Comparison of Credit Default Models

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1. **Credit Default Models**

(a) **Structural (Value of Firm) Models.**

X_i - default index of i^{th} firm

$$dX_i = a_i dt + \sigma_i dW_t^i$$
$$X_i(0) = x_{0i}$$

(e.g., X_i is the log of the value of the firm which satisfies a geometric Brownian motion).

τ_i - (random) default time - the first time t that $X_i(t)$ falls below a preassigned value $b_i(t)$.

$u_i(x,t)$ - survival probability density

$$u_i(x,t)dx = Pr[x < X_i(t) < x + dx \mid t < \tau_i].$$

Kolmogorov forward (Fokker-Planck) equation

$$\begin{aligned}
&u_t = \tfrac{\sigma^2}{2}\, u_{xx} - a u_x,\ b(t) < x < \infty,\ 0 < t < T \\
&u(x,t) = 0,\ x = b(t),\ 0 < t < T \\
&u(x,t) \to 0,\ x \to \infty,\ 0 < t < T \\
&u(x,0) = \delta(x - x_0),\ x > b(0)
\end{aligned}$$

$Pr[\tau > t] = \int_{b(t)}^{\infty} u(x,t)dx = P(t)$ survival probability

R.C. Merton - J. Fin **29** (1974)
F. Black & J.C. Cox, J. Fin. **31** (1976)

$$\begin{aligned}
P'(t) = \tfrac{d}{dt}\, Pr[\tau > t] &= -u(b(t),t)b'(t) + \int_{b(t)}^{\infty} u_t(x,t)dt \\
&= 0 - \frac{\sigma^2}{2}\, u_x(b(t),t) + a u(b(t),t) \\
&= -\frac{\sigma^2}{2}\, u_x(b(t),t).
\end{aligned}$$

Inverse first crossing problem is a free boundary problem with $b(t)$, including $b(0)$, to be determined.

M. Avellaneda & J. Zhu, Risk **4** (2001)
C. Zucca, L. Sacerdote & G. Peskir, preprint (January 2002)
Lan Cheng, X. Chen, J. Chadam & D. Saunders, SIMA **38** (2006)

Theorem: $\exists!$ solution to the inverse first crossing problem. Moreover, if $Q(t) = 1 - P(t)$, the default probability, satisfies

$$\sup_{t \to 0} \frac{Q(t)}{tQ'(t)} < \infty$$

then

$$\lim_{t \to 0} \frac{b(t)}{\sqrt{-4t\ell n Q(t)}} = -1$$

Numerical Scheme: $\Gamma(x, t)$ - Fundamental Solution

$$\Gamma(b(t), t) = \int_0^t \Gamma(b(t) - b(s), t - s) Q'(s) ds$$

$$\frac{1}{2} Q'(t) = \Gamma_x(b(t), t) - \int_0^t \Gamma_x(b(t) - b(s), t - s) Q'(s)) ds$$

Solve $F(x, t) = \Gamma(x, t) - \int_0^t \Gamma(x - b(s), t - s)\ Q'(s) ds$ for $x = b(t)$ using Newton-Raphson

$$b_{n+1}(t) = b_n(t) - \frac{F(b_n(t), t)}{F_x(b_n(t), t) \cong Q'(t)/2}$$

Examples: $x_0 = 0,\ t \simeq 0$

(a) $Q(t) = At^m,\ A,\ m > 0,\ b(t) \simeq -\sqrt{-4mt\ell nt}$

(b) $Q(t) = Ae^{-\mu^2/4t^m},\ A, m, \mu > 0$

$$b(t) \simeq -\mu t^{(1-m)/2} \longrightarrow \begin{cases} -\infty & m > 1 \\ -\mu & m = 1 \\ 0 & 0 < m < 1 \end{cases} \quad \text{as } t \to 0$$

Example (b) with $m = 1$ is associated with the one case that has an explicit solution:

$$\begin{aligned} b(t) &= -b < 0 \text{ for all } t > 0 \\ u(x,t) &= \Gamma(x,t) - \Gamma(x+2b,t) \\ P(t) &= \frac{1}{\sqrt{\pi}} \int_{(-b-at)/\sigma\sqrt{2t}}^{(b-at)/\sigma\sqrt{2t}} e^{-y^2} dy \end{aligned}$$

OPEN PROBLEM: Convexity of default barrier

Default Correlation

Simple model with two firms, no drift

$$dX_i = \sigma_i dW_t^i,\ \ X_i(0) = 0$$
$$E(dW_t^1 dW_t^2) = \rho dt$$

Survival pdf, $u(x_1, x_2, t)$ satisfies

$$u_t = \frac{\sigma_1^2}{2}\ u_{x_1x_1} + \rho\sigma_1\sigma_2 u_{x_1x_2} + \frac{\sigma^2}{2}\ u_{x_2x_2},\ \ b_i(t) < x_i < \infty,\ \ 0 < t < T$$
$$u(b,(t), x_2, t) = 0 = u(x_1, b_2(t), t),\ \ b_i(t) < x_i < \infty,\ \ 0 < t < T$$
$$u(\infty, x_2, t) = 0 = u(x_1, \infty, t),\ \ b_i(t) < x_i < \infty,\ \ 0 < t < T$$
$$u(x_1, x_2, 0) = \delta(x_1)\delta(x_2),\ \ b_1(0) < x_i < \infty$$

where the default barriers $b_1(t)$ and $b_2(t)$ are incorporated into the joint survival distribution function

$$G(t,t) = Pr[\tau_1 \& \tau_2 > t] \text{ through}$$
$$G(t,t) = \int_{b_1(t)}^{\infty} \int_{b_2(t)}^{\infty} u(x_1, x_2, t) dx_2 dx_1$$

One retrieves the single firm (marginal) survival probabilities, $P_i(t)$, by taking limits $b_j(t) \to -\infty (j \neq i)$

$$\lim_{b_2 \to -\infty} \int_{b_1(t)}^{\infty} \int_{b_2(t)}^{\infty} u(x_1, x_2, t; b_1, b_2) dx_2 dx_1 = P_1(t);\ \text{ i.e.,}$$
$$U(x_1, t) = \int_{-\infty}^{\infty} u(x_1, x_2, t; b_1(t), -\infty) dx_2$$

satisfies the $1 - D$ problem with

$$\int_{b_1(t)}^{\infty} U(x_1, t) dx_1 = P_1(t)$$

OPEN PROBLEM: Correspondence between model and $G(t,t), P_1(t), P_2(t)$.

For the explicitly calculable solution:

$$b_i(t) = -b_i < 0, \text{ constant:}$$

$$P_i(t) = \frac{1}{\sqrt{\pi}} \int_{-b_i/\sigma_i\sqrt{2t}}^{b_i/\sigma_i\sqrt{2t}} e^{-y^2} dy$$

$$Q_i(t) = 1 - P_i(t) = \text{ erfc } \left(\frac{b_i}{\sigma_i\sqrt{2t}}\right) = 2\Phi\left(\frac{b_i}{\sigma_i\sqrt{t}}\right)$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$ and $G(t,t)$ can be computed explicitly as an infinite series

$$G(t,t) = \frac{2r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{n=1,3,} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right)\left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4t}\right)\right]$$

where I_ν is the modified Bessel function of order ν

$$\alpha = \begin{cases} \tan^{-1}(-\sqrt{1-\rho^2/\rho}) \text{ if } \rho < 0 \\ \pi + \tan^{-1}(-\sqrt{1-\rho^2/\rho}) \text{ otherwise} \end{cases}$$

and, with $Z_i = b_i/\sigma_i, \; i = 1, 2,$

$$\theta_0 = \begin{cases} \tan^{-1}[Z_2\sqrt{1-\rho^2}/(Z_1 - \rho Z_2)] \text{ if } [\cdots] > 0 \\ \pi + [Z_2\sqrt{1-\rho^2}/(Z_1 - \rho Z_2)] \text{ otherwise} \end{cases}$$

$$r_0 = Z_2/\sin(\theta_0).$$

G. Zhou, Rev. Fin. Studies **14** (2001)
S. Iyengar, SIAM Appl. Math. **45** (1985)
A. Metzler, U. of Waterloo Ph.D. dissertation (2008)

$G(t,t) = Pr[\tau \& \tau_2 > t]$ (related to) distribution of first exit time $F(t,t) = Pr[\tau_1 \text{ or } \tau_2 < t] = 1 - G(t,t)$.

One can also obtain the joint cummulative survival distribution function (J. Huang, B. Shi & L.-K. Tsui; A. Metzler, section 2.4 of thesis)

$$G(t_1, t_2) = Pr(\tau_1 > t_1 \text{ and } \tau_2 > t_2), \ (t_2 > t_1 \text{ say})$$
$$= \int_0^{\alpha} \int_0^{\infty} \text{ erf } \Big(\frac{r \sin \ \theta}{\sqrt{2(t_2 - t_1)}} \Big) \ f(r, \theta, t_1) \ dr \ d\theta$$

in terms of the survival pdf

$$f(r, \theta, t_1) = \frac{r}{\alpha t_1} \ e^{-(r^2 + r_0^2)/2t_1} \sum_{n=1}^{\infty} \ \sin\Big(\frac{n\pi\theta}{\alpha}\Big) \ \sin\Big(\frac{n\pi\theta_0}{\alpha}\Big) \ I_{\frac{n\pi}{\alpha}} \ \Big(\frac{r \ r_0}{t_1}\Big)$$

(solution of $2 - D$ problem with constant boundaries).

The associated joint survival pdf

$$g(t_1, t_2) = Pr(t_1 < \tau_1 < t_1 + dt_1, t_2 < \tau_2 < t_2 + dt_2) = -h(t_1, t_2)$$

$$= -\frac{\pi \sin \alpha}{2\alpha^2 \sqrt{t_1}(t_2 - t_1)\sqrt{t_2 - t_1 \cos^2 \alpha}} \ \exp \ \Big(-\frac{r_0^2}{2t_1} \ \frac{t_2 - t_1 \cos \ 2\alpha}{(t_2 - t_1) + (t_2 - t_1 \cos \ 2\alpha)}\Big)$$
$$\times \sum_{n=0}^{\infty} n \ \sin\Big(\frac{n\pi\theta_0}{\alpha}\Big) I_{\frac{n\pi}{2\alpha}} \ \Big(\frac{r_0^2(t_2 - t_1)/t_1}{2(t_2 - t_1) + 2(t_2 - t_1 \cos 2\alpha)}\Big)$$

Note: Joint default probability,

$$Pr[\tau_2 < t_1 \& \ \tau_2 < t_2] = H(t_1, t_2) = 1 - P_1(t_1) - P_2(t_2) + G(t_1, t_2)$$
$$= Q_1(t_1) + Q_2(t_2) - 1 + G(t_1, t_2)$$

(b) (Gaussian) Copula Models

$$u_i = Q_i(t_i) = Pr(\tau_1 < t_1) : (0, \infty) \to (0, 1)$$
$$U_i = Q_i(\tau_i) \text{ is a uniform random variable.}$$

Gaussian copula for the joint default probability with given marginal default probabilities is

$$H_g(t_1, t_2) = \Phi_\rho(\Phi^{-1}(Q_1(t_1)), \Phi^{-1}(Q_2(t_2))$$
$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\Phi^{-1}(Q_1(t_1))} \int_{-\infty}^{\Phi^{-1}(Q_2(t_2))} e^{-(u_1^2+u_2^2-2\rho u_1 u_2)/2(1-\rho^2)} du_2 du_1$$

Note:

$$H_g(t_1, \infty) = \Phi_\rho(\Phi^{-1}(Q_1(t_1)), \infty)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(Q_1(t_1))} e^{-x^2/2} dx = Q_1(t_1) \text{ as req'd}$$

2. **Comparison of Structural vs. Copula Models**
   (a) Tail dependence (P. Schönbucher, p. 332).

$$\begin{aligned} H_g(t_1,t_2) &= \Phi_\rho(\Phi^{-1}(Q_1,(t_1)),\Phi^{-1}(Q_2(t_2))) \\ &= \Phi_\rho(\Phi^{-1}(u_1),\Phi^{-1}(u_2)) \text{ with } u_i = Q_i(t_i) \\ &= C(u_1,u_2) \end{aligned}$$

$$\lim_{u\to 0} \frac{C(u,u)}{u} = \lambda_L, \text{ lower tail dependence parameter,}$$

$$\lim_{u\to 1} \frac{1+C(u,u)-2u}{1-u} = \lambda_U, \text{ upper tail dependence parameter}$$

For copula models this is independent of the marginals and depends only on the copula chosen; for the Gaussian copula with $-1 < \rho < 1$

$$\lambda_L = 0 \qquad \text{and} \qquad \lambda_U = 0$$

Gaussian copulas are tail independent; extreme events occur almost independently of each other.

In the structural model

$$\begin{aligned} \lambda_L &= \lim_{t\to 0} \frac{H(t,Q_2^{-1}(Q_1(t))}{Q_1(t)} = \lim_{t\to 0} \frac{H(Q_1^{-1}(Q_2(t)),t)}{Q_2(t)} \\ \lambda_U &= \lim_{t\to \infty} \frac{G(t,Q_2^{-1}(Q_1(t))}{1-Q_1(t)} = \lim_{t\to \infty} \frac{G(Q_1^{-1}(Q_2(t)),t)}{1-Q_2(t)} \end{aligned}$$

which clearly depend on the marginals.
In the simplified model with $-1 < \rho < 1$

$$\lambda_L = 0 \qquad \lambda_U = 0$$

Open Problem: Behavior at next, non-trivial, order

(b) Calibration to (Simple) Credit Default Swaps (CDS)

D. Brigo and E. Errais, preprint (June 2005)
J. Hull, M. Predescu and A. White, preprint (November, 2006)
J. Hull and A. White, J. Derivs., **14** (2006)

$$Pr(\tau_1 \text{ or } \tau_2 < t) = Pr(\tau_1 < t) + Pr(\tau_2 < t) - Pr(\tau_1 \ \& \ \tau_2 < t)$$
$$F(t,t) = Q_1(t) + Q_2(t) - H(t,t) = 1 - G(t,t)$$

(i) Merton single horizon pricing
A first-to-default credit default swap, FtD CDS, pays \$1 when either name defaults. In the Merton, single time horizon version, if either fails to redeem a \$1 bond at its maturity T:

Price of FtD CDS is present value of contingent cash flow

$$\begin{aligned} M^{(1)} &= e^{-rT} \ F(T,T) \\ &= e^{-rT} \ (1 - G(T,T)) \end{aligned}$$

Calibrate correlations to get same prices; i.e.,

$$M_s^{(1)}(\rho, T) = M_g^{(1)}(\hat{\rho}, T) \Leftrightarrow G_{s,\rho}(T,T) = G_{g,\hat{\rho}}(T,T)$$

(ii) Dynamic model (barrier crossed)

$$\begin{aligned} M^{(1)} &= \int_0^\infty e^{-rt} \ \frac{d}{dt} \ (F(t,t))dt \\ &= -\int_0^\infty \ e^{-rt} \ \frac{d}{dt} \ (G(t,t))dt \\ &= -(e^{-rt}G(t,t))|_0^\infty - r \ \int_0^\infty \ e^{-rt} \ G(t,t)dt \\ &= 1 - r \ \int_0^\infty \ e^{-rt} \ G(t,t)dt \end{aligned}$$

$$M_s^{(1)}(\rho, r) = M_g^{(1)}(\hat{p}, r) \Leftrightarrow \int_0^\infty \ e^{-rt} G_{s,\rho}(t,t)dt = \int_0^\infty \ e^{-rt} G_{g,\hat{\rho}}(t,t)dt$$

(c) Default clustering

Serial correlation of defaults but not contagion which might require feedback.

$$(i) Pr(\tau_2 = t_2 | \tau_1 = t_1) = Pr(\tau_2 = t_2, \tau_1 = t_1)/Pr(\tau_1 = t_1)$$
$$= h(t_1, t_2)/Q_1^{'}(t_1)$$

$$(ii) P(r)(\tau_2 - \tau_1 = s) = \begin{cases} \int_0^{\infty} h(t_1, t_1 + s) dt_1 & s > 0 \\ \int_0^{\infty} h(t_2 - s, t_2) dt_2 & s < 0 \end{cases}$$

$$(iii) \tau_{\max} = \max(\tau_2, \tau_1), \tau_{\min} = \min(\tau_2, \tau_1).$$

$$Pr(\tau_{\max} - \tau_{\min} = s) = \int_0^{\infty} [h(t, t + s) + h(t + s, t)] dt.$$

Open (Numerical) Problem: Repeat above comparisons for arbitrary marginals.

Open Problem: Structural models with jumps, stochastic volatility or regime switching.