Basic Properties of Rational Expressions

A fraction is not defined when the denominator is zero!

Examples: Simplify and use Mathematics Writing Style.

a) $\dfrac{2x + 8}{2}$ b) $\dfrac{x^2 - 9}{x - 3}$

Solution:

a) $\dfrac{2x + 8}{2} = \dfrac{2(x + 4)}{2}$

$$\boxed{= \quad x + 4}$$

b) $\dfrac{x^2 - 9}{x - 3} = \dfrac{(x + 3)(x - 3)}{x - 3}$

$$\boxed{= \quad x + 3}$$

The Domain of a Rational Function:

Examples: Find the domain of the following functions:

a) $f(x) = 2x + 4$

b) $g(x) = \dfrac{x + 5}{x - 4}$

c) $h(x) = \dfrac{2x + 5}{x^3 - 4x}$

d) $F(x) = \dfrac{x}{x^2 + 9}$

Solution:

a) $f(x) = 2x + 4$

This is a polynomial, and is defined for all values of x. So the domain is the set of real numbers.

$$\boxed{D = \mathbb{R}}$$

b) $g(x) = \dfrac{x + 5}{x - 4}$

This is a rational function and is not defined when the denominator is zero, or when

$$x - 4 = 0$$
$$x = 4$$

So the domain is consists of all real numbers except that.

$$\boxed{D = \{x \quad \mathbb{R} \mid x \neq 4\}}$$

c) $h(x) = \dfrac{2x + 5}{x^3 - 4x}$

This is a rational function and is not defined when the denominator is zero, or when

$$x^3 - 4x = 0$$
$$x(x^2 - 4) = 0$$
$$x(x - 2)(x + 2) = 0$$
$$x = 0, 2, -2$$

So the domain is consists of all real numbers except these.

$$\boxed{D = \{x \quad \mathbb{R} \mid x \neq 0, 2, -2\}}$$

d) $F(x) = \dfrac{x}{x^2 + 9}$

This is a rational function and is not defined when the denominator is zero, or when

$$x^2 + 9 = 0$$

But this expression is never zero for any real value of x. So the domain consists of all real numbers.

$$\boxed{D = \mathbb{R}}$$

Multiplication and Division of Rational Expressions

Example: Perform the indicated operation and simplify:

$$\frac{x^2 + 2x - 3}{x^2 - 3x - 10} \div \frac{4x + 2}{x^2 - x} \bullet \frac{2x^2 - 9x - 5}{x^2 - 2x + 1}$$

$$= \frac{x^2 + 2x - 3}{x^2 - 3x - 10} \bullet \frac{x^2 - x}{4x + 2} \bullet \frac{2x^2 - 9x - 5}{x^2 - 2x + 1}$$

$$= \frac{(x + 3)(x - 1)}{(x - 5)(x + 2)} \bullet \frac{x(x - 1)}{2(2x + 1)} \bullet \frac{(2x + 1)(x - 5)}{(x - 1)^2}$$

$$\boxed{= \frac{x(x + 3)}{2(x + 2)}}$$

Adding or Subtracting Fractions with Equal Denominators

Examples: Perform the indicated operation and simplify:

a) $\dfrac{x}{x^2 - 4} + \dfrac{2}{x^2 - 4}$

$$= \frac{x + 2}{x^2 - 4}$$

$$= \frac{x + 2}{(x + 2)(x - 2)}$$

$$\boxed{= \frac{1}{x - 2}}$$

b) $\dfrac{x^2}{x^2 - 4} - \dfrac{4x - x^2}{x^2 - 4}$

$$= \frac{x^2 - (4x - x^2)}{x^2 - 4}$$

$$= \frac{x^2 - 4x + x^2}{x^2 - 4}$$

$$= \frac{2x^2 - 4x}{x^2 - 4}$$

$$= \frac{2x(x - 2)}{(x + 2)(x - 2)}$$

$$\boxed{= \frac{2x}{(x + 2)}}$$

Finding the LCD

Step 1. Factor each denominator completely, including the prime factors of any constant factor.

Step 2. Form the product of all the factors that appears in the complete factorizations.

Step 3. The number of times any factors appears in the LCD is the most number of times it appears in any one factorization.

Examples: Find the LCD for the given denominators:

a) Denominators are 24, 30, and 36

$24 = 8 \cdot 3 = 2^3 \cdot 3$

$30 = 6 \cdot 5 = 2 \cdot 3 \cdot 5$

$36 = 4 \cdot 9 = 2^2 \cdot 3^2$

$\text{LCD} = 2^3 \cdot 3^2 \cdot 5 = 360$

b) Denominators are $x^3 - x^2$ and $x^3 - x$

$x^3 - x^2 = x^2(x - 1)$

$x^3 - x = x(x^2 - 1) = x(x + 1)(x - 1)$

$\text{LCD} = x^2(x + 1)(x - 1)$

Adding or Subtracting Fractions with Unequal Denominators

(FLEAS)

1. **F**actor the rational expression.

2. Find the **L**east Common Denominator (LCD).

3. **E**qualize each denominators by replacing each fraction with an equivalent one whose denominator is the LCD.

4. **A**dd or **S**ubtract using RASFED.

Example:

$$\frac{1}{2} - \frac{2}{3} + \frac{3}{4}$$

$$= \frac{6}{12} - \frac{8}{12} + \frac{9}{12}$$

$$= \frac{6 - 8 + 9}{12}$$

$$\boxed{= \frac{7}{12}}$$

Examples: Perform the indicated operation and simplify.

a) $\dfrac{2}{x^2 - 1} - \dfrac{1}{x^2 + 2x + 1}$

$$= \frac{2}{(x+1)(x-1)} - \frac{1}{(x+1)^2}$$

$$= \frac{2(x+1)}{(x+1)^2(x-1)} - \frac{x-1}{(x+1)^2(x-1)}$$

$$= \frac{2(x+1) - (x-1)}{(x+1)^2(x-1)}$$

$$= \frac{2x + 2 - x + 1}{(x+1)^2(x-1)}$$

$$\boxed{= \frac{x+3}{(x+1)^2(x-1)}}$$

b) $\dfrac{x+2}{x-2} - \dfrac{x^2 + 2x}{x^2 - 4}$

$$= \frac{x+2}{x-2} - \frac{x^2 + 2x}{(x+2)(x-2)}$$

$$= \frac{(x+2)^2}{(x+2)(x-2)} - \frac{x^2 + 2x}{(x+2)(x-2)}$$

$$= \frac{(x^2 + 4x + 4) - (x^2 + 2x)}{(x+2)(x-2)}$$

$$= \frac{x^2 + 4x + 4 - x^2 - 2x}{(x+2)(x-2)}$$

$$= \frac{2x + 4}{(x+2)(x-2)}$$

$$= \frac{2(x+2)}{(x+2)(x-2)}$$

$$\boxed{= \frac{2}{(x-2)}}$$

Complex Fractions

To simplify complex fractions:

Step 1: Identify all fractions in the numerator and denominator and find the LCD.

Step 2: Multiply the numerator and denominator by the LCD.

Examples:

a) $$\frac{\frac{1}{2} - \frac{1}{3}}{\frac{3}{4} - \frac{1}{6}}$$

$$= \frac{12\left(\frac{1}{2} - \frac{1}{3}\right)}{12\left(\frac{3}{4} - \frac{1}{6}\right)}$$

$$= \frac{6 - 4}{9 - 2}$$

$$\boxed{= \frac{2}{7}}$$

b) $$\frac{x + y}{x^{-1} + y^{-1}}$$

$$= \frac{x + y}{\frac{1}{x} + \frac{1}{y}}$$

$$= \frac{xy\,(x + y)}{xy\left(\frac{1}{x} + \frac{1}{y}\right)}$$

$$= \frac{xy\,(x + y)}{y + x}$$

$$\boxed{= xy}$$

Example: Simplify the following:

a) $$\frac{9 - \frac{1}{y^2}}{3 - \frac{1}{y}}$$

$$= \frac{y^2\left(9 - \frac{1}{y^2}\right)}{y^2\left(3 - \frac{1}{y}\right)}$$

$$= \frac{9y^2 - 1}{3y^2 - y}$$

$$= \frac{(3y + 1)(3y - 1)}{y(3y - 1)}$$

$$\boxed{= \frac{3y + 1}{y}}$$

b) $$\frac{\frac{1}{x + h} - \frac{1}{x}}{h}$$

$$= \frac{x(x + h)\left(\frac{1}{x + h} - \frac{1}{x}\right)}{x(x + h)\,h}$$

$$= \frac{x - (x + h)}{x(x + h)\,h}$$

$$= \frac{x - x - h}{x(x + h)\,h}$$

$$= \frac{-h}{x(x + h)\,h}$$

$$\boxed{= \frac{-1}{x(x + h)}}$$

Long Division of Polynomials

Monomial Denominator: When you divide a polynomial by a monomial, you must divide each term in the numerator by the denominator

Examples: Perform the indicated operation.

a) $(x^3 - 6x^2 + 2x) \div 3x$

$$= \frac{x^3 - 6x^2 + 2x}{3x}$$

$$= \frac{x^3}{3x} - \frac{6x^2}{3x} + \frac{2x}{3x}$$

$$\boxed{= \frac{x^2}{3} - 2x + \frac{2}{3}}$$

b) Divide $15y^3+20y^2-5y$ by $5y$

$$\frac{15y^3 + 20y^2 - 5y}{5y}$$

$$= \frac{15y^3}{5y} + \frac{20y^2}{5y} - \frac{5y}{5y}$$

$$\boxed{= 3y^2 + 4y - 1}$$

c) $(16x^3 - 8x^2 + 3x) \div 2x$

$$= \frac{16x^3 - 8x^2 + 3x}{2x}$$

$$= \frac{16x^3}{2x} - \frac{8x^2}{3x} + \frac{2x}{2x}$$

$$\boxed{= 8x^2 - \frac{8x}{3} + 1}$$

d) $(-16z^4 + 16z^3 + 8z^2 + 64z) \div 8z$

$$= \frac{-16z^4 + 16z^3 + 8z^2 + 64z}{8z}$$

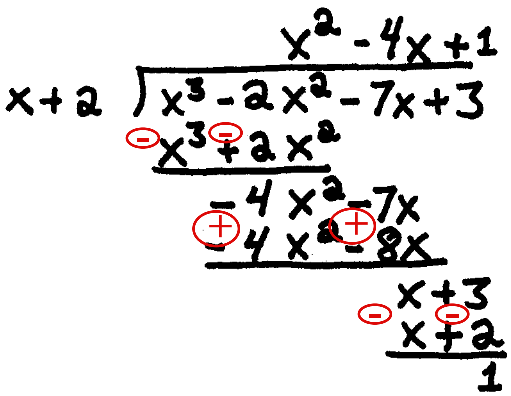
$$= -\frac{16z^4}{8z} + \frac{16z^3}{8z} + \frac{8z^2}{8z} - \frac{64z}{8z}$$

$$\boxed{= -2z^3 + 2z^2 + z - 8}$$

Long Division of Polynomials

Examples: Calculate the indicated quotients by long division:

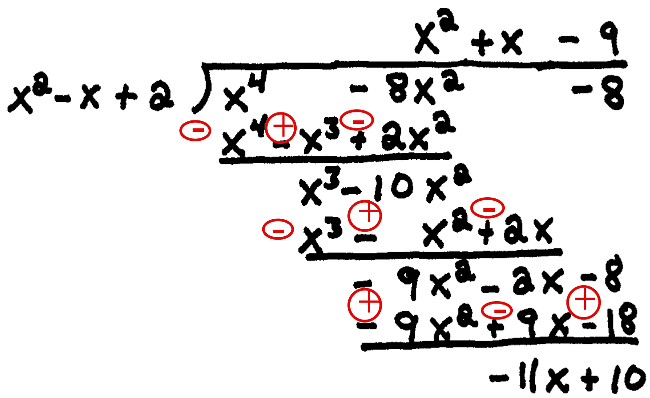
a) $\dfrac{x^3 - 2x^2 - 7x + 3}{x + 2}$



$$\frac{x^3 - 2x^2 - 7x + 3}{x + 2}$$

$$= x^2 - 4x + 1 + \frac{1}{x + 2}$$

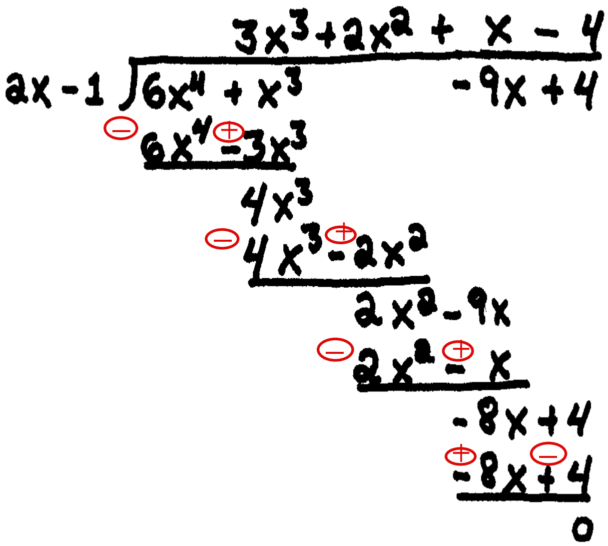
b) $\dfrac{x^4 - 8x^2 - 8}{x^2 - x + 2}$



$$\frac{x^4 - 8x^2 - 8}{x^2 - x + 2}$$

$$= x^2 + x - 9 + \frac{-11x + 10}{x^2 - x + 2}$$

c) $\dfrac{6x^4 + x^3 - 9x + 4}{2x - 1}$



$$\frac{6x^4 + x^3 - 9x + 4}{2x - 1}$$

$$\boxed{= 3x^3 + 2x^2 + x - 4}$$

Synthetic Division of Polynomials

You can only use synthetic division when you divide a polynomial by a **linear polynomial with linear coefficient 1**.

Examples: Calculate the indicated quotients by synthetic division:

a) $\dfrac{x^3 - 2x^2 - 7x + 3}{x + 2}$

–2	1	–2	–7	3
		–2	8	–2
	1	–4	1	1

$$\boxed{\frac{x^3 - 2x^2 - 7x + 3}{x + 2} = x^2 - 4x + 1 + \frac{1}{x + 2}}$$

b) $\dfrac{x^4 - 8x^2 - 8}{x - 3}$

3	1	0	–8	0	–8
		3	9	3	9
	1	3	1	3	1

$$\frac{x^4 - 8x^2 - 8}{x - 3} = x^3 + 3x^2 + x + 3 + \frac{1}{x - 3}$$

c) $\dfrac{x^4 - 81}{x + 3}$

–3	1	0	0	0	–81
		–3	9	–27	81
	1	–3	9	–27	0

$$\frac{x^4 - 81}{x + 3} = x^3 - 3x^2 + 9x - 27$$

Remainder Theorem

When you divide a polynomial P(x) by the factor x – c, the remainder is P(c). Thus we sometimes evaluate a polynomial P(x) when x = c by performing the appropriate synthetic division.

Examples 1: Let $P(x) = 2x^3 - 4x^2 + 5$.

a) By direct substitution, evaluate P(2).

$$P(x) = 2x^3 - 4x^2 + 5$$

$$P(2) = 2(2)^3 - 4(2)^2 + 5$$

$$= 2 \cdot 8 - 4 \cdot 4 + 5$$

$$\boxed{= 5}$$

b) Find the remainder when P(x) is divided by x – 2.

$$\begin{array}{r|rrrr} 2 & 2 & -4 & 0 & 5 \\ & & 4 & 0 & 0 \\ \hline & 2 & 0 & 0 & 5 \end{array}$$

$$\boxed{\text{Remainder is } 5}$$

Examples 2: Let $P(x) = 4x^6 - 25x^5 + 35x^4 + 17x^2$. Find P(4)

$$\begin{array}{r|rrrrrrr} 4 & 4 & -25 & 35 & 0 & 17 & 0 & 0 \\ & & 16 & -36 & -4 & -16 & 4 & 16 \\ \hline & 4 & -9 & -1 & -4 & 1 & 4 & 16 \end{array}$$

$$\boxed{P(4) = 16}$$

Note the problem is easier when we use the Remainder Theorem

Equations Involving Fractions

To solve equations with (simple) fractions:

Step 1: Identify all fractions in the equation and find the LCD.

Step 2: Multiply the both sides of the equation by the LCD.

Step 3: Solve the resulting equation.

Step 4: Check the answer into the original problem.

Examples: Solve the following:

a) $$\frac{2}{x-6} = \frac{3}{x-8}$$

$$(x-6)(x-8)\frac{2}{x-6} = (x-6)(x-8)\frac{3}{x-8}$$

$$2(x-8) = 3(x-6)$$

$$2x - 16 = 3x - 18$$

$$-x = -2$$

$$\boxed{x = 2}$$

b) $$\frac{z-4}{z^2-2z} = \frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3-2z^2}$$

$$\frac{z-4}{z(z-2)} = \frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^2(z-2)}$$

$$z^2(z-2)\frac{z-4}{z(z-2)} = z^2(z-2)\left(\frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^2(z-2)}\right)$$

$$z(z-4) = z(z-2) - 2(z-2) - 4$$

$$z^2 - 4z = z^2 - 2z - 2z + 4 - 4$$

$$z^2 - 4z = z^2 - 4z$$

$$0 = 0$$

The answer would be all real numbers, but we much check!

$$\boxed{\text{all real numbers except } 0,\ 2}$$

c) $$\frac{y-2}{y-3} = 1 - \frac{2}{y^2-9}$$

$$\frac{y-2}{y-3} = 1 - \frac{2}{(y+3)(y-3)}$$

$$(y+3)(y-3)\frac{y-2}{y-3} = (y+3)(y-3)\left(1 - \frac{2}{(y+3)(y-3)}\right)$$

$$(y+3)(y-2) = (y+3)(y-3) - 2$$

$$y^2 + y - 6 = y^2 - 9 - 2$$

$$\boxed{y = -5}$$

d) $$\frac{2}{x^2 - 4} = \frac{1}{x^2} + \frac{1}{x^2 - 2x}$$

$$\frac{2}{(x + 2)(x - 2)} = \frac{1}{x^2} + \frac{1}{x(x - 2)}$$

$$x^2(x + 2)(x - 2)\frac{2}{(x + 2)(x - 2)} = x^2(x + 2)(x - 2)\left(\frac{1}{x^2} + \frac{1}{x(x - 2)}\right)$$

$$2x^2 = (x + 2)(x - 2) + x(x + 2)$$

$$2x^2 = x^2 - 4 + x^2 + 2x$$

$$2x^2 = 2x^2 + 2x - 4$$

$$-x = -2$$

$$x = 2$$

The only possible solution is $x = 2$, but we must check!

no solution

e) $$2 + \frac{10}{x+2} = \frac{3}{x+3}$$

$$(x + 2)(x + 3)\left(2 + \frac{10}{x+2}\right) = (x + 2)(x + 3)\frac{3}{x+3}$$

$$2(x + 2)(x + 3) + 10(x + 3) = 3(x + 2)$$

$$2(x^2 + 5x + 6) + 10x + 30 = 3x + 6$$

$$2x^2 + 10x + 12 + 10x + 30 = 3x + 6$$

$$2x^2 + 17x + 36 = 0$$

$$(2x + 9)(x + 4) = 0$$

$$\boxed{x = -9/2,\ -4}$$

f) $$\frac{y+1}{y+3} + \frac{y+5}{y-2} = 1 + \frac{6y+23}{y^2+y-6}$$

$$(y + 3)(y - 2)\left(\frac{y+1}{y+3} + \frac{y+5}{y-2}\right) = (y + 3)(y - 2)\left(1 + \frac{6y+23}{(y+3)(y-2)}\right)$$

$$(y - 2)(y + 1) + (y + 3)(y + 5) = (y + 3)(y - 2) + (6y + 23)$$

$$y^2 - y - 2 + y^2 + 8y + 15 = y^2 + y - 6 + 6y + 23$$

$$2y^2 + 7y + 13 = y^2 + 7y + 17$$

$$y^2 = 4$$

$$y = \pm 2$$

The solution would be $y = 2$ or $y = -2$, but we must check!

$$\boxed{y = -2}$$

Example: It takes Rosa, traveling at 50 mph, 45 minutes longer to go a certain distance than it takes Maria traveling at 60 mph. Find the distance traveled.

	distance =	rate •	time
Rosa	x	50	x/50
Maria	x	60	x/60

Important: 45 minutes = 3/4 hours. We must use hours here!

$$\frac{x}{50} - \frac{x}{60} = \frac{3}{4}$$

$$300\left(\frac{x}{50} - \frac{x}{60}\right) = 300\left(\frac{3}{4}\right)$$

$$6x - 5x = 225$$

$$x = 225$$

225 miles

Example: Beth can travel 208 miles in the same length of time it takes Anna to travel 192 miles. If Beth's speed is 4 mph greater than Anna's, find both rates.

Solution:

	distance	= rate	• time
Beth	208	$x + 4$	$208/(x+4)$
Anna	192	x	$192/x$

$$\frac{208}{x+4} = \frac{192}{x}$$

$$x(x+4)\left(\frac{208}{x+4}\right) = x(x+4)\left(\frac{208}{x+4}\right)$$

$$208x = 192(x+4)$$

$$208x = 192x + 768$$

$$16x = 768$$

$$x = 48$$

Beth	52 mph
Anna	48 mph

Example: Toni needs 4 hours to complete the yard work. Her husband, Sonny, needs 6 hours to do the work. How long will the job take if they work together?

Toni	4 hours
Sonny	6 hours
together	x hours

$$\frac{1}{4} + \frac{1}{6} = \frac{1}{x}$$

$$12x\left(\frac{1}{4} + \frac{1}{6}\right) = 12x\left(\frac{1}{x}\right)$$

$$3x + 2x = 12$$

$$5x = 12$$

$$x = \frac{12}{5}$$

$$\frac{12}{5} \text{ hours} = 2 \text{ hours} + \frac{2}{5} \text{ hours}$$

$$= 2 \text{ hours} + \frac{2}{5} \bullet 60 \text{ minutes}$$

2 hours 24 minutes

Example: Working together, Rick and Rod can clean the snow from the driveway in 20 minutes. It would have taken Rick, working alone, 36 minutes. How long would it have taken Rod alone?

Rick	36 minutes
Rod	x minutes
together	20 minutes

$$\frac{1}{36} + \frac{1}{x} = \frac{1}{20}$$

$$180x\left(\frac{1}{36} + \frac{1}{x}\right) = 180x\left(\frac{1}{20}\right)$$

$$5x + 180 = 9x$$

$$180 = 4x$$

$$45 = x$$

45 minutes

Example: John, Ralph, and Denny, working together, can clean a store in 6 hours. Working alone, Ralph takes twice as long to clean the store as does John. Denny needs three times as long as does John. How long would it take each man working alone?

John	x hours
Ralph	2x hours
Denny	3x hours
together	6 hours

$$\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = \frac{1}{6}$$

$$6x\left(\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x}\right) = 6x\left(\frac{1}{6}\right)$$

$$6 + 3 + 2 = x$$

$$11 = x$$

John	11 minutes
Ralph	22 minutes
Denny	33 minutes

Example: An inlet pipe on a swimming pool can be used to fill the pool in 12 hours. The drain pipe can be used to empty the pool in 20 hours. If the pool is empty and the drain pipe is accidentally opened, how long will it take to fill the pool?

inlet pipe	12 hours
drain pipe	20 hours
together	x hours

$$\frac{1}{12} - \frac{1}{20} = \frac{1}{x}$$

We subtract because the pipes are working against each other!

$$60x\left(\frac{1}{12} - \frac{1}{20}\right) = 60x\left(\frac{1}{x}\right)$$

$$5x - 3x = 60$$

$$2x = 60$$

$$x = 30$$

30 hours

Example: You can row, row, row your boat on a lake 5 miles per hour. On a river, it takes you the same time to row 5 miles downstream as it does to row 3 miles upstream. What is the speed of the river current in miles per hour?

	distance	= rate	• time
downstream	5	5 + x	5/(5 + x)
upstream	3	5 – x	3/(5 – x)

$$\frac{5}{5+x} = \frac{3}{5-x}$$

$$(5+x)(5-x)\left(\frac{5}{5+x}\right) = (5+x)(5-x)\left(\frac{3}{5-x}\right)$$

$$5(5-x) = 3(5+x)$$

$$25 - 5x = 15 + 3x$$

$$10 = 8x$$

$$\frac{5}{4} = x$$

$$\boxed{\frac{5}{4}\text{ mph}}$$