Integral Equation Methods for Free Boundary Problems*

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Abstract

We outline a unified approach for treating free boundary problems arising in Finance using integral equation methods. Starting with the PDE formulations of the free boundary problems, we show how to derive nonlinear integral equations for the free boundaries in a variety of Finance applications. Methods to treat theoretical (existence, uniqueness) questions and analytical and numerical approximations are sketched in this integral equation context. This article is a summary of joint work with colleagues (Xinfu Chen and David Saunders) and former students (Lan Cheng and Dejun Xie) at the University of Pittsburgh.

Keywords: Integral equations, free boundary problems, partial differential equations, American put option, mortgage prepayment option, default barrier, inverse first crossing problem.

1 Introduction

Free boundary problems (FBPs) are ubiquitous in modern Mathematical Finance. They arise as early exercise boundaries for American style options, as default barriers in structural (value-of-firm) models of credit default, as the optimal strategies for refinancing mortgages, exercising employee stock options and callable convertible bonds, etc. There are many methods for treating the

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FBPs that arises as mathematical models of these Finance problems, including variational inequalities [1], viscosity solutions [2] and the classical PDE approach [3, 4]. In this note we focus on an integral equation (IE) approach that is particularly suited for the types of FBPs that arise in Finance.

In section 2 we will sketch the method in the context of the American put option, arguably the most widely known and best understood FBP in Finance. After deriving an IE problem mathematically equivalent to the original Black, Scholes and Merton PDE FBP for the American put, we will outline how the IE problem can be used to prove existence and uniqueness for the original problem and to derive analytical and numerical estimates for the location of the early exercise boundary [5, 6]. In section 3 we will sketch how this IE approach can be carried over to other FBPs in Finance [7,8, 9] with the goal of indicating that this is a unified approach to a diverse collection of problems.

2 Free Boundary Problems as Integral Equations

In this section we shall outline the IE approach in the context of an American put option on a geometric Brownian motion underlier. Black, Scholes and Merton risk-neutral pricing theory says that the option value, $p(S,t)$, satisfies the FBP

$$p_t + \frac{\sigma^2 S^2}{2} p_{SS} + rSp_S - rp = 0, \ S_f(t) < S < \infty, \quad 0 < t < T \tag{1a}$$

$$p(S,t) = K - S \text{ on } S = S_f(t), \quad 0 < t < T \tag{1b}$$

$$p_S(S,t) = -1 \quad \text{on} \quad S = S_f(t), \quad 0 < t < T, \tag{1c}$$

$$p(S,t) \to 0 \quad \text{as} \quad S \to \infty, \tag{1d}$$

$$p(S,T) = \max(K - S, 0), \quad K = S_f(T) < S < \infty, \tag{1e}$$

where r, σ, K, T have the conventional meanings and $S_f(t)$ is the location of the early exercise, free boundary to be determined along with $p(S,t)$. Letting $\tau = \frac{\sigma^2}{2}(T-t)$ (the scaled time to expiry) and $x = ln(S/K)$, then the scaled option price

$$P_{\text{new}} = \begin{cases} 1 - S/K & S < S_f(t) \\ p/K & S > S_f(t) \end{cases}$$

satisfies the transformed problem (dropping the subscript) in $-\infty < x < \infty,\ 0 < \tau < \sigma^2 T/2$

$$p_\tau - \{p_{xx} + (k-1)p_x - \kappa p\} = kH(x_f(\tau) - x), \tag{2a}$$

$$p(x,0) = \max(1 - e^x, 0), \tag{2b}$$

where $k = 2r/\sigma^2, H$ is the Heaviside function, $x_f(\tau) = ln(S_f/S)$ and the coefficient k appears on the rhs of (2a) because the intrinsic payoff, $p_0(x) = 1 - e^x$ satisfies

$$p_{0\tau} - \{p_{0xx} + (k-1)p_{0x} - kp_0\} = k.$$

The solution to problem (2) can be written in terms of the free boundary, $x_f(\tau)$, using the fundamental solution of the pdo on the lhs of (2a),

$$\Gamma(x,\tau) = \frac{e^{-k\tau}}{2\sqrt{\pi\tau}} e^{-(x+(k-1)\tau)^2/4\tau}, \tag{3}$$

in the form

$$p(x,\tau) = \int_{-\infty}^{0} (1 - e^y)\Gamma(x-y,\tau)dy + k\int_0^\tau \int_{-\infty}^{x_f(u)} \Gamma(x-y,\tau-u)dy\ du. \tag{4}$$

The first term is the price of the European style put while the second is the premium for the American optionality. Integral representations of this sort have been discussed in the Finance literature for some time (see, for example [10]).

For the representation (4) to be useful, one must first determine the unknown location of the boundary, $x_f(\tau)$, which appears in the second integral on the rhs of (4). The usual approach in the free boundary literature proceeds by starting with (4) and evaluating the lhs at $x = x_f(\tau)$ using one or other of the conditions

$$p(x_f(\tau),\tau) = 1 - e^{x_f(\tau)}, \tag{5a}$$

$$p_x(x_f(\tau),\tau) = -e^{x_f(\tau)} \tag{5b}$$

which are the transformed versions of the smooth pasting conditions (1b) and (1c). Instead, we use a trick here, based on financial considerations, to notice that $p_\tau(x_f(\tau),\tau) = 0$. Thus, from (4),

$$p_\tau(x,\tau) = \Gamma(x,\tau) + k\int_0^\tau \Gamma(x - x_f(u), \tau - u)\dot{x}_f(u)du, \tag{6}$$

which, upon evaluation on the early exercise boundary, provides the following nonlinear integral equation for $x_f(\tau)$:

$$\Gamma(x_f(\tau),\tau) = -k\int_0^\tau \Gamma(x_f(\tau) - x_f(u), \tau - u)\dot{x}_f(u)du, \tag{7a}$$

This equation has proven, in our experience, to be much more effective in the mathematical analysis of this problem than the versions obtainable from (5a) and (5b). In addition, we have obtained still other versions, also derivable from the representation (4), whose particular forms have proven useful in various situations [6]:

$$\int_0^\tau \{\Gamma_x(x_f(\tau),u) + k\Gamma(x_f(\tau),u)\}du = k\int_0^\tau \Gamma(x_f(\tau)-x_f(u),\ \tau-u)du \quad (7b)$$

$$\Gamma(x_f(\tau),\tau) = \frac{k}{2} + k\int_0^\tau \{\Gamma(x_f(\tau)-x_f(u),\tau-u) - \Gamma_x(x_f(\tau)-x_f(u),\tau-u)\}du, \quad (7c)$$

$$\dot{x}_f(\tau) = \frac{-2\Gamma_x(x_f(\tau),\tau)}{k} + 2\int_0^t \Gamma_x(x_f(\tau)-x_f(u),\tau-u)\dot{x}_f(u)du. \quad (7d)$$

Using the representation (4) to compute p_x, p_{xx} and $p_{x\tau}$, the above IEs for $x_f(\tau)$ follow (after some rearrangement of terms) by evaluation on the boundary.

The underlying theoretical rationale for using this integral equation approach to treat the original FBPs (1) or (2) is summarized in the following result.

Theorem. (Theorem 3.2 and sections 4, 5, 6 of [6]). Suppose that $x_f \in C^1((0,\infty)) \cap C^0([0,\infty))$ and $\alpha(\tau) = x_f(\tau)^2/4\tau$. Assume that as $\tau \searrow 0$, $\alpha(\tau) = [-1+o(1)]ln\sqrt{\tau}$ and $\tau\dot{\alpha}(\tau) = 0(1)$. Then x_f, together with p defined by (4), solves the (equivalent) FBPs (1) or (2), if and only if x_f satisfies any of the equivalent integro-differential equations (IODEs) (7a), $\cdots$ (7d). Finally, (7d) has a unique solution with the properties listed above.

The equivalence of the IODEs (7) is established in Lemma 3.1 of [6] and the required estimates on α are rigorously derived from (7a). The proof that (7d) has a solution with the required properties is a highly technical analysis [6] based on Schauder's Fixed Point Theorem. A similar existence proof, also based on IEs, but motivated by stopping time arguments, was independently obtained by Peskir [11].

Analytical and numerical estimates for the location of the early exercise boundary, that might be useful to practitioners, can also be obtained from the IODEs (7). For example, if we make the change of variables $\eta = (x_f(\tau) - x_f(u))/2\sqrt{\tau-u}$ in (7a), the rhs for small τ (near expiry) behaves like

$$-k\int_0^{\alpha(\tau)} \left[1 - \frac{x_f(\tau)-x_f(u)}{2\dot{x}_f(u)(\tau-u)}\right]^{-1} \frac{e^{-\eta^2}}{\sqrt{\pi}}d\eta,$$

which tends to k because $\alpha(\tau) \to -\infty$ (above Theorem) and $[\cdots]^{-1} \to 1/2$ uniformly in u because of the convexity of x_f (proved separately in [12] using the method of Friedman and Jensen [13]). Thus, from (7a) with small τ,

$$\Gamma(x_f(\tau), \tau) = \frac{e^{-k\tau}}{2\sqrt{\pi\tau}} e^{-(x_f(\tau)+(k-1)\tau)^2/4\tau} \cong$$
$$= \frac{e^{-x_f(\tau)^2/4\tau}}{2\sqrt{\pi}} \cong k$$

which leads to

$$x_f(\tau) \approx 2\sqrt{\tau}\sqrt{-ln(4\pi k^2 \tau)^{1/2}} \text{ as } \tau \to 0. \tag{8}$$

This implies the first rigorous estimate for the near expiry behavior of the early exercise boundary obtained by Barles et al [14]

$$S_f(t) \simeq k\Big[1 - \sigma\sqrt{-(T-t)ln(T-t)}\Big],\ t \sim T.$$

In addition, it provides the first estimate for $\alpha(\tau)$ in the above existence theorem. Specifically,

$$\alpha(\tau) = x_f(\tau)^2 4\tau \approx -ln(4\pi k^2\tau)^{1/2} = -\frac{\xi}{2}, \tag{9}$$

where $\xi = ln(4\pi k^2 \tau)$.

More precise analytical and numerical estimates can be obtained for $\alpha(\tau)$ (equivalently $x_f(\tau)$ and $S_f(t)$) valid for intermediate and large times as well. For example, using Mathematica to iterate (9) through (7a) one obtains the more accurate estimate [6]

$$\alpha(\tau) = -\frac{\xi}{2} - \frac{1}{\xi} + \frac{1}{2\xi^2} + \frac{17}{3\xi^3} - \frac{51}{4\xi^4} - \frac{1148}{15\xi^5} + \frac{398}{\xi^6} + \cdots \tag{10a}$$

One can also imagine using the integral equation (7a) to express ξ as a function of α. One finds [6], for arbitrary a,

$$-\frac{\xi}{2} = \alpha + ln\Big[1 + \frac{1/2}{\alpha + a} - \frac{a/2}{(\alpha+a)^2} + \frac{(1-a)^2}{2(\alpha+a)^3} + \cdots\Big],$$

or equivalently, on exponentiation,

$$\sqrt{\tau}e^{\alpha(\tau)}\Big[1 - \frac{1}{2(\alpha+a)} - \frac{a}{2(\alpha+a)^2} + \frac{(1-a)^2}{2(\alpha+a)^3} + \cdots\Big] = 1/\sqrt{4\pi k^2} \tag{10b}$$

an implicit estimate (that can be truncated by taking $a = 1$). It also highlights the significance of the constant $4\pi k^2$ that appears in all of these estimates. Because of the previously mentioned convexity of the boundary, these estimates can be interpolated with the Merton infinite horizon solution ($S_f = \frac{k}{k+1}K$) to obtain accurate estimates for all times.

Perhaps even more importantly, a very fast, accurate numerical scheme can be obtained from the IODE (7d) which can be written in the equivalent form

$$\dot{x}_f(\tau) = \frac{x_f(\tau)}{2k\tau}\Gamma(x_f(\tau),\tau)\Big[1 + m(\tau)\Big], \tag{11a}$$

where

$$m(\tau) = k\int_0^\tau \left[\frac{2\tau}{x_f(\tau)}\left(\frac{x_f(\tau) - x_f(u)}{\tau - u}\right) - 1\right]\frac{\Gamma(x_f(\tau) - x_f(u), \tau - u)}{\Gamma(x_f(\tau),\tau)}\ \dot{x}_f(u)du \tag{11b}$$

that is to be solved with initial data $x_f(0) = 0$. Solving this iteratively with $m(\tau) = m_0(\tau) \equiv 0$ initially provides the fastest and most accurate approximation among all our estimates [6].

3 Application of the IE Method to Other FBPs

In the previous section we described how to formulate the American put FBP in terms of IODEs for the early exercise boundary and how to use this formulation to establish theoretical results (existence, uniqueness) as well as analytical and numerical estimates for the original problem. In this section we indicate the wider applicability of the method by briefly discussing several other problems arising from Finance.

(a) Jump-Diffusion Processes. These integral equation methods can be extended to jump-diffusion models. Specifically, letting $x = ln(S/K)$, we now assume that the transformed asset price follows the process

$$X(t) = (\mu - \sigma^2/2)t + \sigma W(t) + N(t), \tag{12}$$

where $N(t)$ is a Poisson process with rate λt and has jumps of size $\pm\epsilon$ with equal probability. In this case, the transformed PDE analogous to (2) is

$$\mathcal{L}p = \mathcal{L}(1 - e^x)H(x_f(\tau) - x) \tag{13a}$$

$$p(x, 0) = \max(1 - e^x, 0) \tag{13b}$$

where $\mathcal{L}$ is the nonlocal pdo

$$\mathcal{L}p = p_\tau - \{p_{xx} + (k-1)p_x - kp\} + \lambda\{p(x+\epsilon,\tau) - 2p + p(x-\epsilon,\tau)\}. \quad (14)$$

This problem is amenable by the methods outlined above because the fundamental solution can be explicitly calculated. Specifically,

$$\Gamma(x,\tau) = \sum_{u=0}^{\infty} \frac{(\lambda\tau)^n}{2^n n!} e^{-\lambda\tau} \Big(\sum_{j=0}^{n} \binom{n}{j} \Gamma_{BS}((2j-u)\epsilon + x, \tau) \Big)$$

where Γ_{BS} is the BSM fundamental solution in (3). Proceeding as in section 2 one obtains the analog to (7a) in the form

$$\Gamma(x_f(\tau),\tau) = -\int_0^\tau (k + \lambda\{2 - e^\epsilon - e^{-\epsilon}\} e^{x_f(u)} \Gamma(x_f(\tau) - x_f(u), \tau - u) \cdot$$

$$\dot{x}_f(u)du] + \lambda \int_0^\epsilon (1 - e^{y-\epsilon}) \Gamma(x_f(\tau) - y, \tau) dy, \quad (15)$$

from which we obtain the near expiry estimate for $\alpha(\tau) = x_f(\tau)^2/4\tau$ (see the analog (10b) with no jumps)

$$\sqrt{\tau}^\alpha \approx 1/\sqrt{4\pi\tilde{k}^2} \text{ as } \tau \to 0 \quad (16)$$

where $\tilde{k} = k + \lambda(1 - e^{-\epsilon})$, agreeing with the result of Pham [15] using other methods.

(b) Interest Rate Processes. These IE methods can also be used to study American style contracts on other underliers. For example, a mortgage prepayment option provides the holder with the right to prepay the outstanding balance of a fixed-rate mortgage

$$M(t) = \frac{m}{c}(1 - e^{-c(T-t)}) \quad (17)$$

where T is the maturity, c is the (continuous) fixed mortgage rate and m is the (continuous) rate of payment of the mortgage (i.e., mdt is the premium paid in any time interval dt). Clearly the value of the prepayment option depends on $M(t)$ and also on the rate of return, $r(t)$, that the mortgage holder (borrower) can obtain by investing $M(t)$. If this short-term rate is assumed to follow the Vasicek model

$$dr = (\eta - \theta r)dt + \sigma dW \quad (18)$$

in a risk-neutral world, then the value of the prepayment option, $V(r,t)$, satisfies [9, 16]

$$V_t + \frac{\sigma^2}{2} V_{rr} + (\eta - \theta r)V_r + m - rV = 0, R(t) < r < \infty, 0 < t < T \quad (19a)$$

$$V(r,t) = M(t),\ r = R(t),\ 0 < t < T \tag{19b}$$

$$V_r(r,t) = 0,\ r = R(t),\ 0 < t < T \tag{19c}$$

$$V(r,t) \to 0,\ \text{as } r \to \infty,\ 0 < t < T \tag{19d}$$

$$V(r,T) = 0,\ c = R(0) < r < \infty \tag{19e}$$

The optimal strategy for the mortgage holder is to exercise the option to pay off the mortgage the first time that the rate r falls below $R(t)$ at time t. Existence and uniqueness for this FBP was proved using variational methods [16].

Because the fundamental solution for the Vasicek "bond pricing equation", (19a), can be explicitly calculated, its form suggests a sequence of changes of dependent and independent variables (not relevant for this summary) that reduces the FBP (19) to the following analog of (2) in $-\infty < x < \infty, s > 1$

$$u_s - \frac{1}{4}u_{xx} = f(x,s)H(x - x_f(s)) \tag{20a}$$

$$u(x,1) = 0 \tag{20b}$$

where $f(x,s)$ is a specific function resulting from the transformations and $x_f(s)$ is the transformed free boundary with $u(x_f(s),s) = 0 = u_x(x_f(s),s)$.

In this form, the procedure outlined in the previous section can be followed to obtain

$$u(x,s) = \int_1^s \Big[\int_{x_f(u)}^{\infty} \Gamma(x-y,s-u)f(y,u)dy\Big]du, \tag{21a}$$

where Γ is the fundamental solution of the heat operator $\partial_s - \frac{1}{4}\partial_{xx}$, and the free boundary can be obtained by solving the integral equation

$$\int_1^s \Big[\int_{x_f(u)}^{\infty} \Gamma(x_f(s)-y,s-u)f(y,u)dy\Big]du = 0. \tag{21b}$$

In his Ph.D dissertation [9], Dejun Xie used the integral representations (21) to obtain near-expiry estimates for the critical rate as well as to obtain a numerical scheme to determine $x_f(s)$ globally. Specifically, if $Q(x,s)$ denotes the integral on the rhs of (21a) he showed that the Newton-Raphson iterative scheme to solve (21b), $Q(x_f(s),s) = 0$, can be written as

$$x_f(s)^{\text{new}} = x_f(s)^{\text{old}} + \frac{Q(x_f(s)^{\text{old}},s)}{2f(x_f(s)^{\text{old}},s)}. \tag{22}$$

where, in the denominator, $Q_x(x_f(s), s)$ is approximated by

$$\frac{1}{2}\{Q_x(x_f(s)+, s) + Q_x(x_f(s)-, s)\} = \frac{1}{2}u_{xx}(x_f(s), s) = -2f(x_f(s), s).$$

(c) Credit Default Processes. As a final example, we outline how these methods can be used to obtain an integral equation formulation for the inverse first crossing problem in a value-of-firm (structural) model for credit default. Suppose the default index of a company, $X(t)$, is a stochastic process following the Uhlenbeck-Ornstein process

$$dX(t) = adt + \sigma dW(t), \quad X(0) = x_0, \tag{23}$$

(equivalently the log of such an index that originally satisfied a geometric Brownian motion). Default of the firm is said to occur the first time τ that $X(t)$ falls below a pre-assigned value, $b(t)$. The survival pdf, $u(x, t)$ defined by

$$u(x, t)dx = Pr[x < x(t) < x + dx|t < \tau]$$

is known to satisfy the following problem for the forward Kolmogorov equation:

$$u_t = \frac{\sigma^2}{2}u_{xx} - au_x, b(t) < x < \infty, 0 < t < T \tag{24a}$$

$$u(x, t) = 0, \; x = b(t), 0 < t < T \tag{24b}$$

$$u(x, t) \to 0 \; x \to \infty, 0 < t < T \tag{24c}$$

$$u(x, 0) = \delta(x - x_0), b(0) < x < \infty, \tag{24d}$$

and the resulting survival probability is given, in terms of the solution $u(x, t)$, by

$$Pr(\tau > t) = P(t) = \int_{b(t)}^{\infty} u(x, t)dx. \tag{24e}$$

Motivated by the work of Avellaneda and Zhu [18], our Ph.D. student, Lan Cheng, studied the inverse first crossing problem in her dissertation [7]: given the survival probability $P(t)$ for $0 < t < T$, find the time dependent absorbing boundary $b(t)$ in (24b), including $b(0)$, such that (24a) ... (24e) are satisfied. The more usual extra Neumann boundary condition appearing in FBPs can be obtained by differentiating (24e):

$$P'(t) = -u(b(t), t)b'(t) + \int_{b(t)}^{\infty} u_t(x, t)dx$$

$$= \frac{-\sigma^2}{2} u_x(b(t), t)$$

using the PDE (24a) and the boundary conditions. With $-P'(t) = (1 - P(t))' = Q'(t) = q(t)$ denoting the default pdf, the extra boundary condition becomes

$$u_x(x,t) = \frac{2}{\sigma^2} q(t), x = b(t), 0 < t < T \tag{24e'}$$

Following the outline in section 2, one can derive integral equations for $b(t)$ in the form:

$$\Gamma(b(t), t) = \int_0^t \Gamma(b(t) - b(s), t - s) q(s) ds \tag{25a}$$

$$\frac{1}{2} q(t) = \Gamma_x(b(t), t) - \int_0^t \Gamma_x(b(t) - b(s), t - s) q(s) ds \tag{25b}$$

where Γ is the fundamental solution of the pdo in (24a). A fast and accurate numerical scheme for solving

$$F(x,t) = \Gamma(x,t) - \int_0^t \Gamma(x - b(s), t - s) q(s) ds = 0$$

for $x = b(t)$ (i.e., solving (25a)) is the Newton-Raphson iteration

$$b(t)^{\text{new}} = b(t)^{\text{old}} - \frac{F(b(t)^{\text{old}}, t)}{q(t)/2}, \tag{26}$$

where in computing F_x in the denominator we have used

$$\frac{1}{2} q(t) = F_x(b(t), t) \simeq F_x(b(t)^{\text{old}}, t) =$$

$$\cong \Gamma_x(b(t)^{\text{old}}, t) - \int_0^t \Gamma_x(b(t)^{\text{old}} - b(s)^{\text{old}}, t - s) q(s) ds$$

Finally, we mention that an IE formulation of the first passage problem for Brownian motion was given by Peskir [19] but the inverse problem described here was not treated. In [7] the proof of existence and uniqueness used viscosity solution methods. A proof using integral equations is still open for this problem as well as the two others listed in this section.

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