Example Environments

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Lucas' (1978 Econometrica) One-Tree Model of Asset Prices

Text reference: Ch. 5.3, pp. 106-116.

Environment:

$$\max_{c_t} \qquad U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

$\beta = \frac{1}{1+\rho}$, $0 < \rho < 1$, subject to

$$c_t + p_t(s_t - s_{t-1}) = d_t s_{t-1} + q_t, \tag{2}$$

s_0 given.

Establish Optimality Conditions

Value function:

$$\begin{aligned} V(s_{t-1}) &= \max_{c_t} E_t \left\{ u(c_t) + \beta V(s_t) \right\} \\ &= \max_{s_t} E_t \left\{ u(d_t s_{t-1} + q_t - p_t(s_t - s_{t-1})) + \beta V(s_t) \right\} \end{aligned}$$

FONC:

$$E_t \left\{ u'(c_t) \left[-p_t \right] + \beta V'(s_t) \right\} = 0$$

Envelope Condition:

$$V'(s_t) = u'(c_{t+1}) \left[d_{t+1} + p_{t+1} \right]$$

Optimality Conditions, cont.

Combining FONC and Env. Cond. Yields Pricing Kernel:

$$p_t = \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (d_{t+1} + p_{t+1}) \right] \tag{3}$$

Equilibrium Conditions

Since agents are identical, market clearing requires $s_t = s_{t-1}$ $\forall t$. Hereafter, we'll normalize: $s_t = 1$ $\forall t$. From the budget constraint (2), this implies

$$c_t = d_t + q_t.$$

Closing the Model

At this point, we have two equations and four unknowns:

$$\begin{aligned} c_t &= d_t + q_t, \\ p_t &= \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (d_{t+1} + p_{t+1}) \right]. \end{aligned}$$

To close the model, we specify stochastic processes for the exogenous state variables $s_t = [d_t \quad q_t]'$:

$$\begin{aligned} d_t &= \overline{d} e^{gt} e^{\omega_{dt}}, \quad \omega_{dt} = \rho_d \omega_{dt-1} + \varepsilon_{dt}, \\ q_t &= \overline{q} e^{gt} e^{\omega_{qt}}, \quad \omega_{qt} = \rho_q \omega_{qt-1} + \varepsilon_{qt}, \end{aligned}$$

with $\upsilon_t \sim iidN(0, \Sigma)$.

Closing the Model, cont.

Exercise:
Defining $\tilde{\tilde{d}}_t = d_t / e^{gt}$, show that the assumed SP for d_t implies

$$\ln \tilde{\tilde{d}}_t = (1 - \rho_d) \ln \overline{d} + \rho_d \ln \tilde{\tilde{d}}_{t-1} + \varepsilon_{dt},$$

and thus:

- dividends feature a deterministic constant-growth component
- logged dividends exhibit AR(1) fluctuations about a linear trend.

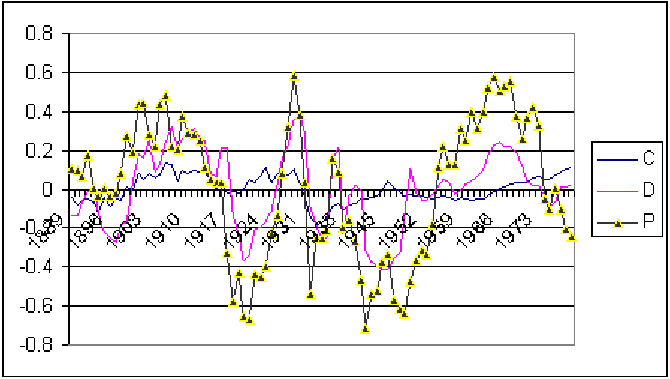
Closing the Model, cont.

In general, when specifying SPs for exogenous forcing variables, it is important that the specifications correspond with their empirical counterparts. (For an analysis of the importance of this issue: Gorodnichenko and Ng, 2007, U. Mich. WP.)

In this case, support for the assumption of trend-stationarity comes from DeJong and Whiteman (1991 *AER;* 1994 *ET*).

The Data

Logged Deviations from Linear Trend



Inducing Stationarity

We seek a specification of the model written in terms of stationary versions of variables. Stationarity will be induced in the data analogously. In this case, conversion to stationarity requires *trend removal*.

From the exercise, we've shown that $\begin{bmatrix} \widetilde{\widetilde{d}}_t & \widetilde{\widetilde{q}}_t \end{bmatrix}'$ is stationary.

So too is $\widetilde{\widetilde{c}}_t$, since $c_t = d_t + q$.

What of p_t?

Inducing Stationarity, cont.

Guess: $\widetilde{\widetilde{p}}_t$ is also stationary. To verify, determine whether this guess is consistent with the pricing kernel. In terms of deterministic components of variables, under the guess we have

$$\overline{p}e^{gt} = \beta\left[\frac{u'(\overline{c}e^{gt+1})}{u'(\overline{c}e^{gt})}(\overline{d}+\overline{p})e^{gt+1}\right],$$

or

$$\overline{p} = \beta\left[\frac{u'(\overline{c}e^{gt+1})}{u'(\overline{c}e^{gt})}(\overline{d}+\overline{p})e^{g}\right].$$

Inducing Stationarity, cont.

Assuming

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma},$$

we have

$$\begin{aligned} \overline{p} &= \beta \left[e^{-\gamma g} (\overline{d} + \overline{p}) e^{g} \right] \\ &= \beta e^{(1-\gamma) g} (\overline{d} + \overline{p}), \end{aligned}$$

and thus the guess is verified.
Solving for $\overline{p}$, we obtain

$$\overline{p} = \frac{1}{(1+\rho)\, e^{(\gamma-1)g} - 1} \overline{d}.$$

The Transformed Model

Hereafter, p_t is in fact $\frac{p_t}{e^{gt}}$, etc. The model is then

$$\begin{aligned}
p_t &= \beta e^{(1-\gamma)g} E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} (d_{t+1} + p_{t+1}) \right] && (4) \\
c_t &= d_t + q_t && (5) \\
d_t &= \overline{d} e^{\omega_{dt}}, \quad \omega_{dt} = \rho_d \omega_{dt-1} + \varepsilon_{dt}, && (6) \\
q_t &= \overline{q} e^{\omega_{qt}}, \quad \omega_{qt} = \rho_q \omega_{qt-1} + \varepsilon_{qt}. && (7)
\end{aligned}$$

State: $s_t = [d_t \quad q_t]'$
Shocks: $\upsilon_t = [\varepsilon_{dt} \quad \varepsilon_{qt}]'$
Controls: $c_t = [c_t \quad p_t]'$
Parameters: $\mu = [\beta \quad \gamma \quad g \quad \overline{d} \quad \overline{q} \quad vec(\Sigma)]'$.

Note that (6) and (7) constitute $s_t = f(s_{t-1}, \upsilon_t)$; (4) and (5) will be used to construct an approximation of $c_t = f(s_t)$. Our goal will be to transform this system into a likelihood function over the observables $X_t = \left[\widetilde{d}_t \quad \widetilde{p}_t \quad \widetilde{c}_t \right]'$.

Stochastic Singularity

As specified, the model features three observable variables, but only two sources of stochastic behavior. This gives rise to an issue known as *stochastic singularity*. Whenever there are more observable variables than structural shocks, various fixed combinations of observable variables are predicted to be deterministic. (See Ingram, Kocherlakota, and Savin, 1994 *JME* for details.)

Possible remedies:

- Augment the model with additional structural shocks.
- Introduce measurement error.

Real Business Cycle Model (Class-Long Exercise)

Text reference: Ch. 5.1, pp. 88-96.

$$\max_{c_t, l_t} \quad U = E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^{\varphi} l_t^{1-\varphi}\right)^{1-\phi}}{1-\phi},$$

subject to

$$\begin{aligned} y_t &= z_t k_t^{\alpha} n_t^{1-\alpha}, \\ 1 &= n_t + l_t, \\ y_t &= c_t + i_t, \\ k_{t+1} &= i_t + (1-\delta) k_t, \\ z_t &= z_0 e^{gt} e^{\omega_t}, \quad \omega_t = \rho \omega_{t-1} + \varepsilon_t. \end{aligned}$$

Exercise:

- Establish associated Value Function
- Derive FONCs
- Establish Non-Linear System
- Derive steady states.