Homework 2

1. Let $K \subseteq L \subseteq M$ be fields with $L$ normal over $K$, and $M$ normal over $L$. Prove that if every automorphism of $L/K$ can be extended to $M$, then $M$ is normal over $K$.

2. Prove that for fields $L$ and $M$ we have $(L \cup M)' = L' \cap M'$ and for groups $H$ and $J$ we have $(H \cup J)' = H' \cap J'$. Extend to arbitrary, even infinite, unions. Prove that any intersection of closed fields or groups is closed.

3. If $K$ is an infinite field and $x$ is an indeterminate, then $K(x)$ is normal over $K$. [Hint: The map $x \mapsto x + a$, for $a \in K$, induces a field automorphism of $K(x)/K$. If the rational function $f/g$ lies in the fixed field $K'$ let $h(x, y) = f(x)g(x + y) - g(x)f(x + y)$. By using that $K$ is infinite, show that $h$ is identically 0. Conclude that $f/g$ is a constant.]

4. With $K$ any field and $x$ an indeterminate, prove that $K(x)$ is finite-dimensional over any intermediate field $L$ different from $K$. [Hint: If $r = f/g \in L$, then $x$ satisfies the equation $rg(x) - f(x) = 0$.]

5. If $K$ is an infinite field and $x$ is an indeterminate, then the only closed subgroups of $G := \text{Gal}(K(x)/K)$ are the finite subgroups of $G$ and $G$ itself.

6. Prove that in the field $Q(x)$, with $x$ an indeterminate, the subfield $Q(x^2)$ is closed but the subfield $Q(x^3)$ is not closed.