Recall the Properties of Exponents

Property:	**Example:**
$b^0 = 1$	$5^0 = 1$
$b^{-n} = \frac{1}{b^n}$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
$b^m \bullet b^n = b^{m+n}$	$2^3 \bullet 2^5 = 2^8$
$\frac{b^m}{b^n} = b^{m-n}$	$\frac{5^9}{5^7} = 5^2 = 25$
$(b^m)^n = b^{m \bullet n}$	$(2^3)^4 = 2^{12} = 4096$
$(a \bullet b)^n = a^n \bullet b^n$	$(4x)^3 = 4^3 \bullet x^3 = 64x^3$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{5}\right)^2 = \frac{3^2}{5^2} = \frac{9}{25}$

If b is any positive number except 1, then we have two very important properties for exponents:

1) If $x = y$, then $b^x = b^y$
2) If $b^x = b^y$, then $x = y$

These properties are extremely useful for solving certain equations.

Examples: Solve the following equations for x using mathematics writing style:

a) $3^{x-2} = \sqrt{3}$

$3^{x-2} = 3^{1/2}$

$x - 2 = 1/2$

$x = 5/2$

b) $2^{x^2-7} = 2^{6x}$

$x^2 - 7 = 6x$

$x^2 - 6x - 7 = 0$

$(x - 7)(x + 1) = 0$

$x = 7$ or $x = -1$

c) $8^{4-2x} = 4^{x+2}$

$\left(2^3\right)^{4-2x} = \left(2^2\right)^{x+2}$

$2^{3(4-2x)} = 2^{2(x+2)}$

$3(4 - 2x) = 2(x + 2)$

$12 - 6x = 2x + 4$

$-8x = -8$

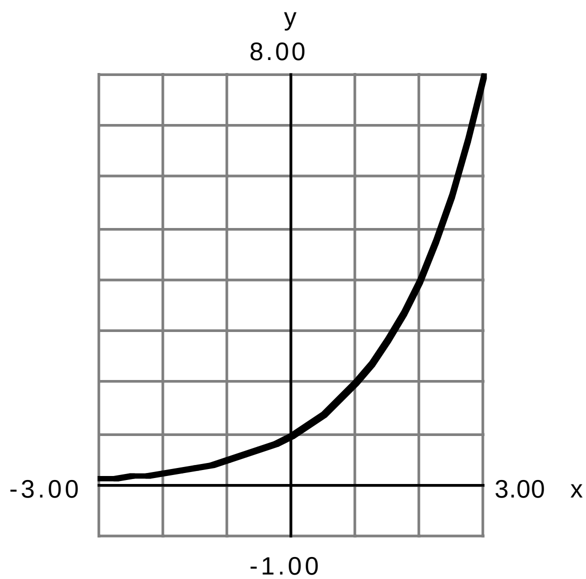
$x = 1$

If b is any positive number except 1, then we can define an exponential function with base b by

$$f(x) = b^x, \; b > 0, \; b \neq 1$$

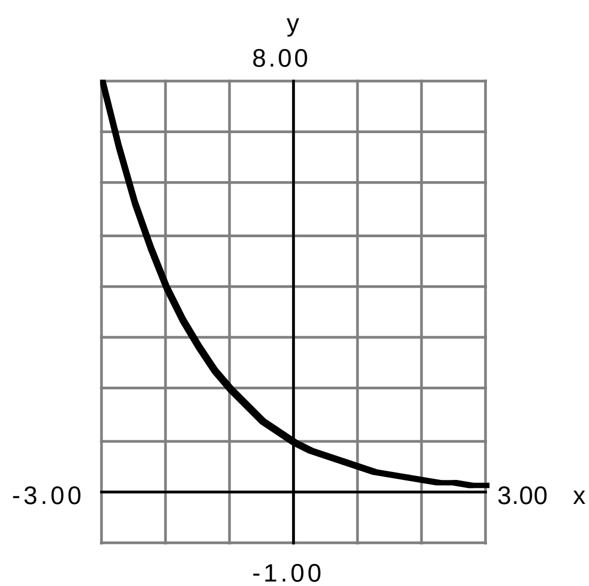
If b > 1, then the graph is positive and increasing and the line y = 0 (x–axis) is a horizontal asymptote.

x	$y = 2^x$
–3	1/8
–2	1/4
–1	1/2
0	1
1	2
2	4
3	8



If 0 < b < 1, then the graph is positive and decreasing and the line y = 0 (x–axis) is a horizontal asymptote.

x	$y = (1/2)^x$
–3	8
–2	4
–1	2
0	1
1	1/2
2	1/4
3	1/8



Introduction to Logarithms

Definition: Let $b > 0$, and $b \neq 1$, and $x = b^y$. Then y is the logarithm of x (base b). That is,

$$x = b^y \text{ means that } y = \log_b x.$$

Two special logarithm bases have special designations. They are common logarithms (base 10) and natural logarithms (base e).

Base	We Write	We Mean
base 10	log x	$\log_{10} x$
base e	ln x	$\log_e x$

Note: Sometimes we simply write $\log_b x$, but the meaning is always $\log_b (x)$, since we are using a function with argument x. If there is any confusion about the argument, then use parentheses.

Exponential Equation	Logarithmic Equation
$5^2 = 25$	$\log_5 25 = 2$
$2^{-3} = \frac{1}{8}$	$\log_2 \frac{1}{8} = -3$
$10^4 = 10{,}000$	$\log 10000 = 4$
$e^0 = 1$	$\ln 1 = 0$

Example 1: Write the following exponential equations into logarithmic equations.

Exponential Equation	Logarithmic Equation
$2^{-2} = 0.24$	
$6^0 = 1$	
$27^{4/3} = 81$	
$10^2 = 100$	
$2^{10} = 1024$	
$10^{-3} = 0.001$	
$e^1 = e$	

Example 2: Write the following logarithmic equations into exponential equations.

Exponential Equation	Logarithmic Equation
	$\log_3 81 = 4$
	$\log_5 0.04 = -2$
	$\log_4 8 = 3/2$
	$\log_8 4 = 2/3$
	$\log \sqrt{10} = 1/2$
	$\log 10 = 1$
	$\ln 1/e = -1$

Properties of Logarithms

Assume that $b > 0$, $b \neq 1$.

PROPERTIES	EXAMPLES
$\log_b 1 = 0$	$\log 1 = 0$, $\ln 1 = 0$, $\log_5 1 = 0$
$\log_b b = 1$	$\log 10 = 1$, $\ln e = 1$, $\log_2 2 = 1$
$\log_b (x \cdot y) = \log_b x + \log_b y$	$\ln 10 = \ln 2 + \ln 5$
$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$	$\log \frac{3}{4} = \log 3 - \log 4$
$\log_b (x^n) = n \cdot \log_b x$	$\log_8 25 = \log_8 5^2 = 2 \cdot \log_8 5$
$\log_b (b^x) = x$	$\log(10^5) = 5$, $\log_8 8^{-3} = -3$
$b^{\log_b x} = x$ (if $x > 0$)	$10^{\log 6} = 6$, $3^{\log_3 4} = 4$

More Logarithmic Expressions

Examples: Use the properties of logarithms to write each expression as a sum and/or difference of logarithmic expressions.

a) $\log 2x^3$

$= \log 2 + \log x^3$

$= \log 2 + 3 \log x$

b) $\log \frac{ab^2}{c}$

$= \log a + \log b^2 - \log c$

$= \log a + 2 \log b - \log c$

c) $\ln (xy)^{-3}$

$= -3(\ln xy)$

$= -3(\ln x + \ln y)$

d) $\log_2 \frac{\sqrt[3]{x^2y}}{zw^5}$

$= \frac{1}{3}(2 \log_2 x + \log_2 y) - \log_2 z - 5 \log_2 w$

Examples: Use the properties of logarithms to write each expression as a single logarithm.

a) $2 \log_b x - 3 \log_b y$

$= \log_b \left(\frac{x^2}{y^3} \right)$

b) $\frac{1}{2} \ln 4 + \ln 5 - \ln y$

$= \ln \left(\frac{\sqrt{4} \cdot 5}{y} \right)$

$= \ln \left(\frac{10}{y} \right)$

c) $\log (x + 1) + \log(x - 1)$

$= \log [(x + 1)(x - 1)]$

$= \log (x^2 - 1)$

d) $2 \log_3 A - 3 \log_3 B + 4 \log_3 C$

$= \log_3 \left(\frac{A^2 C^4}{B^3} \right)$

Using Logarithms to Solve Equations

For $b > 0$, $b \neq 1$, and $x > 0$ and $y > 0$, we get two important logarithmic properties.

1) If $x = y$, then $\log_b x = \log_b y$

2) If $\log_b x = \log_b y$, then $x = y$.

Recall that the argument of a logarithm is always positive. When you solve equations with logarithms, you should always check your answer!

Examples: Solve the following exponential equations.

a) $10^{3x} = 2.1$

$3x = \log 2.1$

$x = \dfrac{\log 2.1}{3}$

b) $e^{0.2x} = 50$

$0.2x = \ln 50$

$x = 5 \ln 50$

c) $6^x = 18$

$\log (6^x) = \log 18$

$x \log 6 = \log 18$

$x = \dfrac{\log 18}{\log 6}$

d) $5^{2x-1} = 10^x$

$\log(5^{2x-1}) = \log(10^x)$

$(2x-1)(\log 5) = x$

$2x \log 5 - \log 5 = x$

$2x \log 5 - x = \ln 5$

$x(2 \log 5 - 1) = \ln 5$

$x = \dfrac{\log 5}{2 \log 5 - 1}$

e) $5^{x+2} = \dfrac{1}{25}$

$5^{x+2} = 5^{-2}$

$x + 2 = -2$

$x = -4$

f) $4^{x+2} = \dfrac{1}{25}$

$4^{x+2} = 5^{-2}$

$(x + 2)\ln 4 = -2 \ln 5$

$x \ln 4 + 2 \ln 4 = -2 \ln 5$

$x = \dfrac{-2 \ln 4 - 2 \ln 5}{\ln 4}$

Examples: Solve the following logarithmic equations.

a) $\log(x - 1) + \log(x - 4) = 1$

$$\log [(x - 1)(x - 4)\} = 1$$

$$(x - 1)(x - 4) = 10^1$$

$$x^2 - 5x + 4 = 10$$

$$x^2 - 5x - 6 = 0$$

$$(x - 6)(x + 1) = 0$$

$$x = 6 \quad \text{or} \quad x = -1$$

$$x = 6$$

b) $\log_2 x - \log_2 (x - 1) = \log_2 3$

$$\log_2 \left(\frac{x}{x - 1}\right) = \log_2 3$$

$$\frac{x}{x - 1} = 3$$

$$x = 3(x - 1)$$

$$x = 3x - 3$$

$$-2x = -3$$

$$x = \frac{3}{2}$$