The Solow Growth Model
Case 1 (no technological progress)

(1) \[ Y = K^\alpha L^{1-\alpha} \quad \text{(note: } A = 1) \]

(2) \[ Y = C + I \]

(3) \[ C = (1-s)Y \quad 0 < (1-s) < 1 \]

(4) \[ \dot{K} = I - \delta K \quad 0 < \delta < 1 \]

(5) \[ \frac{\dot{L}}{L} = n \quad 0 < n \]

Endogenous: \( Y, C, I, K \)
Exogenous: \( L, K_0 \)
Parameters: \( n, (1-s), \delta, \alpha. \)

Solution concept: given an initial value of \( L \) and \( K \), determine entire time paths of \( Y, C, I, K \) such that (1)-(5) hold.

Steady state solution. A solution in which all variables grow at a constant rate (in this case, \( n \)).

Issues: does a steady state exist? If so, if you get there, will you stay (stable)? If you start somewhere else, will you eventually get there?
Goals: search for a steady state, study transition dynamics. To solve, we’ll convert the variables of the model into *per capita* terms, and collapse the model into a single equation.

*Per capita* variables: \( y = Y/L, \ k = K/L, \ c = C/L, \ i = I/L. \)

Collapsing the model:

Substitute \( C = (1-s)Y \) -- eq. 2 -- into \( Y = C + I \) -- eq. 3:

\[ Y = (1-s)Y + I = Y - sY + I, \text{ and thus } S = sY = I. \]

Substituting for \( I \) in (4) gives

\[ \dot{K} = sY - \delta K, \text{ and thus } \]

\[ \frac{\dot{K}}{K} = s - \frac{\delta}{K}. \]

Now, convert to *per capita* terms. Note the relationship between the growth rates of \( k \) and \( K \):

\[ \frac{\dot{k}}{k} = \frac{\partial}{\partial t} \ln(k) = \frac{\partial}{\partial t} \ln\left(\frac{K}{L}\right) = \frac{\partial}{\partial t} (\ln(K) - \ln(L)) = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - n \]

Thus

\[ \frac{\dot{k}}{k} = s - \frac{(n + \delta)}{K}. \]
Note that \( \frac{Y}{K} = \frac{(Y/L)/(K/L)}{= y/k} \), thus

\[
\frac{k}{k} = s \frac{y}{k} - (n + \delta).
\]

Finally, note from the production function that

\[
y = \frac{Y}{L} = K^\alpha L^{1-\alpha} L^{-1} = \left(\frac{K}{L}\right)^\alpha = k^\alpha,
\]

thus

\[
\frac{k}{k} = syk^{-1} - (n + \delta) = sk^{\alpha-1} - (n + \delta).
\]

So, the Solow Model may be summarized by

\[
\frac{k}{k} = sk^{\alpha-1} - (n + \delta),
\]

or equivalently,

\[
k = sk^\alpha - k(n + \delta).
\]
Goals:
♦ characterize the Solow model graphically
♦ examine properties of the steady state solution of the model
♦ examine the behavior of the model away from the steady state
♦ derive the “Golden Rule” savings rate

Steady state condition: all variables grow at the same rate.

\[
\frac{\dot{K}}{K} = \frac{\dot{L}}{L} = n, \text{ which implies }
\]

\[
\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = 0.
\]

Substituting this condition into the Solow equation gives

\[
\frac{\dot{k}}{k} = sk^{\alpha - 1} - (n + \delta) = 0.
\]

Solving for \( k \) yields

\[
k^* = \left( \frac{S}{n + \delta} \right)^{\frac{1}{1 - \alpha}}.
\]

Note that the steady state value of \( k \) is larger the larger is \( s \) and \( \alpha \), and the smaller is \( n \) and \( \delta \).