State Space Representations

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Notation

s_t: state variables

y_t: observed variables

Y_t: $\{y_j\}_{j=1}^t$

Note: variables are expressed in levels (detrended when appropriate)

State Space Representations

State-transition equation:

$$s_t = \gamma(s_{t-1}, Y_{t-1}, \upsilon_t)$$

Associated density:

$$f(s_t | s_{t-1}, Y_{t-1})$$

Measurement equation:

$$y_t = \delta(s_t, Y_{t-1}, u_t)$$

Associated density:

$$f(y_t | s_t, Y_{t-1})$$

Initialization:

$$f(s_0)$$

Likelihood Evaluation and Filtering

- **Filtering objective:** construct $f\left(s_t \middle| Y_t\right)$, which can then be used to approximate $E_t\left(h(s_t) \middle| Y_t\right)$.

- **Likelihood evaluation** obtains as a by-product of the filtering process.

Likelihood Evaluation and Filtering, cont.

- From Bayes' theorem, $f\left(s_t \middle| Y_t\right)$ is given by

$$f\left(s_t \middle| Y_t\right) = \frac{f\left(y_t, s_t \middle| Y_{t-1}\right)}{f\left(y_t \middle| Y_{t-1}\right)} = \frac{f\left(y_t \middle| s_t, Y_{t-1}\right) f\left(s_t \middle| Y_{t-1}\right)}{f\left(y_t \middle| Y_{t-1}\right)},$$

- where $f\left(s_t \middle| Y_{t-1}\right)$ is given by

$$f\left(s_t \middle| Y_{t-1}\right) = \int f\left(s_t \middle| s_{t-1}, Y_{t-1}\right) f\left(s_{t-1} \middle| Y_{t-1}\right) ds_{t-1},$$

- and $f\left(y_t \middle| Y_{t-1}\right)$ is given by

$$f\left(y_t \middle| Y_{t-1}\right) = \int f\left(y_t \middle| s_t, Y_{t-1}\right) f\left(s_t \middle| Y_{t-1}\right) ds_t.$$

Schematic of the Filtering Process

Taking $f\left(s_{t-1}|Y_{t-1}\right)$ as given, initialized with $f\left(s_0|Y_0\right) \equiv f(s_0)$, filtering and likelihood evaluation proceed recursively:

- Prediction: $f\left(s_{t-1}|Y_{t-1}\right)$ combines with $f\left(s_t|s_{t-1}, Y_{t-1}\right)$ to yield

$$f(s_t|Y_{t-1}) = \int f\left(s_t|s_{t-1}, Y_{t-1}\right) f\left(s_{t-1}|Y_{t-1}\right) ds_{t-1} \longrightarrow (4)$$

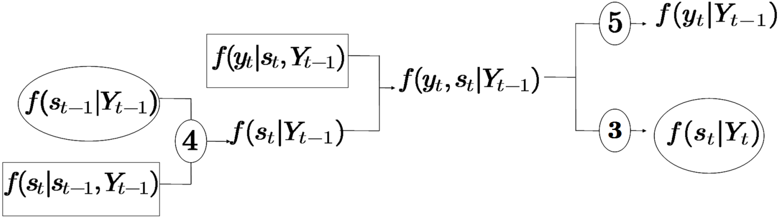
- Forecasting: $f(s_t|Y_{t-1})$ combines with $f\left(y_t|s_t, Y_{t-1}\right)$ to yield

$$f\left(y_t|Y_{t-1}\right) = \int f\left(y_t|s_t, Y_{t-1}\right) f\left(s_t|Y_{t-1}\right) ds_t. \longrightarrow (5)$$

- Updating: Bayes' Rule yields

$$f(s_t|Y_t) = \frac{f\left(y_t|s_t, Y_{t-1}\right) f\left(s_t|Y_{t-1}\right)}{f\left(y_t|Y_{t-1}\right)} \longrightarrow (3)$$

Schematic, cont.



$f(s_{t-1}|Y_{t-1})$

$f(s_t|s_{t-1},Y_{t-1})$

4

$f(s_t|Y_{t-1})$

$f(y_t|s_t,Y_{t-1})$

$f(y_t,s_t|Y_{t-1})$

5

$f(y_t|Y_{t-1})$

3

$f(s_t|Y_t)$

One-Tree Model

Recall that with p_t representing $\frac{p_t}{e^{gt}}$, etc., the model is

$$\begin{aligned}
p_t &= \beta e^{(1-\gamma)g} E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}(d_{t+1}+p_{t+1})\right] && (1)\\
c_t &= d_t + q_t && (2)\\
d_t &= \overline{d}e^{\omega_{dt}}, \quad \omega_{dt} = \rho_d \omega_{dt-1} + \varepsilon_{dt}, && (3)\\
q_t &= \overline{q}e^{\omega_{qt}}, \quad \omega_{qt} = \rho_q \omega_{qt-1} + \varepsilon_{qt}. && (4)
\end{aligned}$$

State: $s_t = \begin{bmatrix} d_t & q_t \end{bmatrix}'$

Shocks: $\upsilon_t = \begin{bmatrix} \varepsilon_{dt} & \varepsilon_{qt} \end{bmatrix}'$

Controls: $c_t = \begin{bmatrix} c_t & p_t \end{bmatrix}'$

One-Tree Model, cont.

- State-transition equations:

$$\begin{aligned} \ln d_t &= (1-\rho_d)\ln \overline{d} + \rho_d \ln d_{t-1} + \varepsilon_{dt}, \\ \ln q_t &= (1-\rho_q)\ln \overline{q} + \rho_q \ln q_{t-1} + \varepsilon_{qt}, \end{aligned}$$

$$\upsilon_t \equiv \left[\varepsilon_{dt} \quad \varepsilon_{qt}\right]' \sim iidN\left(0, \Sigma_\upsilon\right).$$

- Thus the state-transition density $f\left(s_t \middle| s_{t-1}, Y_{t-1}\right)$ is $N\left(0, \Sigma_\upsilon\right)$.
- Measurement equations:

$$\begin{aligned} c_t &= d_t + q_t + u_{ct} \\ p_t &= p(d_t, q_t) + u_{pt} \\ d_t &= d_t, \end{aligned}$$

$$u_t \equiv \left[u_{ct} \quad u_{pt}\right]' \sim iidN\left(0, \Sigma_u\right).$$

- Note that the measurement density $f\left(y_t \middle| s_t, Y_{t-1}\right)$ is partially degenerate.

RBC Model

$$\left(\frac{1-\varphi}{\varphi}\right)\frac{c_t}{l_t} = (1-\alpha)z_t\left(\frac{k_t}{n_t}\right)^{\alpha} \tag{5}$$

$$c_t^{\kappa} l_t^{\lambda} = \beta E_t\left\{*\right\} \tag{6}$$

$$y_t = z_t k_t^{\alpha} n_t^{1-\alpha} \tag{7}$$

$$y_t = c_t + i_t \tag{8}$$

$$\left(1+\frac{g}{1-\alpha}\right)k_{t+1} = i_t + (1-\delta)k_t \tag{9}$$

$$1 = n_t + l_t \tag{10}$$

$$\log z_t = (1-\rho)\log(z_0) + \rho\log z_{t-1} + \varepsilon_t, \tag{11}$$

where $\kappa = \varphi(1-\phi) - 1$ and $\lambda = (1-\varphi)(1-\phi)$.

RBC Model

$$\{*\} = \left\{ \left(1+\frac{g}{1-\alpha}\right)^{\kappa} c_{t+1}^{\kappa} l_{t+1}^{\lambda} \left[\alpha z_{t+1} \left(\frac{n_{t+1}}{k_{t+1}}\right)^{1-\alpha} + (1-\delta) \right] \right\}$$

RBC Model, cont.

A policy function $c(k, z)$ can be obtained by combining (5), (6) and (10) to eliminate (l, n). Given $c(k, z)$, policy functions for (l, n, y, i) obtain from simple algebra. The state-transition equations are then

- $$\begin{aligned} \left(1+\frac{g}{1-\alpha}\right)k_{t+1} &= i\left(k_t, z_t\right)+(1-\delta)k_t \\ \log z_t &= (1-\rho)\log(z_0)+\rho\log z_{t-1}+\varepsilon_t. \end{aligned}$$
- Note that the transition density is partially degenerate.

RBC Model, cont.

Observation equations:

$$\begin{aligned} y_t &= z_t k_t^{\alpha} n(k_t, z_t)^{1-\alpha} + u_{yt} \\ c_t &= c(k_t, z_t) + u_{ct} \\ i_t &= i(k_t, z_t) + u_{it} \\ n_t &= n(k_t, z_t) + u_{nt}, \end{aligned}$$

$u_t \sim N\left(0, \Sigma_u\right).$

Generic Linear State Space Rep.

- State-transition equations:

$$\begin{aligned} x_t &= F x_{t-1} + e_t, \\ e_t &= G v_t, \\ E(e_t e_t') &= G E\left(v_t v_t'\right) G' = Q. \end{aligned}$$

- Measurement equations:

$$X_t = H' x_t + u_t, \qquad E u_t u_t' = \Sigma_u$$

- Note: x_t in general contains state variables s_t and control variables; X_t is directly analagous to y_t.

The Kalman Filter.

Likelihood evaluation and filtering is achieved in the linear-normal case via the Kalman filter. Given linearity/normality, targeted densities are fully characterized by means and covariance matricies.

Notation:

$$\begin{aligned} x_{t|t-j} &= E[x_t|\{X_1, ..., X_{t-j}\}], \\ P_{t|t-j} &= E\left[(x_t - x_{t|t-j})(x_t - x_{t|t-j})'\right], \\ j &= 0, 1. \end{aligned}$$

The Kalman Filter, cont.

Background I: Linear Projections

$$\begin{aligned} P\left(\underbrace{x}_{nx1}\middle|\underbrace{X}_{mx1}\right) &= \underbrace{a}_{nxm}{}'\underbrace{X}_{mx1} \\ &= E\left[x|X\right] \text{ given lin/norm,} \end{aligned}$$

where

$$a = \arg\min E\left[\left(x - a'X\right)^2\right]$$

The Kalman Filter, cont.

FONC for a (Normal Equations/Orthogonality Conditions):

$$E\left[\left(x - a'X\right)X'\right] = 0$$

$$\begin{aligned}
&\rightarrow \quad E\left(xX'\right) = a'E\left(XX'\right) \\
&\rightarrow \quad a' = \left(ExX'\right)\left(E\left(XX'\right)\right)^{-1} \\
&\rightarrow \quad \widehat{x} = \left(ExX'\right)\left(E\left(XX'\right)\right)^{-1}X
\end{aligned}$$

The Kalman Filter, cont.

Background II: Updating

$$P\left(x \middle| \{X_t, X_{t-1}, ...\}\right) = \underbrace{P\left(x \middle| \{X_{t-1}, ...\}\right)}_{\text{old forecast}} +$$

$$P\left(\underbrace{x - P\left(x \middle| \{X_{t-1}, ...\}\right)}_{\text{forecast error}} \middle| \underbrace{X_t - P(X_t| \{X_{t-1}, ...\})}_{\text{new information}}\right)$$

The Kalman Filter, cont.

Kalman Filter I: Initialization $\left(f\left(s_0\right), \text{ or } x_{1|0}, P_{1|0}\right)$

Unconditional mean:

$$\begin{aligned} Ex_t &= FEx_{t-1} = FEx_t \\ &\rightarrow \left(I-F\right)Ex_t = 0 \\ &\rightarrow Ex_t \equiv x_{1|0} = 0 \end{aligned}$$

The Kalman Filter, cont.

Unconditional VCV:

$$\begin{aligned} E\left(x_{1|0}x_{1|0}'\right) &\equiv P_{1|0} \\ &= E\left[\left(Fx_{t-1}+e_t\right)\left(Fx_{t-1}+e_t\right)'\right] \\ &= F\left(E\left(x_{t-1}x_{t-1}'\right)\right)F' + E\left(e_t e_t'\right) \\ &= FP_{1|0}F' + Q \end{aligned}$$

Thus

$$vec(P_{1|0}) = (I - F\otimes F')^{-1} vec(Q)$$

The Kalman Filter, cont.

Kalman Filter II: Forecasting

$\left(f\left(y_t | Y_{t-1}\right), \text{ or } X_{t|t-1}, \Omega_{t|t-1}\right)$

Given $\left(x_{t|t-1}, P_{t|t-1}\right)$ (initially $\left(x_{1|0}, P_{1|0}\right)$):

- $$X_{t|t-1} = H' x_{t|t-1}$$
- MSE:

$$\begin{aligned}
\Omega_{t|t-1} &= E\left[\left(\underbrace{H' \underbrace{\left(x_t - x_{t|t-1}\right)}_{\text{Prediction error}} + \underbrace{u_t}_{\text{Mea. err.}}}_{\text{Forecasting error}}\right)\left(H'\left(x_t - x_{t|t-1}\right) + u_t\right)'\right. \\
&= E\left[H'\left(x_t - x_{t|t-1}\right)\left(x_t - x_{t|t-1}\right)' H + u_t u_t'\right] \\
&= H' P_{t|t-1} H + \Sigma_u
\end{aligned}$$

The Kalman Filter, cont.

Kalman Filter III: Updating $\left(f\left(s_t|Y_t\right), \text{ or } x_{t|t},\ P_{t|t}\right)$

- Using the updating equation from Background II:

$$x_{t|t} = x_{t|t-1} + E\left[\left(x_t - x_{t|t-1}\right) | \left(X_t - X_{t|t-1}\right)\right]$$

- Using the Normal Equations from Background I:

$$E\left[\left(x_t - x_{t|t-1}\right) | \left(X_t - X_{t|t-1}\right)\right] = \underbrace{E\left(x_t - x_{t|t-1}\right)\left(X_t - X_{t|t-1}\right)'}_{P_{t|t-1}H} \times$$

$$\left(\underbrace{E\left(X_t - X_{t|t-1}\right)\left(X_t - X_{t|t-1}\right)'}_{\Omega_{t|t-1}}\right)^{-1} E\left(X_t - X_{t|t-1}\right)$$

The Kalman Filter, cont.

Kalman Filter III: Updating, cont.

- Thus

$$\begin{aligned} x_{t|t} &= x_{t|t-1} + P_{t|t-1} H \Omega_{t|t-1}^{-1} \times \\ &\quad \left(X_t - H' x_{t|t-1} \right) \end{aligned}$$

- MSE:

$$P_{t|t} = E\left[\left(x_t - x_{t|t} \right) \left(x_t - x_{t|t} \right)' \right]$$

The Kalman Filter, cont.

Kalman Filter III: Updating, cont.

$$
\begin{aligned}
P_{t|t} &= E\left[\left(x_t - x_{t|t}\right)\left(x_t - x_{t|t}\right)'\right] \\
&= E\left[\underbrace{\left(x_t - x_{t|t-1}\right)\left(x_t - x_{t|t-1}\right)'}_{P_{t|t-1}}\right] - \\
&\quad \underbrace{E\left(x_t - x_{t|t-1}\right)\left(X_t - X_{t|t-1}\right)'}_{P_{t|t-1}H} \times \\
&\quad \underbrace{\left(E\left(X_t - X_{t|t-1}\right)\left(X_t - X_{t|t-1}\right)'\right)^{-1}}_{\Omega_{t|t-1}^{-1}} \times \\
&\quad \underbrace{E\left(X_t - X_{t|t-1}\right)\left(x_t - x_{t|t-1}\right)'}_{H'P_{t|t-1}} \\
&= P_{t|t-1} - P_{t|t-1}H\Omega_{t|t-1}^{-1}H'P_{t|t-1}
\end{aligned}
$$

The Kalman Filter, cont.

Kalman Filter IV: Prediction

$\left(f\left(s_{t+1}\middle|Y_t\right),\ \text{or}\ x_{t+1|t},\ P_{t+1|t}\right)$

- Plugging $x_{t|t}$ into the state equation:

$$\begin{aligned} x_{t+1|t} &= Fx_{t|t} \\ &= Fx_{t|t-1} + FP_{t|t-1}H\Omega_{t|t-1}^{-1} \times \\ &\quad \left(X_t - H'x_{t|t-1}\right) \end{aligned}$$

- MSE:

$$\begin{aligned} P_{t+1|t} &= E\left[(x_{t+1} - x_{t+1|t})(x_{t+1} - x_{t+1|t})'\right] \\ &= E\left[\left(Fx_t + e_{t+1} - Fx_{t|t}\right)\left(Fx_t + e_{t+1} - Fx_{t|t}\right)'\right] \\ &= FE\left[\left(x_t - x_{t|t}\right)\left(x_t - x_{t|t}\right)'\right]F' + E\left(e_{t+1}e'_{t+1}\right) \\ &= FP_{t|t}F' + Q \end{aligned}$$

The Kalman Filter, cont.

Summary (means and covariances)

- Initialization:
  $x_{1|0} = 0,\ vec(P_{1|0}) = (I - F \otimes F')^{-1} vec(Q)$
- Forecasting: $X_{t|t-1} = H' x_{t|t-1}$,
  $\Omega_{t|t-1} = H' P_{t|t-1} H + \Sigma_u$
- Updating:

$$\begin{aligned} x_{t|t} &= x_{t|t-1} + P_{t|t-1} H \Omega_{t|t-1}^{-1} \times \\ & \left(X_t - H' x_{t|t-1} \right), \end{aligned}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} H \Omega_{t|t-1}^{-1} H' P_{t|t-1}$$

The Kalman Filter, cont.

- Prediction:

$$\begin{aligned} x_{t+1|t} &= Fx_{t|t} \\ &= Fx_{t|t-1} + FP_{t|t-1}H\Omega_{t|t-1}^{-1} \times \\ &\quad \left(X_t - H'x_{t|t-1}\right), \\ P_{t+1|t} &= FP_{t|t}F' + Q \end{aligned}$$

The Kalman Filter, cont.

Code:

- `kalman.prc` $(\Sigma_u = 0)$
- `kalmanm.prc` $(\Sigma_u \neq 0)$

The Kalman Filter, cont.

Exercise:

Consider the AR(p) representation for a generic variable y_t:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + ... + \rho_p y_{t-p} + \varepsilon_t.$$

- Map this into the form of a state-space representation.
- Generate artificial data using the model as a DGP.
- Show that OLS estimates and ML estimates of $\rho(L)$ coincide.