The Solow Growth Model
Case 3: Human Capital

(1) \[ Y = K^{\alpha} (AH)^{1-\alpha} \]

(2) \[ Y = C + I \]

(3) \[ C = (1-s)Y \quad 0 < (1-s) < 1 \]

(4) \[ \dot{K} = I - \delta K \quad 0 < \delta < 1 \]

(5) \[ \frac{\dot{L}}{L} = n \quad 0 < n \]

(6) \[ \frac{\dot{A}}{A} = g \]

(7) \[ H = e^{\psi u} L \]

Endogenous: \ Y, C, I, K \\
Exogenous: \ L, A, K_0, H, u \\
Parameters: \ n, (1-s), \delta, \alpha, g, \psi. \\

\psi: \text{measures return to “schooling” (shown below)}

u: \text{fraction of time spent “studying”. For now, an exogenous variable. Later, a decision variable (endogenous variable).}
Implication of $\psi$: indicates the impact on $H$ of increasing study time $u$. Specifically,

$$\frac{\partial H}{\partial u} = \frac{\partial(\psi u)}{\partial u} e^{\psi u} L = \psi e^{\psi u} L = \psi H,$$

thus

$$\frac{\Delta H}{H} = \psi;$$

i.e., a given $\Delta u$ results in a percentage change in $H$ of $\psi \Delta u$.

Exercise: calculate the marginal product of $u$: $\frac{\partial Y}{\partial u}$.

Caution: $u$ has a positive increase on $Y$, as the MP calculation indicates. But it has an important opportunity cost: time spent studying means time spent outside the work force. This cost is downplayed in the text (buried in footnote 2, and not accounted for directly in the production function).

Goals: search for a steady state, study transition dynamics. To solve, we’ll convert the variables of the model into per effective unit of human capital (AH) terms, and collapse the model into a single equation.
Converted variables: $\tilde{y} = Y/AH$, $\tilde{k} = K/AH$, $\tilde{c} = C/AH$, $\tilde{i} = I/AH$.

Exercise: collapse the model, show that it yields:

\[
\frac{\cdot \tilde{k}}{\tilde{k}} = s \tilde{k} (1 - (n + \delta + g)),
\]

or equivalently,

\[
\frac{\cdot \tilde{k}}{\tilde{k}} = s \tilde{k} (n + \delta + g).
\]

Steady state condition: all variables grow at the same rate, implying:

\[
\tilde{k}^* = \left(\frac{s}{n + \delta + g}\right)^{\frac{1}{1-\alpha}}.
\]
Let $\theta = e^{\psi u}$.

Then note:

$$\tilde{k} = K / (AH) = K / (AL\theta) = (1 / \theta)(K / AL) = (1 / \theta)k .$$

Also,

$$\dot{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{H}}{H} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \dot{k} ,$$

since $\theta$ is a constant. So, this addition to the model merely has a level effect, not a growth effect. It is added to help the model account for differences in steady state output levels resulting from differences in human-capital levels.