<u>Solution</u> We could use methods of lines and planes to solve the problem, but we prefer to use the method of critical point. The closest point exists and it must be at the critical point of the distance function.

The distance is

$$d = \sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$$

where $x-y+z=4$.

Hence $z-3 = 1-x+y$, so

$$d(x,y) = \sqrt{(x-1)^2+(y-2)^2+(1-x+y)^2}$$

But it is more convenient to minimize

$$f(x,y) = d^2(x,y) = (x-1)^2+(y-2)^2+(1-x+y)^2.$$

$$f_x = 2(x-1) - 2(1-x+y) = 4x-2y-4$$

$$f_y = 2(y-2) + 2(1-x+y) = 4y-2x-2$$

$$\begin{cases} 4x-2y-4=0 \\ 4y-2x-2=0 \end{cases}$$

yields

$$(x,y) = \left(\frac{5}{3}, \frac{4}{3}\right)$$

This is the only critical point so it must be the closest one. We still need to find z.

$$z = 4-x+y = \frac{11}{3}.$$

<u>Answer</u> The closest point is

$$\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right).$$

Lagrange Multipliers

In the example on p. 294 we had to find the maximum of
$$V = xyz$$
under the condition that
$$2xz + 2yz + xy = 12.$$

In the example on pp. 305–306 we had to find max./min. of
$$f(x,y) = x^2 + y^2 - 2y + 1$$
on the circle
$$x^2 + y^2 = 9.$$

In Exercise 98 we had to find the maximum of
$$V = xyz$$
under the condition that
$$x + 2y + 3z = 6.$$

Each time we had to invent a different method to deal with a particular problem.

The examples listed above can be formulated as a general problem

Find the local maxima / minima values of

$$f(x, y, z)$$

under the condition that

$$g(x, y, z) = k.$$

Instead of "under the condition" we often say "subject to the constraint" and meanings of the two statements are the same, so the problem is

> Find the local maxima / minima values of
>
> $$f(x, y, z)$$
>
> subject to the constraint
>
> $$g(x, y, z) = k.$$

There is a general method how to solve such problems.

Theorem (The method of Lagrange multipliers)

If $f(x,y,z)$ attains a local maximum or minimum subject to a constraint $g(x,y,z)=k$ at a point (x_0,y_0,z_0) and if $\nabla g(x_0,y_0,z_0)\neq\langle 0,0,0\rangle$, then there is a real number $\lambda\in\mathbb{R}$ such that

$$\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0).$$

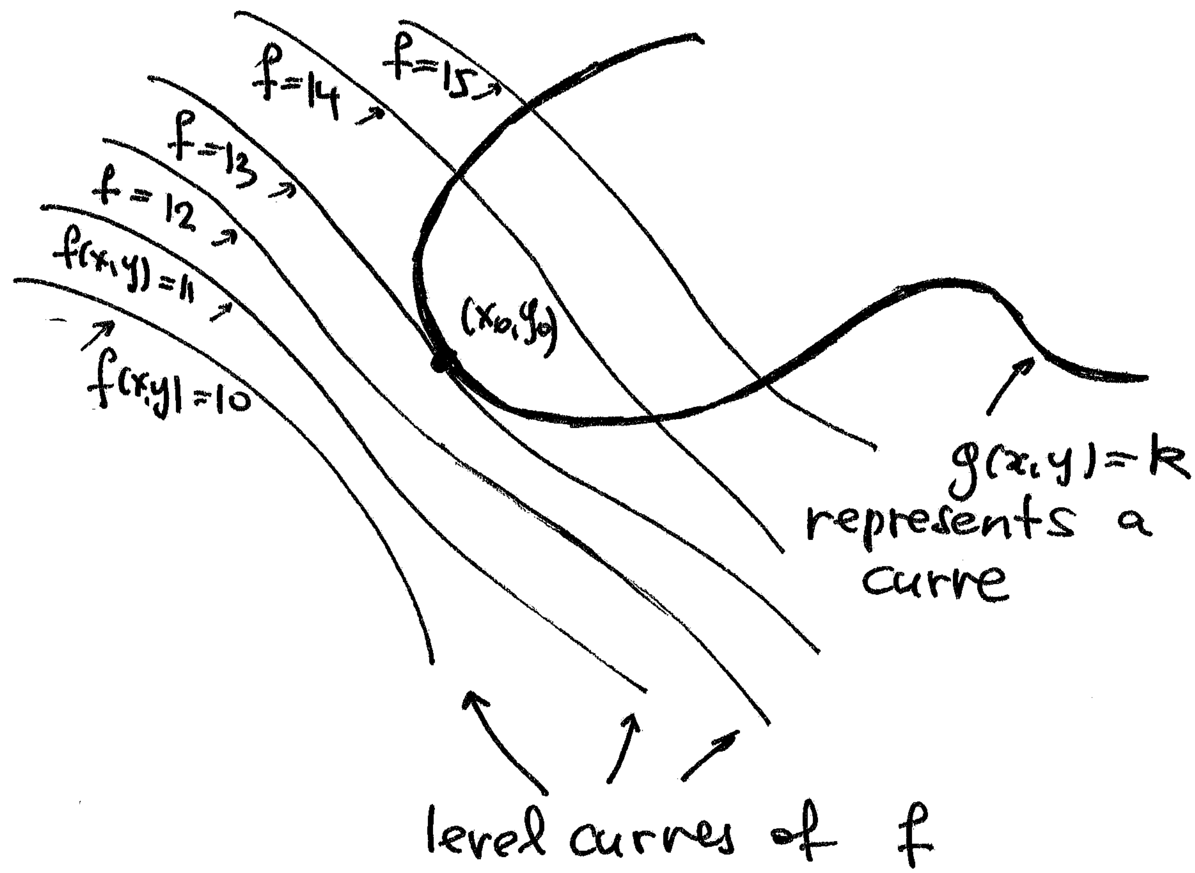
The coefficient λ is called a Lagrange multiplier.

Similar fact is true for functions of two variables

If $f(x,y)$ attains local max./min. at (x_0,y_0) subject to a constraint $g(x,y)=k$ and $\nabla g(x_0,y_0)\neq\langle 0,0\rangle$, then there is λ such that

$$\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$$

This fact has a clear geometric interpretation that we explain on the picture below (324)



Observe that values of f along the curve $g(x,y)=k$ attain (local) minimum at (x_0, y_0).

From the picture we see that the curves

$f(x,y)=13$ and $g(x,y)=k$

are tangent to each other at the

point (x_0, y_0). Hence the normal vectors to the curves

$$f(x,y) = 13 \quad \text{and} \quad g(x,y) = k$$

at the point (x_0, y_0) are parallel, but normal vectors are

$$\nabla f(x_0, y_0) \quad \text{and} \quad \nabla g(x_0, y_0)$$

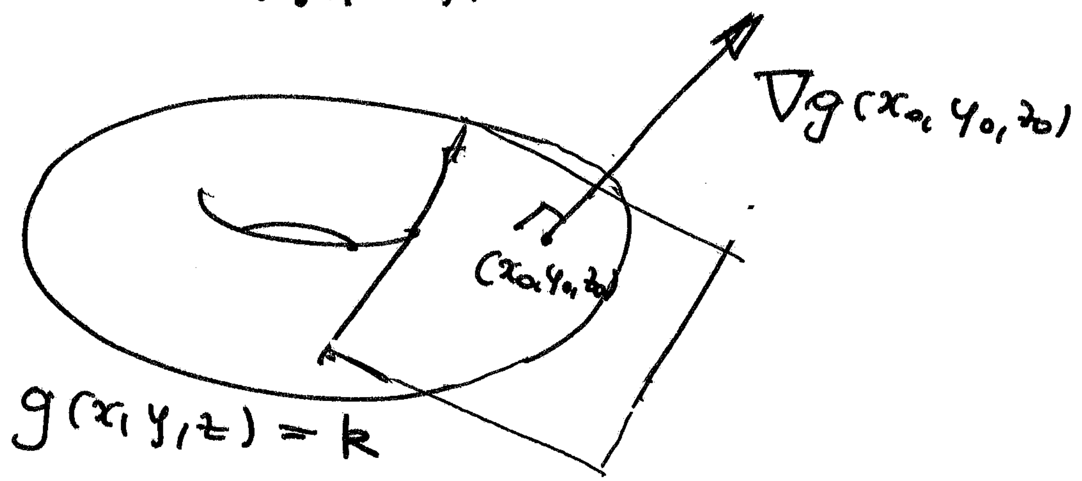
Since the two vectors are parallel, there is λ such that

$$(*) \qquad \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

We need the condition $\nabla g(x_0, y_0) \neq \langle 0,0 \rangle$, because it guarantees that the curve $g(x,y) = k$ is smooth near the point (x_0, y_0). Note also that if $\nabla g(x_0, y_0) = \langle 0, 0 \rangle$, then $\lambda \nabla g(x_0, y_0) = \langle 0, 0 \rangle$ and hence, in general $(*)$ cannot be true.

The argument presented above is intuitive only and we need a rigorous argument. We will show such an argument in the case of functions of three variables.

Recall that on p. 258 we proved that if $\nabla g(x_0, y_0, z_0) \neq \langle 0, 0, 0 \rangle$, then the vector $\nabla g(x_0, y_0, z_0)$ is orthogonal to the tangent plane to the surface $g(x, y, z) = k$ at the point (x_0, y_0, z_0).



Now suppose that $f(x, y, z)$ attains local max. / min. at (x_0, y_0, z_0) subject to the constraint

$$g(x, y, z) = k.$$

That means we consider values of f only for points on the surface $g(x, y, z) = k$.

and among these values $f(x_0, y_0, z_0)$ is a local max / min.

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve that lies on the surface

$$g(x, y, z) = k$$

and $\vec{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, then clearly the function

$$F(t) = f(x(t), y(t), z(t))$$

attains local max./min at $t = t_0$.

Hence

$$0 = F'(t_0) = \frac{d}{dt}\Big|_{t=t_0} f(x(t), y(t), z(t)) =$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt}\Big|_{t=t_0} + \frac{\partial f}{\partial y}\frac{dy}{dt}\Big|_{t=t_0} + \frac{\partial f}{\partial z}\frac{dz}{dt}\Big|_{t=t_0} =$$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle\Big|_{t=t_0} \cdot \langle x'(t_0), y'(t_0), z'(t_0) \rangle$$

$$= \nabla f(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0)$$

$$\nabla f(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0.$$

The vector $\vec{r}\,'(t_0)$ is tangent to
$$g(x,y,z) = k$$
at (x_0, y_0, z_0) and orthogonal to $\nabla f(x_0, y_0, z_0)$. There are infinitely many such curves $\vec{r}$ so the tangent vectors $\vec{r}\,'(t_0)$ can point in different directions of the tangent plane. Always these vectors are orthogonal to $\nabla f(x_0, y_0, z_0)$. That means the vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent plane to the surface $g(x,y,z) = k$ at (x_0, y_0, z_0). But also $\nabla g(x_0, y_0, z_0)$ is orthogonal to the same tangent plane. Hence the vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel, i.e. proportional.

Since $\nabla g(x_0, y_0, z_0) \neq \langle 0, 0, 0 \rangle$, there is a constant λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

In practice, the method of Lagrange Multipliers works as follows

We want to find the absolute maximum and minimum of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$.
We assume that $\nabla g(x,y,z) \neq \vec{0}$ for all points of the surface $g(x,y,z) = k$. Then

① We find all points (x,y,z) such that

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \text{ for some } \lambda \\ g(x,y,z) = k \end{cases}$$

② We take the largest and the smallest value of f among the points from Step ①.

The system of equations in ① can be written as

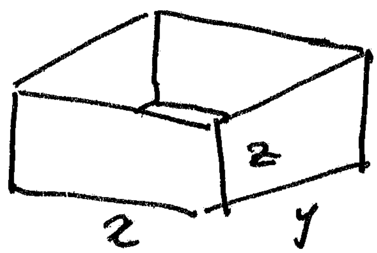
$$\begin{cases} f_x(x,y,z) = \lambda g_x(x,y,z) \\ f_y(x,y,z) = \lambda g_y(x,y,z) \\ f_z(x,y,z) = \lambda g_z(x,y,z) \\ g(x,y,z) = k \end{cases}$$

We have four equations and four unknowns x, y, z, λ. Indeed, the Lagrange multiplier theorem does not tell us what the value of λ is. In general we should expect the system of four equations with four unknowns to have a finite number of solutions.

The first example will be the same as the example on p. 294, but this time we will apply the method of Lagrange multipliers.

Example A rectangular box without a lid is to be made from $12\, m^2$ of cardboard. Find the maximum volume of such a box.

Solution If x, y, z denote sides of the box, then the volume is $V = xyz$, but the surface area is fixed



$$2xz + 2yz + xy = 12.$$

Thus we need to find the maximum of $V = xyz$ subject to the constraint $g(x,y,z) = 2xz + 2yz + xy = 12$.

From geometric considerations we know that the maximum is attained at some point (x_0, y_0, z_0) with $x_0 > 0$, $y_0 > 0$, $z_0 > 0$.

According to the Lagrange Multiplier Theorem

$$\nabla V(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

for some (unknown) constant λ.

We have

$$\nabla V = \langle yz, xz, xy \rangle$$
$$\nabla g = \langle 2z+y, 2z+x, 2x+2y \rangle$$

Note that $\nabla g(x_0, y_0, z_0) \neq \vec{0}$, because $x_0 > 0, y_0 > 0, z_0 > 0$ and indeed we can apply the Lagrange theorem.

Thus we have equations

$$\begin{cases} yz = \lambda(2z+y) & (*) \\ xz = \lambda(2z+x) \\ xy = \lambda(2x+2y) \\ 2xz + 2yz + xy = 12 \end{cases}$$

Note that if we multiply the equations by x, y and z respectively, the left hand sides will be equal to each other

$$\begin{cases} xyz = \lambda(2xz+xy) \\ xyz = \lambda(2yz+xy) \\ xyz = \lambda(2xz+2yz) \end{cases}$$

Hence

$$\lambda(2xz+xy) = \lambda(2yz+xy) + \lambda(2xz+2yz)$$

We have $\lambda \neq 0$. Indeed, $\lambda = 0$ and $(*)$ would imply $yz = 0$ but the solution satisfies $y_0 > 0$, $z_0 > 0$. Thus we can divide by λ

$$(**) \quad 2xz + xy = 2yz + xy = 2xz + 2yz$$

From the first equality

$$2xz + \cancel{xy} = 2yz + \cancel{xy}$$

we obtain $xz = yz$, $x = y$.
Now the equality between the first and the last expression in $(**)$

$$2\cancel{xz} + xy = 2\cancel{xz} + 2yz$$

gives $xy = 2yz$, $z = \frac{x}{2}$. Hence

$$2xz + 2yz + xy = 12$$

becomes $(y = x,\ z = x/2)$:

$$x^2 + x^2 + x^2 = 12$$
$$x^2 = 4$$
$$x = 2 \quad (\text{because } x > 0)$$

Thus $y = 2$, $z = 1$ and the maximal volume is

$$V = 2 \cdot 2 \cdot 1 = 4. \qquad (m^3)$$

The next example is the example from p. 305, but now we will use the Lagrange multipliers

Example Find the absolute maximum and minimum values of
$$f(x,y) = x^2+y^2-2y+1$$
on the disc
$$D = \{(x,y) \mid x^2+y^2 \le 9\}$$

Solution Critical points inside the disc
$$f_x = 2x = 0, \quad f_y = 2y-2 = 0$$
$$(x,y) = (0,1)$$
and this point is in the interior of the disc.
$$f(0,1) = 0.$$
On the boundary we have to find maximum and minimum of
$$f(x,y) = x^2+y^2-2y+1$$
subject to the constraint
$$g(x,y) = x^2+y^2 = 9.$$
According to the Lagrange multiplier theorem there is λ such that

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 9 \end{cases}$$

i.e.

$$(*) \quad \begin{cases} 2x = \lambda \cdot 2x \\ 2y-2 = \lambda \cdot 2y \\ x^2+y^2 = 9 \end{cases}$$

$$2xy = \lambda \cdot 2xy$$
$$2yx - 2x = \lambda \cdot 2yx$$

The right hand sides are equal, so

$$\cancel{2xy} = \cancel{2yx} - 2x$$
$$x = 0$$

From the third equation in (*) we obtain

$$0^2 + y^2 = 9$$
$$y = \pm 3$$

Thus we have two points on the circle $(0,3)$ and $(0,-3)$

$$f(0,3) = 4, \quad f(0,-3) = 16.$$

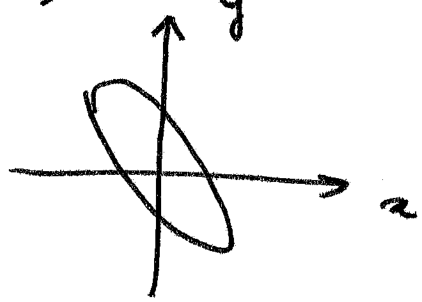
Thus the absolute maximum is $f(0,-3) = 16$ and the absolute minimum is $f(0,1) = 0$.

Example The curve

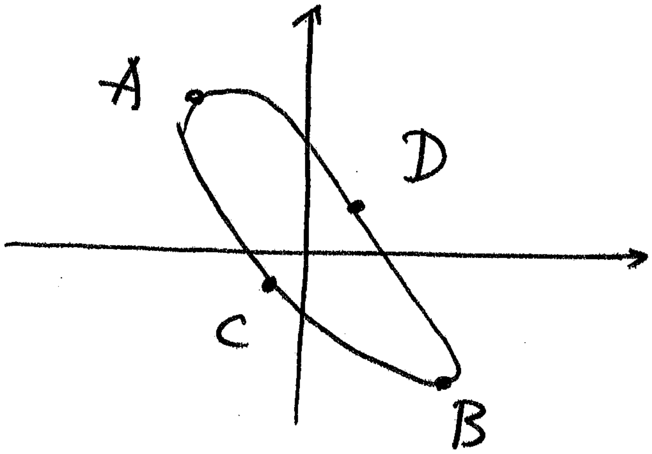
$$x^2 + xy + y^2 = 1$$

is an ellipse, but its axes are not parallel to the coordinate system, i.e., the ellipse is rotated. We will use the Lagrange multiplier theorem to sketch the ellipse.

First observe that if (x, y) solves the equation, then also $(-x, -y)$ solves the equation which means, the ellipse is symmetric with respect to $(0,0)$, i.e., it is centered at $(0,0)$



Now we will use the Lagrange multiplier theorem to find the vertices of the ellipse, i.e. the points A, B, C, D shown on the picture



The points A, B are the points on the ellipse that are furthest away from the origin and the points C, D are closest to the origin.

Thus in order to find points A, B, C, D we need to find maximum and minimum of

$$f(x,y) = x^2 + y^2$$

(the square of the distance of (x,y) to $(0,0)$) subject to the condition

$$g(x,y) = x^2 + xy + y^2 = 1.$$

According to the Lagrange multiplier theorem we have

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases}$$

i.e.

$$\begin{cases} 2x = \lambda(2x+y) \\ 2y = \lambda(2y+x) \\ x^2+xy+y^2=1 \end{cases}$$

Note that $\lambda \neq 0$, because for $\lambda = 0$ we would have $(x, y) = (0,0)$, but this point does not satisfy the equation of the ellipse.

From the first two equations we have

$$2xy = \lambda(2xy+y^2)$$
$$2yx = \lambda(2yx+x^2)$$

The left hand sides are equal so

$$\lambda(2xy+y^2) = \lambda(2yx+x^2)$$
$$\cancel{2xy}+y^2 = \cancel{2yx}+x^2 \qquad (\lambda \neq 0)$$
$$x^2 = y^2$$
$$y = \pm x$$

<u>Case I</u> $\quad x = y$

$$x^2+x^2+x^2=1 \qquad \text{(ellipse's equation)}$$
$$x^2 = \frac{1}{\sqrt{3}}$$
$$x = \pm\frac{1}{\sqrt{3}}$$

$(x,y) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ or $(x,y) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ 339

<u>Case II</u> $\quad y = -x$

$x^2 - x^2 + x^2 = 1$ (ellipse's equation)

$x^2 = 1$

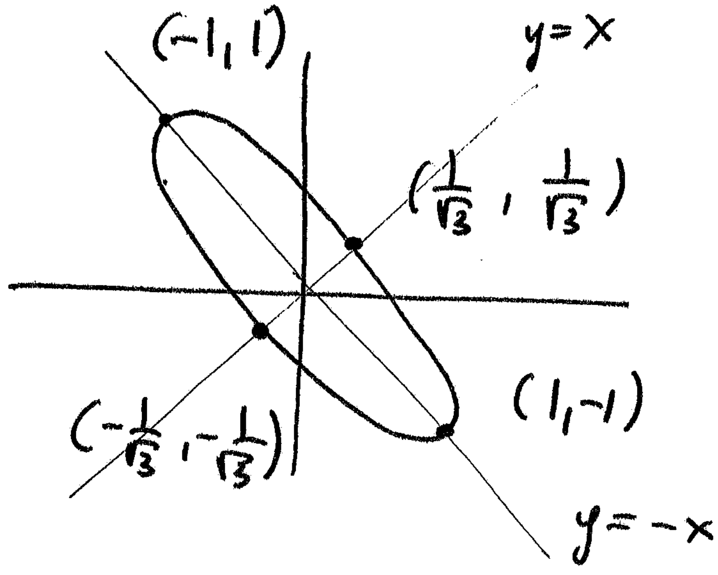
$x = \pm 1$

$(x,y) = (1,-1)$ or $(x,y) = (-1,1)$.

Hence the distance to the origin attains maxima and minima at four points

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), (1,-1), (-1,1)$

Two points for maximum and two points for minimum. These four points are the vertices of the ellipse. Hence the ellipse looks as follows



The lines $y = x$ and $y = -x$ pass through the vertices so they form axes of the ellipse – the ellipse is rotated by $45°$ clockwise.

Problems

Exercise 100 Find the area of the largest rectangle with pairs of sides parallel to the coordinate axes that is inscribed in the ellipse $x^2 + 4y^2 = 1$.

Solution If (x, y) is the vertex of the rectangle that is located in the first quadrant, then the sides have length $2x$ and $2y$ and hence the area equals

$$f(x,y) = (2x)\cdot(2y) = 4xy.$$

Note that $x>0$, $y>0$. Since the point (x,y) is on the ellipse we also have $x^2+4y^2=1$. Thus we are asked to find maximum of

$$f(x,y) = 4xy$$

subject to the condition

$$g(x,y) = x^2+4y^2=1.$$

We have

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1, \end{cases}$$

i.e.,

$$\begin{cases} 4y = \lambda\cdot 2x \\ 4x = \lambda\cdot 8y \\ x^2+4y^2=1 \end{cases}$$

Clearly $\lambda \neq 0$, because for $\lambda = 0$ we would have $(x,y)=(0,0)$ which is not a point on the ellipse.

We have

$$4yx = \lambda \cdot 2x^2$$
$$4xy = \lambda \cdot 8y^2$$

$$\lambda \cdot 2x^2 = \lambda \cdot 8y^2$$
$$x^2 = 4y^2 \qquad (\lambda \neq 0)$$
$$y = \frac{x}{2} \qquad (x, y > 0)$$

Now the equation of the ellipse gives

$$x^2 + 4\left(\frac{x}{2}\right)^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \frac{1}{\sqrt{2}} \qquad (x > 0)$$
$$y = \frac{1}{2\sqrt{2}}$$

and the largest area is

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = 4 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} = 1.$$

<u>Exercise 10)</u> Using the method of Lagrange multipliers find the distance of the point $A(17, -4, -3)$ to the plane $6x - 3y + 2z = 10$.

Solution An equivalent problem is to find the minimum of the square of the distance

$$f(x,y,z) = (x-17)^2 + (y+4)^2 + (z+3)^2$$

between $(17,-4,-3)$ and a point (x,y,z) on the plane $6x-3y+2z=10$, i.e., we want to find the minimum of

$$f(x,y,z) = (x-17)^2 + (y+4)^2 + (z+3)^2$$

subject to the constraint

$$g(x,y,z) = 6z - 3y + 2z = 10.$$

It follows from geometric considerations that the minimum is attained and then by the Lagrange multiplier theorem at the point of minimum we have

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 10, \end{cases}$$

i.e.

$$\begin{cases} 2(x-17) = \lambda \cdot 6 \\ 2(y+4) = \lambda \cdot (-3) \\ 2(z+3) = \lambda \cdot 2 \\ 6x - 3y + 2z = 10 \end{cases}$$

Solving the first three equations yields

$$(*)\quad \begin{cases} x = 3\lambda + 17 \\ y = -\frac{3}{2}\lambda - 4 \\ z = \lambda - 3 \end{cases}$$

Hence the equation of the plane gives

$$6(3\lambda+17) - 3\left(-\frac{3}{2}\lambda - 4\right) + 2(\lambda - 3) = 10$$

$$\frac{49}{2}\lambda = -98$$

$$\lambda = -4,$$

So from $(*)$ we obtain

$$x = 5, \quad y = 2, \quad z = -7$$

Hence the minimal distance is

$$d = \sqrt{f(5,2,-7)} = \sqrt{196} = 14.$$

Also the point on the plane that is closest to $(17,-4,-3)$ is $(5,2,-7)$.