Adding and Subtracting Polynomials

When you add polynomials, simply combine all like terms. When subtracting polynomials, **do not forget to use parentheses when needed!**

Recall the distributive property:

$$a(b + c) = ab + ac.$$

In the particular case when $a = -1$, we get

$$-(b + c) = -b - c.$$

This suggests that when you subtract polynomials, change all signs of the terms of the polynomial immediately following the minus sign.

Example: Perform the indicated operation and / or simplify.

$$(2x^2 + 4x - 10) - (6x^2 - x - 1) + (5x^2 - 6x + 3)$$

$$= \quad 2x^2 + 4x - 10 \quad - \; 6x^2 + x + 1 \; + \; 5x^2 - 6x + 3$$

$$= \quad (2x^2 - 6x^2 + 5x^2) + (4x + x - 6x) + (-10 + 1 + 3)$$

$$\boxed{= \quad x^2 - x - 6}$$

Multiplying Polynomials

The product of two polynomials can be found by using some form of the distributive property $a(b + c) = ab + ac$.

Example: Multiply the given polynomials and simplify.

a) $(2x - 5)(3x - 2)$

$= 2x(3x - 2) - 5(3x - 2)$

$= 6x^2 - 4x - 15x + 10$

$\boxed{= 6x^2 - 19x + 10}$

b) $(z + 2)(4z - 1)$

$= z(4z - 1) + 2(4z - 1)$

$= 4z^2 - z + 8z - 2$

$\boxed{= 4z^2 + 7z - 2}$

c) $(x + 2y - 3)(2x - y - 2)$

$= x(2x - y - 2) + 2y(2x - y - 2) - 3(2x - y - 2)$

$= 2x^2 - xy - 2x + 4xy - 2y^2 - 4y - 6x + 3y + 6$

$\boxed{= 2x^2 - 2y^2 + 3xy - 8x - y + 6}$

Special Products of Polynomials

Take time and learn these formulas. You will need them later and you will be tested on them!

$(F + S)(F - S)$	$=$	$F^2 - S^2$	Difference of squares
$(F + S)^2$	$=$	$F^2 + 2FS + S^2$	Perfect square trinomial
$(F - S)^2$	$=$	$F^2 - 2FS + S^2$	Perfect square trinomial
$(F + S)(F^2 - FS + S^2)$	$=$	$F^3 + S^3$	Sum of cubes
$(F - S)(F^2 + FS + S^2)$	$=$	$F^3 - S^3$	Difference of cubes

WARNING:

Unless x or y is zero, $(x + y)^2$ does not equal $x^2 + y^2$!! For example $(3 + 4)^2 = 7^2 = 49$, but $3^2 + 4^2 = 9 + 16 = 25$. (Note that 25 is less than 49 by $2(3)(4) = 24$, the missing middle term.)

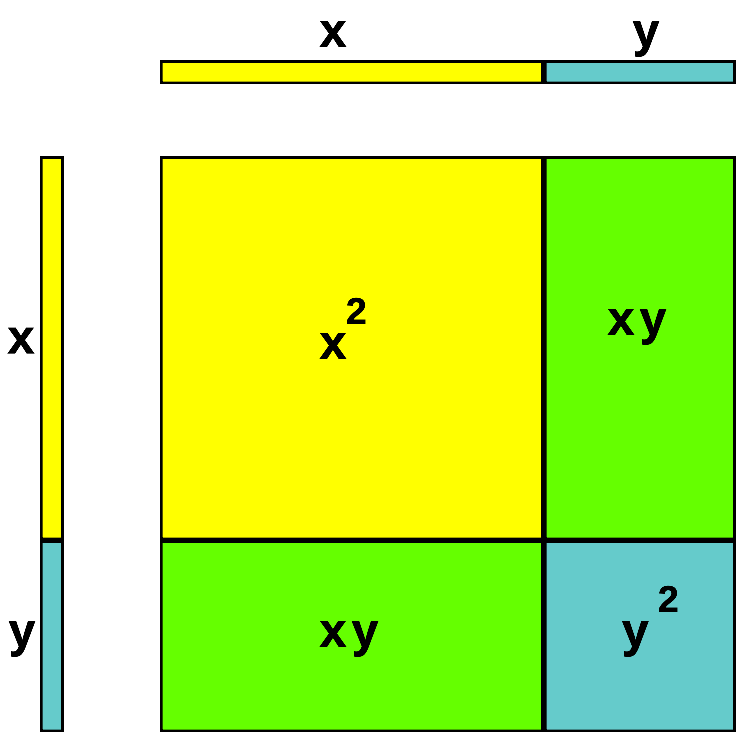

Table to demonstrate that $\mathbf{(x + y)^2 = x^2 + 2xy + y^2}$

Nonsense like thinking that $(x + y)^2$ equals $x^2 + y^2$ will not be tolerated!!

We use the table below for reference to help with multiplication.

$(F + S)(F - S)$	$=$	$F^2 - S^2$	Difference of squares
$(F + S)^2$	$=$	$F^2 + 2FS + S^2$	Perfect square trinomial
$(F - S)^2$	$=$	$F^2 - 2FS + S^2$	Perfect square trinomial

Example: Perform the indicated products and simplify.

Difference of squares

Let $F = 3xy$

$S = 4z$

a) $(3xy - 4z)(3xy + 4z)$

$= (F - S)(F + S)$

$= F^2 - S^2$

$\boxed{= 9x^2y^2 - 16z^2}$

Perfect square trinomial

Let $F = x$

$S = 5$

b) $(x + 5)^2$

$= (F + S)^2$

$= F^2 + 2FS + S^2$

$\boxed{= x^2 + 10x + 25}$

Perfect square trinomial

Let $F = 2s$

$S = 3t$

c) $(2s - 3t)^2$

$= (F - S)^2$

$= F^2 - 2FS + S^2$

$\boxed{= 4s^2 - 12st + 9t^2}$

FACTORING

Factoring is the inverse operation to multiplying polynomials.

Remember: In order to be good at factoring polynomials, you must be good at multiplying polynomials. Make sure you are very good at multiplying polynomials and using the special products.

Special Polynomial Factors

Common factor	$ax + ay$	$=$	$a(x + y)$
Sum/Diff. of squares	$\begin{cases} x^2 + y^2 \\ x^2 - y^2 \end{cases}$	$=$	$\begin{cases} \text{does not factor} \\ (x - y)(x + y) \end{cases}$
Sum/Diff. of cubes	$\begin{cases} x^3 + y^3 \\ x^3 - y^3 \end{cases}$	$=$	$\begin{cases} (x + y)(x^2 - xy + y^2) \\ (x - y)(x^2 + xy + y^2) \end{cases}$
Perfect square trinomial	$\begin{cases} x^2 + 2xy + y^2 \\ x^2 - 2xy + y^2 \end{cases}$	$=$	$\begin{cases} (x + y)^2 \\ (x - y)^2 \end{cases}$

Common Factors & Negative Exponents

Anytime you factor, remember to **take out the common factor first!!** Also, the biggest factor uses the smallest exponent.

Examples: Factor the following and use mathematics writing style:

GCF: $4z^2$

z smallest exponent: 2

a) $12x^6 - 16x^2$

$\boxed{= 4z^2 (3x^4 - 4)}$

GCF: $6A^{-8}B^{-9}$

A smallest exponent: -8

B smallest exponent: -9

h) $30A^{-5}B^{-9} - 24A^{-8}B^{-5}$

$\boxed{= 6A^{-8}B^{-9} (5A^3 - 4B^4)}$

Factoring By Grouping

Examples: Factor the following and use mathematics writing style:

a) $ax + ay + bx + by$

$= a(x + y) + b(x + y)$

$\boxed{= (a + b)(x + y)}$

b) $ax + ay - x - y$

$= a(x + y) - 1(x + y)$

$\boxed{= (a - 1)(x + y)}$

c) $x^2 + 4xy - 2x - 8y$

$= x(x + 4y) - 2(x + 4y)$

$\boxed{= (x - 2)(x + 4y)}$

d) $xy - 3x - 4y + 12$

$= x(y - 3) - 4(y - 3)$

$\boxed{= (x - 4)(y - 3)}$

e) $x^3 - 4x^2 + 3x - 12$

$= x^2(x - 4) + 3(x - 4)$

$\boxed{= (x^2 + 3)(x - 4)}$

f) $2x^2 - 7xy + 6y^2$

$= 2x^2 - 4xy - 3xy + 6y^2$

$= 2x(x - 2y) - 3y(x - 2y)$

$\boxed{= (2x - 3y)(x - 2y)}$

Factoring Polynomials Using The ac–Method

$$ax^2 + bx + c$$

We assume that a, b, and c are integers.

Step 1. Multiply $a \bullet c$

Step 2. If you can, find two integers such that:

a. their product is ac

b. their sum is b

Step 3. If two integers don't exist, STOP because the problem cannot be factored. Otherwise, move on to Step 4.

Step 4. Rewrite the middle term (bx) using the two integers from Step 2 as coefficients.

Step 5. Factor by grouping (like when there are 4 terms).

Examples: Factor the following using the ac–method:

a) $4x^2 - x - 18$

$= 4x^2 - 9x + 8x - 18$

$= x(4x - 9) + 2(4x - 9)$

$\boxed{= (x + 2)(4x - 9)}$

b) $12x^2 - 23x - 24$

$= 12x^2 - 32x + 9x - 24$

$= 4x(3x - 8) + 3(3x - 8)$

$\boxed{= (4x + 3)(3x - 8)}$

Sum / Difference of Squares & Cubes

You are required to determine at a quick glance any special product polynomial.

$F^2 + S^2$	does not factor,	$F^3 + S^3$	$= (F + S)(F^2 - FS + S^2)$
$F^2 - S^2$	$= (F - S)(F + S)$	$F^3 - S^3$	$= (F - S)(F^2 + FS + S^2)$

Examples: Is the polynomial is the sum of squares, difference of squares, difference of cubes, or the sum of cubes? If so, then factor it, if possible.

a) $A^2 + 64$ s.s. does not factor

b) $w^2 - 25$ d.s. $(w - 5)(w + 5)$

c) $z^2 - 8$ no

d) $z^3 - 8$ d.c $(z - 2)(z^2 + 2x + 4)$

e) $4x^2 - 9y^4$ d.s $(2x - 3y^2)(2x + 3y^2)$

f) $z^3 + 125$ s.c. $(z + 5)(z^2 - 5z + 25)$

g) $B^2 + 9$ s.s. does not factor

h) $c^2 - 16$ d.s. $(c - 4)(c + 4)$

i) $c^3 - 16$ no

j) $x^2y^6 - 81z^{10}$ d.s. $(xy^3 - 9z^5)(xy^3 + 9z^5)$

k) $8y^3z^6 + 27w^3$ s.c. $(2yz^2 + 3w)(4y^2z^4 - 6yz^2w + 9w^2)$

Perfect Square Trinomials

You are required to determine at a quick glance any special product polynomial.

$$F^2 + 2FS + S^2 = (F + S)^2$$
$$F^2 - 2FS + S^2 = (F - S)^2$$

Examples: Is the polynomials a perfect square trinomial? If so, then factor it.

$x^2 - x + 1$	no	
$x^2 - 2x + 1$	yes	$(x - 1)^2$
$x^2 + 6x + 9$	yes	$(x + 3)^2$
$x^2 - 4x - 4$	no	
$x^2 - 4x + 4$	yes	$(x - 2)^2$
$a^2 - 10a + 25$	yes	$(a - 5)^2$
$9B^2 - 12BC + 4C^2$	yes	$(3B - 2C)^2$
$z^2 + 13z + 36$	no	
$25z^2 + 60z + 36$	yes	$(5z + 6)^2$

Factoring With Substitution

Examples: Factor the following and use mathematics writing style:

a) $x^2y^6 - 81z^{10}$

$= F^2 - S^2$

$= (F - S)(F + S)$

$\boxed{= (xy^3 - 9z^5)(xy^3 + 9z^5)}$

b) $x^3 + 8$

$= F^3 + S^3$

$= (F + S)(F^2 - FS + S^2)$

$\boxed{= (x + 2)(x^2 - 2x + 4)}$

c) $x^2 - (y + 2)^2$

$= F^2 - S^2$

$= (F - S)(F + S)$

$= [x - (y + 2)][x + (y + 2)]$

$\boxed{= (x - y - 2)(x + y + 2)}$

d) $(x + 2y)^2 - 25$

$= F^2 - 25$

$= (F - 5)(F + 5)$

$= [(x + 2y) - 5][(x + 2y) + 5]$

$\boxed{= (x + 2y - 5)(x + 2y + 5)}$

e) $(2x - 5)^2 - 2(2x - 5) - 8$

$= F^2 - 2F - 8$

$= (F - 4)(F + 2)$

$= [(2x - 5) - 4][(2x - 5) + 2]$

$= [2x - 5 - 4][2x - 5 + 2]$

$\boxed{= (2x - 9)(2x - 3)}$

f) $(x - 3)^2 - 9(y + 2)^2$

$= F^2 - 9S^2$

$= (F - 3S)(F + 3S)$

$= [(x-3) - 3(y+2)][(x-3) + 3(y+2)]$

$= [x - 3 - 3y - 6)][x - 3 + 3y + 6]$

$\boxed{= (x - 3y - 9)(x + 3y + 3)}$

Factoring Completely

You are always required to continue to factor until every expression can be factored no further.

Steps For Factoring Polynomials

1. **Always** take out the greatest common factor **first**!!!
   (This includes the case with negative exponents.)

2. See if you can use one of the special factors. Check how many terms are present.

   A. If there are 2 terms:

      a. See if it is a difference of squares
      b. See if it is a sum or difference of cubes

   B. If there are 3 terms:

      See if it is a perfect square trinomial

   C. If there are 4 terms:

      See if you can factor by grouping.

3. Try factoring by any other method you can.
   (Substitution, trial and error, ac method, etc.)

Hint: Whenever you have to factor, **DO NOT MULTIPLY OUT** the terms unless it is absolutely necessary!! First consider using a substitution.

Examples: Factor completely the following and use mathematics writing style:

a) $w^{-7} - 4w^{-9}$

$= w^{-9}(w^{22} - 4)$

$\boxed{= w^{-9}(w - 2)(w + 2)}$

b) $4 + 4z^{-1} + z^{-2}$

$= z^{-2}(4z^2 + 4z + 1)$

$\boxed{= z^{-2}(2z + 1)^2}$

c) $(x^2 - x)^2 - 18(x^2 - x) + 72$

$= F^2 - 18F + 72$

$= (F - 6)(F - 12)$

$= (x^2 - x - 6)(x^2 - x - 12)$

$\boxed{= (x - 3)(x + 2)(x - 4)(x + 3)}$

d) $x^8 - y^8$

$= (x^4 - y^4)(x^4 + y^4)$

$= (x^2 - y^2)(x^2 + y^2)(x^4 + y^4)$

$\boxed{= (x-y)(x+y)(x^2+y^2)(x^4+y^4)}$

e) $(x^2 + 3x - 10)^2 - (x - 2)^2$

$= F^2 - S^2$

$= (F + S)(F - S)$

$= [(x^2 + 3x - 10) + (x - 2)][(x^2 + 3x - 10) - (x - 2)]$

$= [x^2 + 3x - 10 + x - 2][x^2 + 3x - 10 - x + 2]$

$= (x^2 + 4x - 12)(x^2 + 2x - 8)$

$= (x + 6)(x - 2)(x + 4)(x - 2)$

$\boxed{= (x + 6)(x + 4)(x - 2)^2}$

Solving Polynomial Equations

Zero Product Property

If a and b are any real numbers whose product is zero, then we know that a is zero or b is zero. That is

$a \cdot b = 0$ $\qquad\qquad$ $a = 0$ or $b = 0$

This property allows us to solve many different types of equations.

Ex 1: $(x - 3)(x + 7) = 0$

$x - 3 = 0 \qquad x + 7 = 0$

$x = 3 \qquad x = -7$

$\boxed{x = 3, -7}$

Ex 2: $5x(x + 1)(x - 2) = 0$

$5x = 0 \qquad x + 1 = 0 \qquad x - 2 = 0$

$x = 0 \qquad x = -1 \qquad x = 2$

$\boxed{x = 0, -1, 2}$

Notice that if $a \cdot b = 1$, then we know nothing about a nor b except that neither one is equal to zero. This is a unique property of zero.

W A R N I N G :

This property is only true for factors whose product is zero. Improper use of the Zero Product Property will not be tolerated!!

Solving Equations
By Factoring

(SFSC Method)

1. Put the equation in Standard form. This means you must get a zero on one side of the equation.

2. Factor the one side.

3. Solve by using the zero product property. (Set each variable factor equal to zero and solve.)

4. Check your answers.

Examples: Solve the following equations by factoring:

Step 1: Standard form

a) $4x^3 = 100x$

$4x^3 - 100x = 0$

Step 2: Factor completely

$4x(x^2 - 25) = 0$

$4x(x - 5)(x + 5) = 0$

Step 3: Set factors to 0, solve

$4x = 0 \quad x-5 = 0 \quad x+5 = 0$

$x = 0 \quad x = 5 \quad x = -5$

$\boxed{x = 0, 5, -5}$

Step 1: Standard form

b) $(3z + 6)(4z + 12) = -3$

$12z^2 + 60z + 72 = -3$

$12z^2 + 60z + 75 = 0$

Step 2: Factor completely

$3(4z^2 + 20z + 25) = 0$

$3(2z + 5)^2 = 0$

Step 3: Set factors to 0, solve

Don't need to set 3 = 0

Don't need 2z+5 = 0 twice

$2z + 5 = 0$

$\boxed{z = -5/2}$