Autumn 2007

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ALGEBRA – Final exam

Provide full justification for all assertions made.

1. Assume all field extensions are finite.

(a) If L is normal over K, M is normal over L, and any automorphism of L/K is extendable to an automorphism of M/K, then M is normal over K.

(b) If L and M are each normal over F, then $L \cap M$ and $L \cup M$ are normal over F.

2. If $f \in F[X]$ is a polynomial of degree n (> 1) whose Galois group has order $n!$, then f is irreducible over F. Is the converse statement true?

3. If $L = Q(\zeta, \sqrt[5]{3})$, with ζ a primitive 5^{th} root of unity, then the Galois group of L over Q is of order 20. Describe this group as explicitly as possible.

4. Exhibit in full detail the Galois correspondence in the case of the extension $Q \subseteq Q(i, \sqrt[4]{2})$. [To learn something, do this on your own. Forget about textbooks.]

5. With t an indeterminate, consider the automorphisms of $F(t)$ over F defined by

$$\sigma(\alpha(t)) = \alpha(t^{-1}) \text{ and } \tau(\alpha(t)) = \alpha(t-1).$$

(a) $G = <\sigma, \tau>$ is isomorphic to S_3.

(b) The fixed field of G is $G' = F(\frac{(t^2-t+1)^3}{t^2(t-1)^2})$.

(c) $G' \subseteq F(t)$ is a Galois extension with Galois group S_3.

6. If F is an infinite field and t is an indeterminate, then $F(t)$ is normal over F.

7. With t an indeterminate and L an intermediate field other than F in the field extension $F \subset F(t)$, prove that $F(t)$ is finite dimensional over L.

8. If F is infinite and t is an indeterminate, then the only closed subgroups of the $G = Gal(F(t)/F)$ are G itself and its finite subgroups.