ALGEBRA 2 – EXAM

1. If \(|L : K|\) is prime prove that there are no fields between \(K\) and \(L\).
2. If the degree of \(u\) over \(K\) is odd prove that \(K(u) = K(u^2)\).
3. Let \(u\) and \(v\) be algebraic over \(K\) of degree \(m\) and \(n\) respectively. Show that \(u\) has degree \(m\) over \(K(v)\) if and only if \(v\) has degree \(n\) over \(K(u)\) and both statements are true if \(m\) and \(n\) are coprime.
4. Give an example where \(K \subseteq L \subseteq M; M\) is normal over \(K; L\) is closed and normal over \(K\); yet \(L\) is not stable. [\textit{Hint}: Take \(K\) infinite, \(M = K(x, y), L = K(x),\) with \(x\) and \(y\) indeterminates.]
5. Let \(K \subseteq L \subseteq M\) be fields with \(L\) a splitting field over \(K\). Prove that \(L\) is stable.
6. If \(M\) is a finite-dimensional normal extension of \(K\), then any element of \(K\) is the trace of a suitable element of \(M\). Is an analogous statement true about the norm?
7. Let \(K\) be a field of characteristic \(p\) and \(a\) and \(b\) nonzero elements of \(K\). Prove that \(x^p - x - a\) is either irreducible or it factors completely in \(K\). [\textit{Hint}: If \(u\) is a root, examine \(u + i, i \in F_p\).] Prove that \(x^p - b^{p-1}x - a\) and \(x^p + b^{p-1}a^{-1}x^{p-1} - a^{-1}\) have the same property.
8. Prove that any element of a finite field can be written as the sum of two squares.
9. Determine the Galois group of the cubic \(x^3 - 3x + 1\) over \(Q\). If \(\theta\) is a root of this polynomial, express the other roots in the form \(a + b\theta + c\theta^2\) for some \(a, b, c \in Q\).
10. Prove that the polynomial \(x^4 + px + p\) is irreducible over \(Q\) for every prime \(p\). Show that for \(p = 3\) its Galois group over \(Q\) is \(D_3\), for \(p = 5\) it is \(C_4\); otherwise its Galois group is \(S_4\).
11. Let \(p\) be a prime and let \(a, b \in Q\) such that \(a^2 + pb^2 = 1\). Show that there exist \(c, d \in Q\) such that \(a = \frac{c^2 + pd^2}{c^2 - pd^2}\) and \(b = \frac{2cd}{c^2 - pd^2}\). [\textit{Hint}: Apply Hilbert’s Thorem 90 to the field \(Q(\sqrt{p})\).]
12. Find the splitting field and the Galois group of the polynomial \(x^3 - 5\) over the field \(Q(\sqrt{2})\).
13. Factorize \(x^4 + 1\) over the fields \(F_5\), and \(F_{25}\). Find the splitting field in each case.