NON-CONVEXITY OF THE OPTIMAL EXERCISE BOUNDARY FOR AN AMERICAN PUT OPTION ON A DIVIDEND-PAYING ASSET

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Abstract. We prove that when the dividend rate of the underlying asset following a geometric Brownian motion is slightly larger than the risk-free interest rate, the optimal exercise boundary of the American put option is not convex.

1. Introduction

Recently we provided a rigorous proof that the early exercise boundary for an American put option on an asset whose price followed a geometric Brownian motion was convex when the dividend rate was zero [12]; an independent proof was obtained by Ekstrom [13]. To date the convexity of the early exercise boundary is an open problem when the dividend rate is non-zero. Numerical experiments suggest that convexity obtains for dividend rates, $D \leqslant r$, the risk-free rate, and convexity breaks down when D becomes larger than r (D. Chakraborty [9] confirming independent private communications from J. Detemple, P. Duck and G. Meyer). In this note we provide a rigorous proof that for $0 < D - r \ll 1$ the early exercise boundary loses convexity.

The proof is based on careful estimates obtained from a pair of integro-differential equations ((4.6), (4.7) in §4) for the optimal exercise boundary. Since these equations involve the derivative of the boundary, complete rigor necessitates a proof of its regularity. To this end we provide a short, direct proof that the boundary is C^∞. We note that the regularity of the boundary was recently established for more general underliers (Bayraktar and Xing [5] proved C^∞ for jump-diffusion processes and Lamberton and Mikou [21] proved continuity of the boundary for general Lévy processes) but these proofs are necessarily considerably more technical than the proof presented here.

The paper is organized as follows. In the next section we provide a direct and rigorous formulation of the American put problem as a variational inequality. We then provide mathematically precise statements of the main results. In §3 we prove the regularity of the early exercise boundary. An outline of the derivation of the integro-differential equations central to our proof of the non-convexity is provided in §4. The details of the proof of the non-convexity of the optimal exercise boundary when $0 < D - r \ll 1$ are provided in §5. Finally, in §6 we provide a rigorous proof of the (time-to-expiry)$^{1/2}$ near expiry asymptotic behavior of the early exercise boundary formally derived in [27]. Combining these results confirms the generally accepted belief that for $0 < D - r \ll 1$ the boundary begins convex at expiry and loses convexity as the time-to-expiry increases. Analytic and numerical estimates for the location of the non-convex region are provided in §2 and §5. We also provide numerical evidence that convexity returns when D becomes sufficiently large (e.g., $D > e^{0.4} r \cong 1.5r$, when $r = 0.05$ and the volatility, $\sigma = 0.25$).

2. Background, Notation, and Main Results

We consider a financial market consisting of a bond and a stock, whose time t prices, $\mathbf{B}_t$ and $\mathbf{S}_t$, are stochastic processes defined by the stochastic differential equations

$$d\mathbf{S}_t = \mu_t \, \mathbf{S}_t \, dt + \sigma \, \mathbf{S}_t d\mathbf{W}_t, \qquad d\mathbf{B}_t = r\mathbf{B}_t \, dt \tag{2.1}$$

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where σ and r are positive constants and $\{\mathbf{W}_t\}$ is the standard Brownian motion (Wiener process). In the time interval $[t, t+dt)$, the stock pays $D\mathbf{S}_t dt$ dividend at time $t+dt$. An American put option with strike price E and expiry T is a guaranteed right to sell a stock at price E at any time on or before expiry. One wants to know the value of the option and the optimal time to exercise the right of the option.

The Black–Scholes model is widely used to value options. An important advantage of the model is that European options (no early exercise) can be valued analytically by the Black–Scholes formula [17, 24]. The situation is quite different, however, for American put options with early exercise. While considerable progress has been made, no completely satisfactory analytic solution has been found. As a result, people resort routinely either to numerical methods or to analytic approximations. There is a considerable literature in these fields; see, for example, [1–13, 17, 19–23, 25–27] and the references therein.

2.1. **The Black–Scholes Theory.** We denote the Black–Scholes operator by

$$\mathcal{L}^* P = \frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + (r-D) S \frac{\partial P}{\partial S} - rP.$$

Proposition 1 (Black-Scholes Theory)**.** *Consider the American put option with strike price E and expiry T, on a stock that pays dividends at a constant rate, $D \geqslant 0$, in a system where the bond and stock prices obey* (2.1)*, with positive constants σ and r. Let (B, P) be the classical solution of*

$$\begin{cases} \max\{\mathcal{L}^* P, (E-S)^+ - P\} = 0 & \text{in } (0,\infty)\times(-\infty, T), \\ P(S,T) = (E-S)^+ := \max\{E-S, 0\} & \text{on } (0,\infty)\times\{T\}, \\ B(t) := \inf\{S>0 \mid P(S,t) > (E-S)^+\} & \text{on } (-\infty, T]. \end{cases} \tag{2.2}$$

Then the no-arbitrage price of the option at time t is $P(\mathbf{S}_t, t)$; more precisely, the following holds:

1. *For any $t_0 < T$, there exists a self-financing portfolio starting at time t_0 with a value $\mathbf{\Pi}_{t_0} = P(\mathbf{S}_{t_0}, t_0)$ such that at any time $t \in (t_0, T]$ its value, $\mathbf{\Pi}_t$, is at least as large as $P(\mathbf{S}_t, t)$.*

   Consequently, if the option is sold at t_0 at a price, $\bar{p}$, higher than $P(\mathbf{S}_{t_0}, t_0)$, then the seller can make a profit, $\bar{p} - P(\mathbf{S}_{t_0}, t_0)$, at time t_0 and use $P(\mathbf{S}_{t_0}, t_0)$ to form the above self-financing portfolio, cashing it to pay the option obligation whenever the option is exercised, making an additional profit $\mathbf{\Pi}_\tau - (E - \mathbf{S}_\tau)^+$ at the exercise time, τ.
2. *Define the optimal exercise time, τ^*, by*

   $$\tau^* := \sup\{t \leqslant T \mid \mathbf{S}_s > B(s)\ \forall\, s \in [t_0, t)\}.$$

   Then $\tau^* \leqslant T$ and $\mathbf{\Pi}_{\tau^*} = (E - \mathbf{S}_{\tau^*})^+$.

   Consequently, if at time t_0 the option can be bought at a price, $\underline{p}$, lower than $P(\mathbf{S}_{t_0}, t_0)$, then the buyer can make the profit, $P(\mathbf{S}_{t_0}, t_0) - \underline{p}$, at time t_0 by short selling the above portfolio at time t_0 and clearing it at the optimal exercise time at which the payment, $(E - \mathbf{S}_{\tau^*})^+ = \mathbf{\Pi}_{\tau^*}$, from the option is just enough to clear the short position.

Proof. The self-financing portfolio is maintained in each time interval $(t, t+dt]$ with

$$\frac{\partial P(S,t)}{\partial S}\Big|_{S=\mathbf{S}_t} \text{ shares of stock, and the remainder, } \mathbf{\Pi}_t - \mathbf{S}_t \frac{P(\mathbf{S}_t, t)}{\partial S}, \text{ in the bond.}$$

The stochastic change, $d\mathbf{\Pi}_t$, of the value of the portfolio from t to $t+dt$ can be calculated by

$$d\mathbf{\Pi}_t = \frac{\partial P(\mathbf{S}_t, t)}{\partial S}\Big\{d\mathbf{S}_t + D\mathbf{S}_t\, dt\Big\} + \Big\{\mathbf{\Pi}_t - \mathbf{S}_t \frac{\partial P(\mathbf{S}_t, t)}{\partial S}\Big\} r dt.$$

Note that, by Itô's Lemma,

$$dP(\mathbf{S}_t, t) = \frac{\partial P(\mathbf{S}_t, t)}{\partial S}\, d\mathbf{S}_t + \Big\{\frac{\partial P(\mathbf{S}_t, t)}{\partial t} + \frac{\sigma^2 \mathbf{S}_t^2}{2} \frac{\partial P(\mathbf{S}_t, t)}{\partial S^2}\Big\} dt.$$

Taking the difference of $d\mathbf{\Pi}_t$ and $dP(\mathbf{S}_t, t)$ we find that

$$d[\mathbf{\Pi}_t - P(\mathbf{S}_t, t)] = r[\mathbf{\Pi}_t - P(\mathbf{S}_t, t)]dt - \mathcal{L}^* P(\mathbf{S}_t, t) dt \geqslant r[\mathbf{\Pi}_t - P(\mathbf{S}_t, t)]dt$$

by the variational inequality. Thus,

$$\mathbf{\Pi}_t - P(\mathbf{S}_t, t) \geqslant [\mathbf{\Pi}_{t_0} - P(\mathbf{S}_{t_0}, t_0)]e^{r[t-t_0]} = 0 \quad \forall\, t \in [t_0, T].$$

This completes the proof. □

2.2. **The Change of Variables.** It is mathematically convenient to use the dimensionless quantities:

$$x := \ln \frac{S}{E}, \quad s := \frac{\sigma^2}{2}(T-t), \quad k := \frac{2r}{\sigma^2}, \quad \ell := \frac{2D}{\sigma^2}, \quad \alpha := k - \ell - 1,$$
$$P(S,t) = E\, p(x,s) = E\, p\Big(\ln \frac{S}{E}, \frac{\sigma^2}{2}(T-t)\Big), \qquad B(t) = Ee^{b(s)}.$$

In a typical financial situation we might have $\sigma = 20\%$ $(\text{year}^{-1/2})$, $T - t = 0.5$ (year), with $\mathbf{S}_t$ fluctuating in $(40\%E, 250\%E)$. The resulting range of interest for the dimensionless variable (x,s) would be $x \in (-1,1), s \in [0, 0.01]$.

With

$$p_0(x) := \max\{1 - e^x, 0\}, \qquad \mathcal{L}p := p_{xx} + \alpha\, p_x - kp,$$

the variational problem for (B, P) is transformed to

$$\begin{aligned} &\max\{\mathcal{L}p - p_s, p_0 - p\} = 0 \ \text{ in } \ \mathbb{R} \times (0,\infty), \qquad p(\cdot, 0) = p_0; \\ &b(s) := \inf\{x \mid p(x,s) > p_0(x)\} \qquad \forall\, s > 0. \end{aligned} \tag{2.3}$$

We shall call $x = b(s)$ the free boundary and use the default extension $b(0) := \lim_{s \searrow 0} b(s)$.

2.3. **The Main Results.** The well-posedness, i.e., the existence of a unique solution (P, B) of (2.2) follows from the standard theory of variational inequalities; see for example, [10, 14]. Here we are concerned with the behavior of the free boundary (the optimal exercise boundary). We will give a self-contained proof, of the C^∞ regularity of the boundary, that focuses the presentation on the key ingredients in the case of the classical American put option.

Theorem 1. *There exists a unique solution of* (2.3). *In addition,* $b \in C^\infty((0,\infty)) \cap C([0,\infty))$.

As mentioned earlier, numerical experiments suggest that there is a breakdown of the convexity of the early exercise boundary as the dividend rate, D, increases past the risk-free interest rate, r. Figure 1 shows the loss of convexity as $\varepsilon := \ln(D/r) = \ln(\ell/k)$ crosses zero. These numerics were carried out using an integro-differential equation derived in §4. It is clear that at its onset when $0 < D - r \ll 1$, the non-convex region occurs close to expiry. On the other hand, the formal expansion of Wilmott et. al. [27, p121], for $s \searrow 0$,

$$b(s) = \ln \frac{r}{D} - [A + o(1)]\sqrt{s}, \quad A = 0.9034...,$$

does not capture this behavior. More specifically,

$$B''(t) = \frac{Ee^{b(s)}\sigma^4}{4}\Big[\ddot{b}(s) + (\dot{b}(s))^2\Big]$$

is positive for the above near-expiry asymptotics agreeing with the numerics in Figure 1 that the boundary begins convex. To observe the loss of the convexity, more precise estimates are required. In particular, we will prove

Theorem 2. *When $0 < D - r \ll 1$, the optimal exercise boundary is not convex. More precisely, when $\varepsilon := \ln(D/r) = \ln(\ell/k)$ is positive and sufficiently small, neither $S = B(t)$ nor $x = b(s)$ is convex. In particular, there exist a $\hat{t}$ for which $B''(\hat{t}) < 0$ and hence $\ddot{b}(\hat{s}) < 0$, where*

$$0 < \hat{s} \leqslant \frac{\varepsilon^2}{6|\ln \varepsilon|} \quad \textit{and} \quad \hat{t} = T - \frac{2\hat{s}}{\sigma^2}.$$

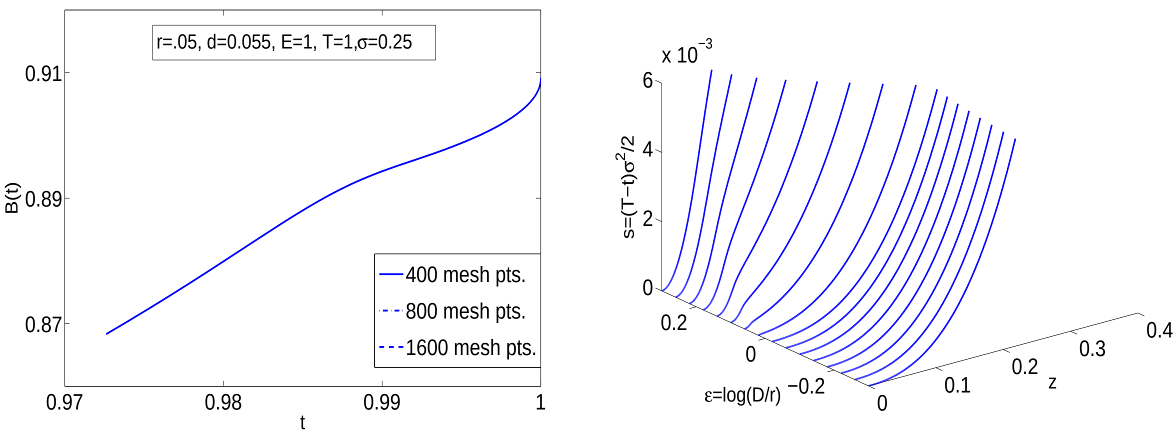

FIGURE 1. Loss of Convexity as $\varepsilon = \ln(D/r)$ crosses zero
The figure on the left, in the original S, t variables, shows that the optimal free boundary is not convex. Numerical accuracy is demonstrated by the overlap of the curves produced from 400,800, and 1600 mesh points. The figure on the right shows the variation of the free boundary as ε increases. Here $z = b(0) - b(t)$; for $\varepsilon < 0$, all free boundaries are convex. For ε positive and small, the free boundary loses its convexity near $s = 0$, i.e., near expiry.

The proof of Theorem 2 is given in §5. The intuition for the upper bound on $\hat{s}$, the location of the non-convexity, comes from a formal argument giving

$$\lim_{\varepsilon\to 0} \frac{\hat{s}|\ln\varepsilon|}{\varepsilon^2} = \frac{1}{8}.$$

See Figure 2.

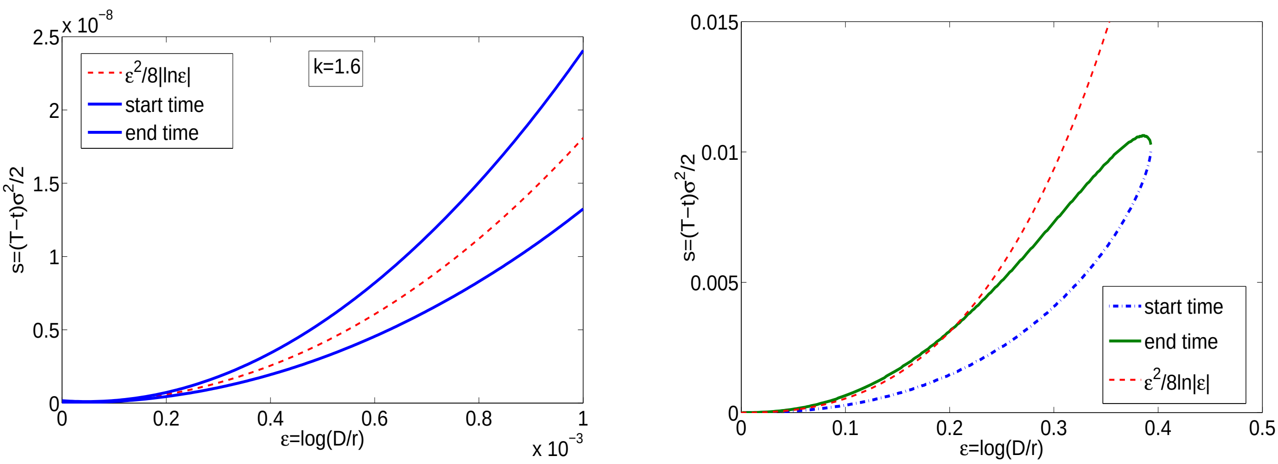

FIGURE 2. Time interval for which the free boundary is non-convex
The solid curves in the left figure are inflection points of the free boundary curves. The dashed curve is the analytic estimate for the location of the non-convexity. The figure on the right shows that when ε is large enough (> 0.4), the free boundary regains convexity, the teardrop region in the $\varepsilon - s$ domain indicating where the free boundary is concave.

In the course of this analysis, some of the estimates can be used to provide a rigorous proof of the previously mentioned near-expiry expansion [27].

Theorem 3. *Assume that $D > r$. Let $A = 0.903446597884....$ Then*

$$b(s) = \ln\frac{r}{D} - [A + o(1)]\sqrt{s}, \quad \dot{b}(s) = -\frac{A + o(1)}{2\sqrt{s}}, \quad \ddot{b}(s) = \frac{A + o(1)}{4s^{3/2}} \quad \forall\, s > 0$$

where $\lim_{s\searrow 0} o(1) = 0$. *Consequently, the inverse, $s = s(z)$, of $z = \ln(D/r) - b(s)$ satisfies*

$$s(z) = \frac{z^2}{A^2 + o(1)}, \quad s'(z) = \frac{2z}{A^2 + o(1)}, \qquad s''(z) = \frac{2}{A^2 + o(1)} \quad \forall\, z \geqslant 0.$$

Note that the above asymptotic behavior implies that very near expiry, the optimal exercise boundary begins convex. The location of the non-convex region occurs a little farther from expiry near $\hat{t}$ given in Theorem 2.

3. Regularity of The Free Boundary

3.1. **Basic Properties of the Solution.** Note that

$$\mathcal{L}p_0(x) = \delta(x) + (\ell e^x - k)H(-x)$$

where H is the Heaviside function, $H(z) = 1$ for $z > 0$ and $H(z) = 0$ for $z < 0$, and $\delta(x) = H'(x)$ is the Delta function. It is critical here that $\mathcal{L}p_0$ changes sign only once:

$$\mathcal{L}p_0 \geqslant 0 \ \text{ in } \ [b_0, \infty), \quad \mathcal{L}p_0 < 0 \ \text{ in } \ (-\infty, b_0); \qquad b_0 := \min\{0, \ln(k/\ell)\}.$$

This property implies that the free boundary in (2.3b) is well-defined. Indeed, for each $s > 0$,

$$p(x,s) > p_0(x) \quad \forall\, x > b(s), \qquad p(x,s) = p_0(x) \quad \forall\, x \leqslant b(s).$$

Later we shall show that $b \in C^\infty((0,\infty))$, from which we can derive the following: for each $s > 0$,

$$\begin{aligned}
p(b(s),s) &= 1 - e^{b(s)}, \\
p_x(b(s),s) &= -e^{b(s)}, \\
p_s(b(s),s) &= 0, \\
p_{xx}(b(s)\pm,s) &= -e^{b(s)} + (\tfrac{1}{2} \pm \tfrac{1}{2})(k - \ell e^{b(s)}), \\
p_{sx}(b(s)\pm,s) &= (\tfrac{1}{2} \pm \tfrac{1}{2})\dot{b}(s)(\ell e^{b(s)} - k).
\end{aligned}$$

The first and second equations are a direct consequence of $p(x,s) = p_0(x) = 1 - e^x$ for $x \leqslant b(s)$. The equation $p_s(b(s),s) = 0$ is derived by differentiating $p(b(s),s) = p_0(b(s))$ and using $p_x(b(s),s) = p_{0x}(b(s))$. The value $p_{xx}(b(s)+,s)$ is obtained from the differential equation $p_s = \mathcal{L}p$ and $p_s(b(s)\pm,s) = 0$. Similarly, differentiating $p_x(b(s)\pm,s) = p_{0x}(b(s))$ we obtain the expression for $p_{sx}(b(s)\pm,s)$.

To see the monotonicity of the free boundary b, it is useful to notice that $q := p_s$ satisfies the free boundary problem

$$\left\{\begin{array}{ll}
q_s(x,s) = \mathcal{L}q(x,s) & \forall\, x > b(s), s > 0, \\
q(x,s) = 0 & \forall\, x \leqslant b(s), s > 0, \\
q(x,0) = \max\{\mathcal{L}p_0(x), 0\} & \forall\, x \in \mathbb{R}, \quad s = 0, \\
\Pi(b(s)) = \int_0^s q_x(b(t),t)dt & \forall\, s \geqslant 0
\end{array}\right. \tag{3.1}$$

where

$$\Pi(z) = \int_\infty^z \min\{\mathcal{L}p_0(x), 0\}dx = \left\{\begin{array}{ll}
0 & \text{if } \ z \geqslant b_0, \\
\int_{b_0}^z (\ell e^x - k)dx & \text{if } \ z < b_0.
\end{array}\right.$$

Here the last equation in (3.1) is a weak formulation of the free boundary condition

$$b(0) = b_0, \quad \dot{b}(s)[\ell e^{b(s)} - k] = q_x(b(s),s), \quad b(s) \leqslant b_0 \quad \forall\, s > 0. \tag{3.2}$$

Formally, $p \geqslant p_0$ implies $q(\cdot,0) = p_s(\cdot,0) \geqslant 0$. Since $\mathcal{L}p_0(x) < 0$ when $x < b_0$, one readily sees that $b(0) \geqslant b_0$. In the case $b_0 = 0$, we have $q(\cdot,0) = \delta(x)$ so clearly $b(0) \leqslant b_0$. In the case $b_0 < 0$, we have $q(\cdot,0) > 0$ in $(b_0, 0]$ so we also have $b(0) \leqslant b_0$. Thus, we should have $b(0) = b_0$. We denote

$$Q_b = \{(x,s) \mid s > 0, x > b(s)\}.$$

Note that $q \geqslant 0$ on the parabolic boundary of Q_b, so by the maximum principle and Hopf's Lemma, $q > 0$ in Q_b and $q_x(b(s), s) > 0$ for each $s > 0$. Thus,

$$\dot{b}(s)[\ell e^{b(s)} - k] > 0, \quad \dot{b}(s) < 0, \quad \ell e^{b(s)} - k < 0 \qquad \forall\, s > 0.$$

We now turn to the proof of Theorem 1. Existence and uniqueness of weak solutions of (2.3) follows from the standard theory of variational inequalities (see [14] and [10] for more details). The key consideration here is the regularity. As remarked in [12], for the American put option, it is convenient to carry out the analysis in terms of the function $q := p_s$; i.e., using the free boundary problem (3.1), for (q, b). This is a Stefan type free boundary problem which has been well-studied (see, for example, [16, 18]). The existence of a smooth classical solution would be standard if the free boundary condition is not degenerate; i.e., the coefficient of $\dot{b}$ at $s = 0$ in (3.2) is not zero. We shall treat the degeneracy that arises here in (3.2) by using the initial value $b(0) = b_0 - \epsilon$ for positive ϵ and then sending $\epsilon \searrow 0$. (Note that this ϵ is unrelated to $\varepsilon := \ln(\ell/k)$).

3.2. **The Bootstrap Argument.** We begin with the bootstrap argument which shows that $b \in C^\infty((0,\infty))$ if $b \in C^\gamma((0,\infty))$ for some $\gamma > 1/2$.

For this, we denote $D = \{(x,t) \mid x \geqslant b(s), s > 0\}$. Suppose that $b \in C^{\beta/2}((0,\infty))$ for some $\beta > 1$ that is not an integer. Then by standard local regularity theory, e.g. the potential theory in [15], the solution of $q_s = \mathcal{L}q$ in Q_b with zero boundary value on $x = b(s)$ has the regularity $q \in C^{\beta,\beta/2}(D)$ and $q_x \in C^{\beta-1,(\beta-1)/2}(D)$. Consequently, $q_x(b(\cdot),\cdot) \in C^{(\beta-1)/2}((0,\infty))$. Since $\ell e^{b(s)} - k < 0$ for all $s > 0$, the last equation in (3.1) can be differentiated to give $\dot{b}[\ell e^b - k] = q_x(b,s)$, from which we conclude that $b \in C^{(\beta+1)/2}((0,\infty))$. Thus, by induction, $b \in C^\infty((0,\infty))$.

3.3. **An Upper Bound for q.** Let q_0 be the solution of

$$q_{0s} = \mathcal{L}q_0 \;\text{ on }\; \mathbb{R}\times(0,\infty), \qquad q_0(\cdot,0) = \max\{\mathcal{L}p_0, 0\}.$$

Then $q_0 > 0$ on $\mathbb{R}\times(0,\infty)$ so by comparison, $q \leqslant q_0$ on $\mathbb{R}\times[0,\infty)$.

For the subsequent analysis, we consider the function $\tilde{q}_0 := q_0(x,t)e^{\alpha x/2 + (k+\alpha^2/4)t}$ which satisfies $\tilde{q}_{0s} = \tilde{q}_{0xx}$ on $\mathbb{R}\times(0,\infty)$ with initial data $\delta(x) + e^{\alpha x/2}(\ell e^x - k)H(-x)H(x-b_0)$. Hence,

$$\tilde{q}_0(x,s) = \frac{e^{-x^2/(4s)}}{\sqrt{4\pi s}} + \int_{b_0}^{0} \frac{e^{-(x-y)^2/(4s)}}{\sqrt{4\pi s}} e^{\alpha y/2}(\ell e^y - k)\,dy.$$

Note that $\|\tilde{q}(x,\cdot)\|_{L^\infty([0,\infty))}$ is finite for every $x \neq 0$.

3.4. **Long Time Behavior.** Let $(p^*(\cdot), b^*) \in C^1(\mathbb{R})\times\mathbb{R}$ be the solution of

$$p^* = p_0 \;\text{ on }\; (-\infty, b^*], \qquad p^{*\prime\prime} + \alpha p^{*\prime} - kp^* = 0 \;\text{ in }\; (b^*,\infty), \qquad p^*(\infty) = 0. \tag{3.3}$$

The solution is given by

$$p^*(x) := \max\Big\{1 - e^x, \frac{e^{-\lambda(x-b^*)}}{1+\lambda}\Big\}, \quad b^* := \ln\frac{\lambda}{1+\lambda}, \quad \lambda := \frac{\alpha + \sqrt{\alpha^2 + 4k}}{2}.$$

One can use a comparison argument to show that $p \leqslant p^*, b^* \leqslant b$. Since $p_s \geqslant 0 > \dot{b}$, the $\lim_{t\to\infty}(p(\cdot,t), b(s))$ exists and must be the solution of (3.3). Hence,

$$p(x,s) \leqslant p^*(x), \quad b(0) > b(s) > b^*, \quad \lim_{t\to\infty}(p(x,t), b(t)) = (p^*(x), b^*) \qquad \forall\, x \in \mathbb{R}, s > 0.$$

3.5. **An Upper Bound for $|\dot b|$.** Let η be any positive constant such that $\eta \geqslant -b_0$. We define

$$
\begin{aligned}
M_0 &:= \max_{x\in[b_0,0]}\Big|\frac{d[e^{\alpha x/2}(\ell e^x-k)]}{dx}\Big|,\\
M(\eta) &:= \max\Big\{M_0,\ \sup_{t>0}\frac{\tilde q_0(-\eta/2,t)}{\eta/2}\Big\},\\
b^\eta(s) &:= \min\{-\eta, b(s)\},\\
q_1(x,s) &:= M(\eta)\ (x-b^\eta(s))e^{-\alpha x/2-(k+\alpha^2/4)s}.
\end{aligned}
\tag{3.4}
$$

Note that b^η is a continuous decreasing function. The following is the key to our analysis:

$$q_{1s}-\mathcal{L}q_1=-M(\eta)[x-b^\eta(s)]\dot b^\eta(s)e^{-\alpha x/2-(k+\alpha^2/4)s}\geqslant 0 \qquad \forall\, x\geqslant b^\eta(s), s\geqslant 0.$$

Now we compare q_1 with q on $\overline{Q}$ where $Q=\{(x,t)\mid b(s)<x<-\eta/2, s>0\}$. Since $q\leqslant q_0$ on $\mathbb{R}\times(0,\infty)$ and $q=0$ when $x\leqslant b(s)$, our choice of $M(\eta)$ implies that $q_1\geqslant q$ on the parabolic boundary of Q so $q<q_1$ in Q.

Suppose $b(s)\leqslant -\eta$. Then $q(b(s),s)=q_1(b(s),s)=0$ and $q(x,s)<q_1(x,s)$ for $x\in(b(s),-\eta/2]$. Hence,

$$0\leqslant q_x(b(s),s)\leqslant q_{1x}(b(s),s)=M(\eta)e^{-\alpha b(s)/2-(k+\alpha^2/4)s} \quad \text{if}\ \ b(s)\leqslant -\eta.$$

It then follows from the last equation in (3.1) or from (3.2) that

$$0\leqslant e^{\alpha b(s)/2}[\ell e^{b(s)}-k]\dot b(s)\leqslant M(\eta)e^{-[k+\alpha^2/4]s} \qquad \text{if}\ \ b(s)\leqslant -\eta. \tag{3.5}$$

As $b^*<b(s)<b_0$ for all $s>0$, for any $\delta>0$, by taking $\eta=-b(\delta)$, we see that b is Lipschitz continuous on $[\delta,\infty)$. Hence, b is locally Lipschitz on $(0,\infty)$, so by the bootstrap argument, we have $b\in C^\infty((0,\infty))$.

Remark 3.1. *When $\ell>k$, we have $-b_0=\varepsilon:=\ln(\ell/k)>0$. Hence, we can set $\eta=-b_0=\varepsilon$ in the above analysis and use $b^*<b\leqslant b_0$ to derive that there exists a constant $C>0$ such that*

$$
\begin{cases}
0<[b(s)-b_0]\dot b(s)\leqslant Ce^{-(k+\alpha^2/4)s} & \forall\, s>0,\\
0\leqslant q(x,s)\leqslant C[x-b(s)]e^{-(k+\alpha^2/4)s} & \forall\, s>0, x\in[b(s),-\varepsilon/2].
\end{cases}
\tag{3.6}
$$

3.6. **An Approximation.** Once we have the a priori estimate, the existence will be standard. There are many ways to do it. We find the following interesting. Let $\epsilon>0$ be a fixed small number. We remove the degeneracy of the free boundary condition by requiring $b(0)=b_0-\epsilon$. Hence, we consider the following problem, for (q^ϵ,b^ϵ):

$$
\begin{cases}
q^\epsilon_s(x,s)=\mathcal{L}q^\epsilon(x,s) & \forall\, x>b^\epsilon(s), s>0,\\
q^\epsilon(x,s)=0 & \forall\, x\leqslant b^\epsilon(s), s>0,\\
q^\epsilon(x,0)=\max\{\mathcal{L}p_0(x),0\} & \forall\, x\in\mathbb{R},\quad s=0,\\
\Pi(b^\epsilon(s))=\Pi(b_0-\epsilon)+\int_0^s q_x(b(t),t)dt & \forall\, s\geqslant 0.
\end{cases}
\tag{3.7}
$$

This is a standard Stefan problem, for which the existence of a classical solution can be proven as follows. We establish the existence of a solution in a time interval $[0,h]$ for an arbitrary large h. We define a function space

$$\mathbf{B}=\Big\{b\in C^1([0,h])\ \Big|\ b(0)=b_0-\epsilon;\ 0\leqslant e^{\alpha b/2}[\ell e^b-k]\dot b\leqslant M(\epsilon-b_0)\ \text{in}\ \ [0,h]\Big\}$$

where $M(\cdot)$ is defined in (3.4). Clearly, $\mathbf{B}$ is a closed subset of $C^1([0,h])$.

For each $b\in\mathbf{B}$, we let q be the solution of the following initial–boundary value problem

$$
\begin{cases}
q_s(x,s)=\mathcal{L}q(x,s) & \forall\, x>b(s), s\in(0,h],\\
q(b(s),s)=0 & \forall\, s>0,\\
q(x,0)=\max\{\mathcal{L}p_0(x),0\} & \forall\, x\geqslant b_0-\epsilon.
\end{cases}
$$

This problem admits a unique classical solution q. As the boundary is C^1, we see that $q \in \cap_{0<\beta<1} C^{2\beta,\beta}(\overline{Q_b} \setminus ([b_0, 0] \times \{0\}))$. This implies $q_x(b(\cdot), \cdot) \in \cap_{0<\beta<1/2} C^{\beta}([0, h])$.

From the solution, we define $\tilde{b} = \mathbf{T}[b]$ by solving the ode

$$e^{\alpha\tilde{b}(s)/2}[\ell e^{\tilde{b}(s)} - k]\frac{d\tilde{b}(s)}{ds} = q_x(b(s), s)e^{\alpha b(s)/2} \quad \forall\, s \in (0, h], \qquad \tilde{b}(0) = b_0 - \epsilon.$$

Since $q > 0$ in Q_b, we have $q_x(b(s), s) \geqslant 0$. Hence, $\tilde{b}$ is well-defined and $\tilde{b}(s) \leqslant b_0 - \epsilon$ for all $s \in [0, h]$. We want to show that $\mathbf{T}$ has a fixed point in $\mathbf{B}$. This fixed point is a solution of (3.7).

First of all, since $\dot{b} \leqslant 0$, following the same line of proof as in the previous subsection, we find that $0 \leqslant q \leqslant q_0$ and $0 \leqslant q_x(b(s), s) \leqslant M(\epsilon - b_0)e^{-\alpha b(s)/2 - (k+\alpha^2/4)s}$. Hence,

$$0 \leqslant e^{\alpha\tilde{b}/2}[\ell e^{\tilde{b}} - k]\dot{\tilde{b}} \leqslant M(\epsilon - b_0)e^{-(k+\alpha^2/4)s}.$$

Thus, $\mathbf{T}$ maps $\mathbf{B}$ to itself. In addition $\{\mathbf{T}[b] \mid b \in \mathbf{B}\}$ is a bounded set in $C^{1+1/4}([0, h])$ which is a compact subset of $C^1([0, h])$. Hence, by the Schauder's fixed point theorem, $\mathbf{T}$ admits a fixed point, b^ϵ, in $\mathbf{B}$. Extend the corresponding q by zero for $x < b^\epsilon$ and denoting it by q^ϵ, we see that (q^ϵ, b^ϵ) is a classical solution of (3.7) in $\mathbb{R} \times [0, h]$. Since $M(\epsilon - b_0)$ does not depend on h, we can let $h \to \infty$ to obtain a solution of (3.7) on $\mathbb{R} \times (0, \infty)$.

Remark 3.2. *When $\alpha > 0$, we may need extra care to prevent b^ϵ from being unbounded from below. Since we know the limit $b := \lim_{\epsilon \searrow 0} b^\epsilon$ is bounded from below by b^*, this technicality can be treated by replacing $e^{\alpha b/2}$ by $\max\{e^{\alpha b/2}, e^{\alpha b^*/2}\}$. We omit the details.*

3.7. **Monotonicity of the Approximation Sequence.** Let $0 < \epsilon_1 < \epsilon_2$. Then $b^{\epsilon_2}(0) < b^{\epsilon_1}(0)$. We claim that $b^{\epsilon_2} < b^{\epsilon_1}$ on $[0, \infty)$. Suppose this is not true. Then $t^* := \sup\{t > 0 \mid b^{\epsilon_2} < b^{\epsilon_1} \text{ in } [0, t]\}$ is finite and we have $b^{\epsilon_2} < b^{\epsilon_1}$ in $[0, t^*)$ and $b^{\epsilon_2}(t^*) = b^{\epsilon_1}(t^*)$.

Now by comparison on $D = \{(x, t) \mid x \geqslant b^{\epsilon_1}(s), s \in [0, t^*]\}$ we see that $q^{\epsilon_2} \geqslant q^{\epsilon_1}$ on D. Since b^{ϵ_1} is C^1, we obtain from Hopf's lemma that $q_x^{\epsilon_2}(b^{\epsilon_1}(t^*), t^*) > q_x^{\epsilon_1}(b^{\epsilon_1}(t^*), t^*)$. Consequently, since $b^{\epsilon_1}(t^*) = b^{\epsilon_2}(t^*)$, we find from the boundary condition that

$$-\dot{b}^{\epsilon_2}(t^*) = \frac{q_x^{\epsilon_2}(b^{\epsilon_1}(t^*), t^*)}{k - e^{b^{\epsilon_1}(t^*)}} > \frac{q_x^{\epsilon_1}(b^{\epsilon_1}(t^*), t^*)}{k - e^{b^{\epsilon_1}(t^*)}} = -\dot{b}^{\epsilon_1}(t^*).$$

That is $(b^{\epsilon_2} - b^{\epsilon_1})'|_{s=t^*} < 0$. But this implies $b^{\epsilon_2}(s) - b^{\epsilon_1}(s) > 0$ when $0 < t^* - s \ll 1$, contradicting the definition of t^*. Hence, we must have $b^{\epsilon_2} < b^{\epsilon_1}$ on $[0, \infty)$. Consequently, by comparison, we have $q^{\epsilon_1} < q^{\epsilon_2}$ in $Q_{b^{\epsilon_1}}$.

3.8. **The Limit of the Approximation Sequence.** Now we define $(q, b) = \lim_{\epsilon \searrow 0}(q^\epsilon, b^\epsilon)$. We shall show that (q, b) solves (3.1) and $b \in C^\infty((0, \infty)) \cap C([0, \infty))$.

First of all, $q \in C^\infty(Q_b)$ and $q_s = \mathcal{L}q$ in $Q_b := \{(x, t) \mid s > 0, x > b(s)\} = \cap_{\epsilon>0} Q_{b^\epsilon}$.

Next we establish the regularity of b. As the limit of a sequence of decreasing functions, b is also decreasing. Next we claim that $b(s) < b_0$ for every $s > 0$. Indeed, if this is not true, then we have $b(s) = b_0$ for all $s \in [0, \delta]$ for some $\delta > 0$. Note that for the ϵ problem, integrating $q_s^\varepsilon = \mathcal{L}q^\epsilon$ over Q_{b^ϵ} we have the following identity, for $t_2 > t_1 \geqslant 0$,

$$\int_{\mathbb{R}} q^\epsilon(x, t_2)dx - \int_{\mathbb{R}} q^\epsilon(x, t_1)dx + k\int_{t_1}^{t_2}\int_{\mathbb{R}} q^\epsilon(x, t)dxdt = -\int_{t_1}^{t_2} q_x^\epsilon(b^\epsilon(t), t)dt = \Pi(b^\epsilon(t_1)) - \Pi(b^\epsilon(t_2).$$

Sending $\epsilon \searrow 0$ and using Lebesgue's dominated theorem we obtain

$$\int_{\mathbb{R}} q(x, t_2) - \int_{\mathbb{R}} q(x, t_1)dx + k\int_{t_1}^{t_2}\int_{\mathbb{R}} q(x, t)dxdt = \Pi(b(t_1) - \Pi(b(t_2)) \quad \forall\, t_2 > t_1 \geqslant 0.$$

Now if $b \equiv b_0$ on $[0, \delta]$, we can integrate $q_s = \mathcal{L}q$ over $(0, \infty) \times (\delta/2, \delta)$ to derive

$$\int_{\delta/2}^{\delta} q_x(b_0, t)dt = \Pi(b(\delta) - \Pi(b(\delta/2)) = 0,$$

which is impossible since Hopf's maximum principle implies that $q_x(b_0,s)>0$ for each $s\in(0,\delta)$. In conclusion, $b(s)<b_0$ for every $s>0$.

Now let $\eta\geqslant -b_0$ be any small positive constant. Passing the estimate of $\dot b^\epsilon$ to the limit we find that (3.5) holds. This implies that b is Lipschitz continuous on $[\delta,\infty)$ where $\delta:=\inf\{s>0\mid b(s)<-\eta\}$. A bootstrap argument shows that $b\in C^\infty((\delta,\infty))$. Now we can use $(q^\varepsilon,b^\epsilon)\to(q,b)$ in the needed norm to conclude that $q(b(s),s)=0$ and $q_x(b(s),s)=\dot b[\ell e^{b(s)}-k]$ for all $s\in(\delta,\infty)$. Sending $\eta\searrow -b_0$, we must have $\delta\to 0$ since $b(s)<b_0$ for each $s>0$. Thus, $b\in C^\infty((0,\infty))$ and (q,s) is a solution of (3.1).

Finally, let $\delta^\epsilon:=\inf\{s>0\mid b^\epsilon(s)>b_0-2\epsilon\}$. Then $\delta^\epsilon\in(0,\infty]$ and $b(s)\geqslant b^\epsilon>b_0-2\epsilon$ for all $s\in(0,\delta^\epsilon)$. Hence, $\underline{\lim}_{s\searrow 0}b(s)\geqslant b_0-2\epsilon$. Sending $\epsilon\searrow 0$ we find $\underline{\lim}_{s\searrow 0}b(s)\geqslant b_0$. As $b(s)<b_0$ for $s>0$, we conclude that $\lim_{s\to 0}b(s)=b_0$, so $b\in C([0,\infty))$.

3.9. **Recovering p from q.** Once we have a classical solution of (q,b) of (3.1), we can define

$$p(x,s):=p_0(x)+\int_0^s q(x,t)dt \qquad \forall\, x\in\mathbb{R}, t\geqslant 0.$$

Since $q(x,t)=0$ for $x\leqslant b(t)$ and since $\dot b<0$, we see that $q=0$ on $(-\infty,b(s)]\times[0,s]$ for each $s>0$. Hence, $p(x,s)\equiv p_0(x)$ when $x\leqslant b(s)$ and $p_x(b(s),s)=p_{0x}(b(s))$ for each $s>0$. Also, $p_s=q$ on $\mathbb{R}\times(0,\infty)$. In addition, when $x\geqslant b_0$,

$$\mathcal{L}p(x,s)=\mathcal{L}p_0(x)+\int_0^s\mathcal{L}q(x,t)dt=\mathcal{L}p_0+\int_0^s q_t(s,t)dt=q(x,s),$$

since $q(\cdot,0)=\max\{\mathcal{L}p_0,0\}=\mathcal{L}p_0$ when $x\geqslant b_0$.

When $x\in(b(s),b_0)$, write $s=\hat s(x)$ the inverse of $x=b(s)$. Then

$$p(x,t)=p_0(x)+\int_{\hat s(x)}^s q(x,t)dt.$$

Consequently,

$$\begin{aligned}\mathcal{L}p(x,s) &= \mathcal{L}p_0(x)-\hat s'(x)q_x(x,\hat s(x))+\int_{\hat s(x)}^s\mathcal{L}q(x,t)dt\\ &= \mathcal{L}p_0(x)-\frac{1}{\dot b(\hat s(x))}q_x(x,\hat s(x))+q(x,s)=q(x,s)=p_s.\end{aligned}$$

Thus, (p,s) is a solution of the variational inequality (2.3).

Since the solution of variational inequality (2.3) is unique [10, 14], the assertion of Theorem 1 thus follows.

Remark 3.3. *Let $p^\epsilon:=p_0+\int_0^t q^\epsilon(x,t)dt$. Then (p^ϵ,b^ϵ) does not solve the original problem; one finds that $p^\epsilon_s-\mathcal{L}p^\epsilon=-\mathcal{L}p_0(x)$ in $(b_0-\epsilon,b_0)\times(0,\infty)$. Indeed, as $\epsilon\searrow 0, p^\epsilon\searrow p$ and $b^\epsilon\nearrow b$.*

4. Integral Formulation

Introduce $\phi(x,s)=p(x,s)-p_0(x)$. Then

$$\phi_s-\mathcal{L}\phi(x,s)=H(x-b(s))\,\mathcal{L}p_0(x) \quad\text{on}\quad \mathbb{R}\times(0,\infty),\qquad \phi(\cdot,0)=0\quad\text{on}\quad\mathbb{R}\times\{0\}.$$

Hence, we can use Green's formula to write

$$\phi(x,s)=\int_0^s\int_{b(s-t)}^\infty\Gamma(x-y,t)\mathcal{L}p_0(y)dydt\quad\forall\,(x,s)\in\mathbb{R}\times[0,\infty),$$

where Γ is the fundamental solution given by

$$\Gamma(x,s):=K(x+\alpha s,s)e^{-ks},\qquad K(z,t):=(4\pi t)^{-1/2}e^{-z^2/4t}.$$

We therefore obtain the following, for every $(x,s)\in\mathbb{R}\times(0,\infty)$,

$$\begin{aligned}
\phi(x,s) &= \int_0^s \Gamma(x,t)dt + \int_0^s\int_{b(s-t)}^0 [\ell e^y - k]\Gamma(x-y,t)dydt,\\
\phi_x(x,s) &= \int_0^s \Gamma_x(x,t)dt + \int_0^s\int_{b(s-t)}^0 [\ell e^y - k]\Gamma_x(x-y,t)dydt,\\
\phi_s(x,s) &= \Gamma(x,s) + \int_{b(0)}^0 [\ell e^y - k]\Gamma(x-y,s)dy - \int_0^s \dot b(s-t)[\ell e^{b(s-t)} - k]\Gamma(x-b(s-t),t)\,dt\\
&= \Gamma(x,s) + \int_{b_0}^0 [\ell e^y - k]\Gamma(x-y,s)dy - \int_0^s \dot b(t)[\ell e^{b(t)} - k]\Gamma(x-b(t),s-t)\,dt.
\end{aligned}$$

Also, for every $s>0$ and $x\neq b(s)$,

$$\phi_{sx}(x,s) = \Gamma_x(x,s) + \int_{b_0}^0 [\ell e^y - k]\Gamma_x(x-y,s)dy - \int_0^s \dot b(t)[\ell e^{b(t)} - k]\Gamma_x(x-b(t),s-t)\,dt.$$

Evaluating these expressions at $x=b(s)$ we then obtain the following.

Theorem 4. *Let (b,p) be the solution of the variational inequality (2.3). Then b satisfies the following integral identities:*

$$0 = \int_0^s \Gamma(b(s),t)\,dt + \int_0^s\int_{b(s-t)}^0 [\ell e^y - k]\Gamma(b(s)-y,t)dydt, \tag{4.1}$$

$$0 = \int_0^s \Gamma_x(b(s),t)dt + \int_0^s\int_{b(s-t)}^0 [\ell e^y - k]\Gamma_x(b(s)-y,t)dydt, \tag{4.2}$$

$$\begin{aligned}
0 &= \Gamma(b(s),s) + \int_{b_0}^0 [\ell e^y - k]\Gamma(b(s)-y,s)dy\\
&\quad - \int_0^s \dot b(t)[\ell e^{b(t)} - k]\Gamma(b(s)-b(t),s-t)\,dt,
\end{aligned}\tag{4.3}$$

$$\begin{aligned}
\dot b(s)[\ell e^{b(s)} - k] &= 2\Gamma_x(b(s),s) + 2\int_{b_0}^0 [\ell e^y - k]\Gamma_x(b(s)-y,s)dy\\
&\quad -2\int_0^t \dot b(t)[\ell e^{b(t)} - k]\Gamma_x(b(s)-b(t),s-t)\,dt.
\end{aligned}\tag{4.4}$$

Also, for any $\theta\in\mathbb{R}$,

$$\begin{aligned}
\dot b(s)[\ell e^{b(s)} - k] &= \Big(\theta - \frac{b(s)}{s}\Big)\Gamma(b(s),s) + \int_{b_0}^0 [\ell e^y - k]\Big(\theta - \frac{b(s)-y}{s}\Big)\Gamma(b(s)-y,s)dy\\
&\quad - \int_0^s \dot b(t)[\ell e^{b(t)} - k]\Big(\theta - \frac{b(s)-b(t)}{s-t}\Big)\Gamma(b(s)-b(t),s-t)\,dt\,.
\end{aligned}\tag{4.5}$$

With the particular choice of $\theta=0$ and $\theta=[b(s)-b_0]/s$ we have

$$\begin{aligned}
\dot b(s)[\ell e^{b(s)} - k] = \frac{-b(s)\Gamma(b(s),s)}{s} &+ \frac{1}{s}\int_{b_0}^0 [\ell e^y - k][y-b(s)]\Gamma(b(s)-y,s)dy\\
&+ \int_0^s \dot b(t)[e^{b(t)-b_0} - 1]\,\frac{b(s)-b(t)}{s-t}\,\Gamma(b(s)-b(t),s-t)\,dt,
\end{aligned}\tag{4.6}$$

$$\begin{aligned}
\dot b(s)[\ell e^{b(s)} - k] = \frac{-b_0\Gamma(b(s),s)}{s} &+ \frac{1}{s}\int_{b_0}^0 [\ell e^y - k][y-b_0]\Gamma(b(s)-y,s)dy\\
&- \int_0^s \dot b(t)[\ell e^{b(t)} - k]\Big(\frac{b(s)-b_0}{s} - \frac{b(s)-b(t)}{s-t}\Big)\Gamma(b(s)-b(t),s-t)\,dt.
\end{aligned}\tag{4.7}$$

Here (4.5) is obtained by adding a $\theta+\alpha$ multiple of (4.3) to (4.4) and using

$$\Gamma_x(x,s)=\frac{-x-\alpha s}{2s}\,\Gamma(x,s).$$

We shall use (4.6) and (4.7) for our analysis. For numerical simulation we use (4.5) with $\theta=(b(s)-b_0)/(2s)$, since a formal asymptotic expansion suggests that for $\ell>k$

$$b(s)-b_0\propto\sqrt{s},\quad \dot b(s)\sim\frac{b(s)-b_0}{2s}\quad\text{when}\ \ 0<s\ll 1.$$

Hence, taking $\theta=(b(s)-b_0)/(2s)$ in (4.5) removes the singularity of the last integral in (4.5) in a natural way.

5. Non-Convexity

From now on we assume that $\ell>k$. We shall prove Theorem 2.

5.1. **The Idea.** It is convenient to visualize graphs in the first quadrant. Hence, we introduce

$$z(s)=b_0-b(s)=-\varepsilon-b(s)\qquad\forall\, s\geqslant 0.$$

Since $z'(s)>0$ for all $s>0$, $z=z(s)$ admits an inverse, which we denote by $s=s(z)$.

We can rewrite (4.6) and (4.7) as follows. Note that $\ell e^y-k=k[e^{y-b_0}-1]$. Hence we can use the change of variable $\tilde y=y-b_0$ in the first integral and the change of variable $y=b_0-b(t)$ in the second integral in (4.6) and (4.7) to obtain

$$[1-e^{-z}]\,\frac{dz}{ds}\quad=I_1+I_2-I_3=J_1+J_2+J_3 \tag{5.1}$$

where

$$\begin{aligned}
I_1 &:= \frac{(\varepsilon+z)\Gamma(-\varepsilon-z,s)}{k\,s},\\
I_2 &:= \frac{1}{s}\int_0^{\varepsilon}[e^y-1][z+y]\Gamma(-y-z,s)dy,\\
I_3 &:= \int_0^z[1-e^{-y}]\Big(\frac{z-y}{s(z)-s(y)}\Big)\Gamma(y-z,s(z)-s(y))\,dy,\\
J_1 &:= \frac{\varepsilon\,\Gamma(-\varepsilon-z,s)}{k\,s},\\
J_2 &:= \frac{1}{s}\int_0^{\varepsilon}y[e^y-1]\Gamma(-z-y,s)dy,\\
J_3 &:= \int_0^z[1-e^{-y}]\Big(\frac{z}{s(z)}-\frac{z-y}{s(z)-s(y)}\Big)\Gamma(y-z,s(z)-s(y))\,dy.
\end{aligned}$$

Since the coefficient of dz/ds at $s=0$ is zero, it may be better to consider the inverse function, which is smooth on $[0,\infty)$. Hence we write (5.1) as

$$\frac{ds}{dz}=\frac{1-e^{-z}}{I_1+I_2-I_3}=\frac{1-e^{-z}}{J_1+J_2+J_3}\ . \tag{5.2}$$

Recall that

$$B(t)=Ee^{b(s)}=Ee^{-\varepsilon-z(s)}\Big|_{s=\frac{\sigma^2}{2}(T-t)}.$$

It follows that

$$\begin{aligned}\frac{4e^{\varepsilon}}{\sigma^4 E}\frac{d^2B(t)}{d^2t} &= \frac{d^2e^{-z(s)}}{ds^2} = -\frac{dz}{ds}\frac{d}{dz}\Big(\frac{1}{e^z\frac{ds}{dz}}\Big)\\ &= e^{-2z}\Big(\frac{dz}{ds}\Big)^3\frac{d}{dz}\Big(e^z\frac{ds}{dz}\Big)\\ &= e^{-z}\Big(\frac{dz}{ds}\Big)^3\Big(\frac{d^2s}{dz^2}+\frac{ds}{dz}\Big).\end{aligned}$$

Thus, to show that B is not convex ($B''(t)\geqslant 0$ not true), we need only show that $e^z ds/dz$ is not an increasing function. For this, we compare the values of $e^z ds/dz$ at two points:

$$z_1 := z(s_1),\ s_1 := \frac{\varepsilon^2}{[8+\nu]\,|\ln\varepsilon|},\qquad z_2 := z(s_2),\ s_2 := \frac{\varepsilon^2}{[8-\nu]\,|\ln\varepsilon|} \tag{5.3}$$

where ν can be any constant in $(0,8)$. For definiteness, we fix $\nu=2$.

For convenience, we introduce

$$z^* := \sup\Big\{z>0\ \Big|\ \frac{d^2s}{dz^2}+\frac{ds}{dz}\geqslant 0\ \text{ in }\ (0,z]\Big\},\qquad s^* := s(z^*). \tag{5.4}$$

Note that the optimal exercise boundary is convex if and only if $s^*=\infty$. In the sequel, we shall show that when ε is small, $s^*<s_2$; that is, the convexity changes at time before s_2.

We remark that the convexity of $s(\cdot)$ implies the convexity of $B(\cdot)$ since B'' is proportional to $s''+s'$ with $s'>0$. Hence, if $B(\cdot)$ is not convex, then neither is $s(\cdot)$.

In the sequel, $\varepsilon=\ln(D/r)=\ln(\ell/k)$ is a small positive constant. We use the standard notation $o(1)$ to denote a generic small quantity that approaches zero as $\varepsilon\searrow 0$.

5.2. **The General Lower Bound of ds/dz.** We shall utilize the first equation in (5.2) to estimate the positive lower bound of ds/dz. The key here is that all terms I_1, I_2 and I_3 are positive, so we have the basic estimate

$$(1-e^{-z})\frac{dz}{ds}\leqslant I_1+I_2,\qquad \frac{ds}{dz}\geqslant\frac{1-e^{-z}}{I_1+I_2}.$$

We can estimate I_2 as follows. Notice that $\alpha=k-\ell-1<-1$. Hence, when $0\leqslant y\leqslant z$,

$$\Gamma(-y-z,s) = \frac{e^{-(y+z)^2/(4s)+[y+z]\alpha/2-(k+\alpha^2/4)s}}{\sqrt{4\pi s}}\leqslant\frac{e^{-(y+z)^2/(4s)-y-(k+\alpha^2/4)s}}{\sqrt{4\pi s}}.$$

It then follows, using $0\leqslant e^y-1\leqslant ye^y\leqslant\frac12(y+z)e^y$, that

$$\begin{aligned}I_2 &:= \frac1s\int_0^{\varepsilon}[e^y-1][z+y]\Gamma(-y-z,s)dy\\ &\leqslant \frac{e^{-(k+\alpha^2/4)s}}{4\sqrt{\pi s^3}}\int_0^{\varepsilon}(y+z)^2e^{-(y+z)^2/(4s)}dy=\frac{e^{-(k+\alpha^2/4)s}}{4\sqrt{\pi}}\int_{z/\sqrt{s}}^{(z+\varepsilon)/\sqrt{s}}\eta^2e^{-\eta^2/4}d\eta\\ &\leqslant \frac{e^{-(k+\alpha^2/4)s}}{4\sqrt{\pi}}\int_0^{\infty}\eta^2e^{-\eta^2/4}d\eta=\frac{e^{-(k+\alpha^2/4)s}}{2}\leqslant\frac12\qquad\forall\, s>0.\end{aligned}$$

Next, we consider

$$I_1 := \frac{(\varepsilon+z)\Gamma(-\varepsilon-z,s)}{k\,s}=\frac{(\varepsilon+z)e^{-(\varepsilon+z-\alpha s)^2/(4s)-ks}}{\sqrt{4\pi k^2s^3}}.$$

Since $\alpha<0$, we see that

$$\frac{\partial I_1}{\partial z}=\frac{I_1}{\varepsilon+z}\Big\{1-\frac{(\varepsilon+z)(\varepsilon+z-\alpha s)}{2s}\Big\}<0\quad\text{when } s<\frac{\varepsilon^2}{2}.$$

It then follows that when $s\in(0,\varepsilon^2/2]$,

$$I_1\leqslant\frac{\varepsilon\,\Gamma(-\varepsilon,s)}{k\,s}\leqslant\frac{\varepsilon\, e^{-\varepsilon^2/(4s)}}{\sqrt{4\pi k^2s^3}}.$$

Combing these estimates, we then obtain

$$\frac{ds}{dz}=\frac{1-e^{-z}}{I_1+I_2-I_3}\geqslant\frac{1-e^{-z}}{I_1+I_2}\geqslant\frac{z\,e^{-z}}{\frac{\varepsilon\,e^{-\varepsilon^2/(4s)}}{\sqrt{4\pi k^2 s^3}}+\frac{1}{2}}\quad\forall\, s\in\Big[0,\frac{\varepsilon^2}{2}\Big].\tag{5.5}$$

5.3. **The Lower Bound of ds/dz on $[0,s_2^*]$ where $s_2^*=\min\{s_2,s^*\}$.**

(1) First we consider $s\in[0,s_1]$ where $s_1:=\varepsilon^2/(10|\ln\varepsilon|)$, $0<\varepsilon\ll 1$. It is easy to verify that $I_1=o(1)$ when $s\in(0,s_1]$, so that

$$\frac{ds}{dz}\geqslant\frac{ze^{-z}}{o(1)+\frac{1}{2}},\qquad s\geqslant\frac{z^2e^{-z}}{1+o(1)}\qquad\forall\, z\in[0,z_1].\tag{5.6}$$

This also implies that when $s\in[0,s_1]$, $z(s)=o(\varepsilon)$. In particular, $z_1:=z(s_1)=o(\varepsilon)$.

To extend the estimate beyond s_1, we notice that $s''+s'\geqslant 0$ implies that $(e^z s')'\geqslant 0$ so integrating it over $[x,x+h]\subset(0,z^*]$ we obtain

$$s'(x+h)\geqslant e^{-h}s'(x)\quad\forall\, 0<x<x+h\leqslant z^*.\tag{5.7}$$

(2) With $z_1:=z(s_1)$, $s^*:=s(z^*)$, $s_2^*:=\min\{s_2,s^*\}$, and $z_2^*:=z(s_2^*)$, we claim that $z_2^*<2z_1$. Suppose not. Then we take $h=z_1$ and integrate (5.7) over $[0,z_1]$ to obtain $s(2z_1)-s(z_1)\geqslant e^{-z_1}s(z_1)$, so $s(2z_1)\geqslant[1+e^{-z_1}]s(z_1)=[2-o(\varepsilon)]s(z_1)$. However, since $s_2=\frac{5}{3}s_1$, we obtain $s(2z_1)\geqslant[2-o(\varepsilon)]s_1>s_2\geqslant s(z_2^*)$, contradicting the assumption $z_2^*\geqslant 2z_1$. Hence, we must have $z_2^*<2z_1$.

(3) We now consider the lower bound of ds/dz in $[s_1,s_2^*]$. From (5.7) and (5.6) we derive, for any $z\in[z_1,z_2^*]\subset[z_1,2z_1]$,

$$\frac{ds(z)}{dz}\geqslant e^{-z_2^*}\frac{ds(z_1)}{dz}\geqslant\frac{e^{-z_2^*}z_1e^{-z_1}}{o(1)+\frac{1}{2}}=\frac{e^{-z_1-z_2^*}z_1}{[o(1)+\frac{1}{2}]z}\,z\geqslant\frac{z}{2}.$$

In conclusion, when ε is positive and sufficiently small,

$$\frac{ds(z)}{dz}\geqslant\frac{z}{2},\quad s(z)\geqslant\frac{z^2}{4},\quad s(z)-s(y)\geqslant\frac{z^2-y^2}{4}\quad\forall\, 0\leqslant y\leqslant z\leqslant z_2^*.\tag{5.8}$$

5.4. **Upper Bounds of ds/dz.** The optimal exercising boundary is convex if and only if $s^*=\infty$. In what follows, we shall show that when ε is small positive, $s^*<s_2$. To do this, we show that the value $e^z ds/dz$ at z_2 is much smaller than at z_1, so it cannot be an increasing function and therefore the free boundary cannot be convex. In (5.8), we already have a lower bound of $e^z dz/ds$ at z_1 if $s^*\geqslant s_1$ (if $s^*<s_1$, the non-convexity is established). Here what we need is an upper bound at $z=z_2$. The basic idea is to use the second equation in (5.2).

5.4.1. *Estimate of J_3.* In the case $s''\geqslant 0$, it is easy to show that J_3 is positive. Under the weaker condition $s''+s'\geqslant 0$ in $(0,s^*]$, we shall show that J_3 is almost positive. For this purpose, we write

$$J_3=\int_0^z[1-e^{-y}]R(z,y)\Gamma(y-z,s(z)-s(y))\,dy$$

where

$$\begin{aligned}R(z,y)\;:=\;&\frac{z}{s(z)}-\frac{z-y}{s(z)-s(y)}=\frac{ys(z)-zs(y)}{s(z)[s(z)-s(y)]}\\=\;&\frac{zy}{s(z)[s(z)-s(y)]}\Big(\frac{s(z)}{z}-\frac{s(y)}{y}\Big)\\=\;&\frac{zy}{s(z)[s(z)-s(y)]}\int_y^z\frac{s'(x)x-s(x)}{x^2}dx\\=\;&\frac{zy}{s(z)[s(z)-s(y)]}\int_y^z\frac{\int_0^x\hat{x}s''(\hat{x})d\hat{x}}{x^2}dx.\end{aligned}$$

Since $s''+s'\geqslant 0$ in $(0,z^*]$, when $\hat{x}\in(0,z^*]$, we have $s''(\hat{x})\geqslant -s'(\hat{x})$. Also $s'(\hat{x})\leqslant e^{x-\hat{x}}s'(x)\leqslant e^z s'(x)$ when $0<\hat{x}\leqslant x\leqslant z\leqslant z^*$. Hence $s''(\hat{x})\geqslant -s'(\hat{x})\geqslant -e^z s'(x)$. Thus, when $0<y<z<z_2^*$,

$$R(z,y)\geqslant -\frac{zy}{s(z)[s(z)-s(y)]}\int_y^z \frac{\int_0^x \hat{x}e^z s'(x)d\hat{x}}{x^2}dx=-\frac{zye^z}{2\,s(z)}.$$

Next using (5.8) we derive that

$$\Gamma(y-z,s(z)-s(y))<\frac{1}{\sqrt{4\pi[s(z)-s(y)]}}\leqslant\frac{1}{\sqrt{\pi[z^2-y^2]}}.$$

Hence,

$$\begin{aligned} J_3 &= \int_0^z [1-e^{-y}]R(z,y)\Gamma(y-z,s(z)-s(y))\,dy \\ &\geqslant -\int_0^z y\,\frac{zye^z}{2\,s(z)}\,\frac{1}{\sqrt{\pi(z^2-y^2)}}dy=-\frac{\sqrt{\pi}z^3e^z}{8\,s(z)}\geqslant -z \quad \forall\, z\in(0,z_2^*]. \end{aligned}$$

5.4.2. *Estimate of* J_2. One can estimate the lower bound of J_2 as follows:

$$\begin{aligned} J_2=\frac{1}{s}\int_0^\varepsilon y[e^y-1]\Gamma(-z-y,s)dy &\geqslant \frac{e^{\alpha z-(k+\alpha^2/4)s}}{\sqrt{4\pi s^3}}\int_0^\varepsilon y^2e^{-(z+y)^2/(4s)}dy \\ &= \frac{e^{\alpha z-(k+\alpha^2/4)s}}{\sqrt{4\pi}}\int_{z/\sqrt{s}}^{(z+\varepsilon)/\sqrt{s}} e^{-\eta^2/4}d\eta. \end{aligned}$$

In view of (5.8), we see that when $s\in(0,s_2^*]$, we have

$$J_2\geqslant \frac{e^{o(\varepsilon)}}{\sqrt{4\pi}}\int_2^{\varepsilon/\sqrt{s}} e^{-\eta^2/4}d\eta\geqslant\frac{1}{4}\int_2^3 e^{-\eta^2/4}d\eta:=c>0.$$

It then follows that

$$J_2+J_3\geqslant c-z>0\quad \forall\, s\in(0,s_2^*]\quad (\Leftrightarrow\ \forall\, z\in(0,z_2^*]).$$

5.5. **Completion of the Proof of Theorem 2.** Now suppose $z^*>z_2$. Then $z_2^*=z_2$ and $s_2^*=s_2$. Hence we can use the second equation in (5.2) to conclude that

$$\begin{aligned} \frac{1}{z}\frac{ds(z)}{dz}\Big|_{z=z_2} &= \frac{1-e^{-z}}{z}\frac{1}{J_1+J_2+J_3}\leqslant\frac{1}{J_1} \\ &= \frac{k}{\varepsilon}\sqrt{4\pi s^3}e^{(\varepsilon+z(s)-\alpha s)^2/(4s)+ks}\Big|_{s=\frac{\varepsilon^2}{-6\ln\varepsilon}}\leqslant\varepsilon^{1/4}. \end{aligned}$$

In comparing with (5.8), we see that

$$e^{z_2}\frac{ds(z_2)}{dz}-e^{z_1}\frac{ds(z_1)}{dz}\leqslant z_2e^{z_2}\varepsilon^{1/4}-\frac{1}{2}e^{z_1}z_1\leqslant 2z_1e^{2z_1}\varepsilon^{1/4}-\frac{e^{z_1}z_1}{2}<0.$$

This implies that $e^z ds(z)/dz$ is not an increasing function on $[z_1,z_2]$. Consequently, we must have $z^*<z_2$. This completes the proof of Theorem 2.

6. Expansion Near Expiry

6.1. **A Formal Expansion.** In [27], the ideas were presented for obtaining a formal expansion for the leading order behavior, as $s\searrow 0$. Here we extend these ideas to a full expansion. Recall that $\phi:=p-p_0$ satisfies $\phi_s-\mathcal{L}\phi=\delta(x)+k[e^{x-b_0}-1]H(x-b(t))H(-x)$, with zero initial value. When $\varepsilon=-b_0=\ln(\ell/k)>0$, the Delta function will not interfere much with the solution near $(b_0,0)$. Hence, we expect the following expansion

$$\phi(x,s)=\Phi(\xi,s)\Big|_{\xi=\frac{x-b(0)}{\sqrt{s}}},\quad \Phi(\xi,s)\sim s\sum_{n=1}^\infty \phi_n(\xi)s^{n/2},\quad b(s)\sim b_0+\sum_{n=1}^\infty A_ns^{n/2}.$$

Note that Φ satisfies

$$\Phi_s - \frac{\xi}{2s}\Phi_\xi - \frac{1}{s}\Phi_{\xi\xi} = \frac{\alpha}{\sqrt{s}}\Phi_\xi - k\Phi + k\sum_{n=1}^{\infty}\frac{\xi^n s^{n/2}}{n!} \quad \forall\,\xi \in \Big(\frac{b(s)+\varepsilon}{\sqrt{s}}, \frac{\varepsilon}{\sqrt{s}}\Big),\ s>0.$$

This leads to the following

$$\mathcal{L}_n\phi_n = \frac{k\,\xi^n}{n!} + \alpha\phi_{n-1} - k\phi_{n-2} \qquad \forall\,\xi \in (A_1,\infty)$$

where

$$\mathcal{L}_n\psi := \Big(1 + \frac{n}{2} - \frac{\xi}{2}\frac{d}{d\xi} - \frac{d^2}{d\xi^2}\Big)\psi.$$

Here we have used the extension $\phi_0 = \phi_{-1} \equiv 0$.

The boundary conditions for ϕ_n and the unknown A_n will be derived from

$$0 = \phi(b(s),s) = \Phi(\xi(s),s), \quad 0 = \phi_x(b(s),s) = s^{-1/2}\Phi_\xi(\xi(s),s),$$

$$\xi(s) = [b(s)-b_0]s^{-1/2} \sim A_1 + \sum_{n=2}^{\infty} A_n s^{n-1/2}.$$

Using the asymptotic expansion, we have

$$\begin{aligned}
0 &\sim \sum_{n=1}^{\infty} s^{n/2}\phi_n\Big(A_1 + \sum_{m=2}^{\infty} A_m s^{(m-1)/2}\Big)\\
&\sim \sum_{n=1}^{\infty} s^{n/2}\sum_{i=0}^{\infty}\frac{\phi_n^{(i)}(A_1)}{i!}\Big(\sum_{m=2}^{\infty} A_m s^{(m-1)/2}\Big)^i\\
&\sim \phi_1(A_1)s^{1/2} + [\phi_2(A_1) + \phi_1'(A_1)A_2]s + \sum_{n=3}^{\infty}[\phi_n(A_1) + \phi_1'(A_1)A_n + \cdots]s^{n/2}\\
0 &\sim \sum_{n=1}^{\infty} s^{n/2}\phi_n'\Big(A_1 + \sum_{m=2}^{\infty} A_m s^{(m-1)/2}\Big)\\
&\sim \phi_1'(A_1)s^{1/2} + [\phi_2'(A_1) + \phi_1''(A_1)A_2]s + \sum_{n=3}^{\infty}[\phi_n'(A_1) + \phi_1''(A_1)A_n + \cdots]s^{n/2}.
\end{aligned}$$

Hence, we obtain the boundary conditions and the free boundary conditions

$$\begin{array}{lll}
\phi_1(A_1) = 0, & \phi_1'(A_1) = 0, & \phi_1(\xi) = O(\xi) \text{ as } \xi\to\infty,\\
\phi_2(A_1) = 0, & \phi_2'(A_1) + \phi_1''(A_1)A_2 = 0, & \phi_2(\xi) = O(\xi^2) \text{ as } \xi\to\infty,\\
\phi_n(A_1) = a_{n-1}, & \phi_n'(A_1) + \phi_1''(A_1)A_n = b_{n-1}, & \phi_n(\xi) = O(\xi^n) \text{ as } \xi\to\infty
\end{array}$$

where a_m, b_m are constants depending only expansions of order up to m.

For the homogeneous equation $\mathcal{L}_n\psi = 0$, one can verify that the following are two linear independent solutions:

$$\psi_n(\xi) = \int_\xi^\infty (\eta-\xi)^{n+2}e^{-\eta^2/4}d\eta, \quad \tilde\psi_n(\xi) = \int_{\mathbb{R}}(\eta-\xi)^{n+2}e^{-\eta^2/4}d\eta.$$

Here $\tilde\psi_n$ is a polynomial of degree $n+2$. It is easy to verify that

$$\phi_1(\xi) = k\Big\{\xi - \frac{A_1\psi_1(\xi)}{\psi_1(A_1)}\Big\}$$

where A_1 is the solution of the transcendental equation

$$\int_{A_1}^\infty (\eta-A_1)^2(\eta+2A_1)e^{-\eta^2/4}d\eta = 0 \quad\Rightarrow\quad A_1 = -0.9034465978843... \tag{6.1}$$

6.2. **An Alternative Formal Derivation.** We can also use the first or second equation in (5.2) to derive the asymptotic behavior. Assume that $0 < s < \varepsilon^3$. Also assume that $z(s) = [A + O(\sqrt{s})]\sqrt{s}$. Then one finds $J_1 = O(e^{-\varepsilon^2/(4s)})$ can be neglected in the expansion for small s. Also,

$$J_2 = \frac{1}{\sqrt{4\pi}} \int_A^\infty (A-\eta)^2 e^{-\eta^2/4} d\eta + O(z).$$

For J_3, one uses the change of variable

$$\eta = \frac{(z-y)}{\sqrt{s(z) - s(y)}} \approx A\sqrt{\frac{z-y}{z+y}}.$$

Then one derives that

$$J_3 = \frac{A^2}{\sqrt{4\pi}} \int_0^A \Big(\frac{A^2 - \eta^2}{A^2 + \eta^2}\Big)^2 e^{-\eta^2/4} d\eta + O(z).$$

Thus the differential equation (5.2) gives the following equation for A:

$$A^2 = \frac{1}{\sqrt{\pi}} \int_A^\infty (A-\eta)^2 e^{-\eta^2/4} d\eta + \frac{A^2}{\sqrt{\pi}} \int_0^A \Big(\frac{A^2 - \eta^2}{A^2 + \zeta^2}\Big)^2 e^{-\eta^2/4} d\eta$$

$$\Longrightarrow \qquad A = 0.9034465978843....$$

Numerically it is evident that $A = -A_1$ and is precisely the value obtained in [27].

6.3. **Proof of Theorem 3.** The leading order expansion can be made rigorous. Instead of considering the premium $\phi = p - p_0$, we consider the rate, $q = p_s = \phi_s$, of the premium change. We study the blow-up family $\{q^L, b^L\}_{L>0}$ defined by

$$q^L(x,s) := L\, q(b_0 + L^{-1}x, L^{-2}s), \qquad b^L(s) = L\, [b(L^{-2}s) - b_0\,].$$

Information obtained from b^L will be cycled back to b through the identity

$$b(s\theta) = b_0 + \sqrt{s}\; b^L(\theta)\Big|_{L=1/\sqrt{s}} \qquad \forall\, \theta \in \Big[\frac{1}{2}, 2\Big], s > 0. \tag{6.2}$$

We shall show that $\lim_{L\to\infty} b^L(\theta) = A_1\sqrt{\theta}$ in $C^2([\delta, h])$ for any $h > \delta > 0$.

For each $L > 0$, (q^L, b^L) satisfies

$$\begin{cases} q^L_s = q^L_{xx} + \alpha L^{-1} q^L_x - kL^{-2} q^L & \forall\, x > b^L(s),\ s > 0, \\ q^L(x,s) = 0 & \forall\, x \leqslant b^L(s),\ s > 0, \\ kL[e^{L^{-1}b^L(s)} - 1]\dot b^L(s) = q^L_x(b^L(s), s) & \forall\, s > 0, \\ q^L(x, 0) = \delta(x - \varepsilon L) + k \max\{\frac{e^{x/L} - 1}{L}, 0\} & \forall\, x \in \mathbb{R}. \end{cases}$$

From (3.6) we derive that

$$0 < b^L(s)\, \dot b^L(s) \leqslant C, \quad 0 > b^L(s) \geqslant -\sqrt{2Cs} \qquad \forall\, s > 0,$$

$$\begin{aligned} 0 \leqslant q^L(x,s) &= L\, q(b_0 + xL^{-1}, sL^{-2}) \leqslant LC\{b_0 + xL^{-1} - b(L^{-2}s)\} \\ &= C[x - b^L(s)] \leqslant C[x + \sqrt{2Cs}] \quad \forall\, s > 0, x \in [b^L(s), \varepsilon L/2]. \end{aligned}$$

Since the bounds of the above estimates for q^L and b^L are independent of L, one can show that $\{(q^L, b^L)\}_{L>1}$ is locally compact and we can select a subsequence along which (q^L, b^L) approaches a limit, (Ψ, ζ). The limit satisfies

$$\begin{cases} \Psi_s = \Psi_{xx} & \forall\, x > \zeta(s), s > 0, \\ \Psi(x,s) = 0, & \forall\, x \leqslant \zeta(s), s > 0, \\ k\zeta(s)\dot\zeta(s) = \Psi_x(\zeta(s), s) & \forall\, s > 0, \\ \Psi(x, 0) = k \max\{0, x\} & \forall\, x \in \mathbb{R}, \\ 0 \leqslant \Psi(x,s) \leqslant C[x - \zeta(s)] & \forall\, x \geqslant \zeta(s), s \geqslant 0. \end{cases}$$

Since the solution is unbounded, the last condition imposes a constraint on the growth of the solution.

This problem admits a unique solution and the solution is self-similar, given by

$$\begin{aligned}\zeta(s) &= A_1\sqrt{s} \qquad \forall\, s>0,\\ \Psi(x,s) &= k\Big(x-\frac{A_1\sqrt{s}\int_{x/\sqrt{s}}^{\infty}(\eta-\frac{x}{\sqrt{s}})e^{-\eta^2/4}d\eta}{\int_{A_1}^{\infty}(\eta-A_1)e^{-\eta^2/4}}\Big) \quad \forall\, x\geqslant\zeta(s), s>0,\end{aligned}$$

where A_1 is the solution of the equation

$$1+\frac{A_1\int_{A_1}^{\infty}e^{-\eta^2/4}d\eta}{\int_{A_1}^{\infty}(\eta-A_1)e^{-\eta^2/4}d\eta}=\frac{A_1^2}{2}.$$

It is easy to verify that this equation is equivalent to (6.1).

Once we know the uniqueness of the limit, we then know that the whole sequence (q^L, b^L) converges. In addition, by compactness, $\lim_{L\to\infty} b^L=\zeta$ in $C^2([1/2,2])$. Using (6.2) and its differentiation with respective θ, we obtain the assertion of Theorem 3. This completes the proof. $\square$

References

[1] F. AitSahlia & T. Lai, *Exercise boundaries and efficient approximations to American option prices and hedge parameters,* J. Computational Finance, **4** (2001), 85–103.

[2] G. Barone-Adesi & R. E. Whaley, *Efficient analytic approximation of American option values,* Journal of Finance, **42** (1987), 301–320.

[3] G. Barone-Adesi, & R. Elliott, *Approximations for the values of American options,* Stochastic Analysis and Applications, **9** (1991), 115–131.

[4] G. Barles, J. Burdeau, & M. Romano, & N. Samsoen, *Critical stock price near expiration,* Mathematical Finance, **5** (1995), 77-95.

[5] E. Bayraktar & Hao Xing, *Analysis of the optimal exercise boundary of American option for jump diffusions*, preprint.

[6] M. Broadie & J. Detemple, *American option valuation: New bounds, approximations and comparison of existing methods,* Review of Financial Studies, **9** (1996), 1121–1250.

[7] D.S. Bunch & H. Johnson, *The American put option and its critical stock price,* J. Finance, **55** (2000), 2333–2356.

[8] P. Carr, R. Jarrow, & R. Myneni, *Alternative characterization of American put option,* Mathematical Finance, **2** (1992), 87–105.

[9] D. Chakraborty, *Numerical study of the convexity of the exercise boundary of the American put option on a devidend-paying asset,* MS. Thesis, Department of Mathematics, University of Pittsburgh, December (2008).

[10] Xinfu Chen & J. Chadam, *A mathematical analysis of the optmal boundary for American put options*, SIAM J. Math. Anal., **38** (2006), 1613–1641.

[11] Xinfu Chen & J. Chadam *Analytic and numerical approximations for the early exercise boundary for American put options,* Dyn. Cont. Disc. and Impulsive Sys., **10** (2003), 649–657.

[12] Xinfu Chen, J. Chadam, L. Jiang, & W. Zheng, *Convexity of the exercise boundary of the American put option on a zero dividend asset.* Math. Finance **18** (2008), 185–197.

[13] E. Ekstrom, *Convexity of the optimal stopping boundary for the American put option*, J. Math. Anal. Appl. **299** (2004), 147–156.

[14] A. Friedman, VARIATIONAL PRINCIPLES AND FREE BOUNDARY PROBLEMS, John Wiley & Sons, New York, 1982.

[15] A. Friedman, PARTIAL DIFFFERENTIAL EQUATIONS OF PARABOLIC TYPE, Prentice-Hall, 1964.

[16] A. Friedman, *Analyticity of the free boundary for the Stefan problem,* Arch. Rational Mech. Anal. **61** (1976), 97–125.

[17] J. Hull, OPTIONS, FUTURES AND OTHER DERIVATIVE SECURITIES, , 3rd Ed., Prentice-Hall, New York, 1997.

[18] L. Jiang, *Existence and differentiability of the solution of a two phase Stefan problem for quasi-linear parabolic equations,* Chinese Math. Acta **7** (1965), 481–496.

[19] I. J. Kim, *The analytic evaluation of American Options,* Review of Financial Studies, **3** (1990), 547–572.
[20] R. A. Kuske & J. B. Keller, *Optimal exercise boundary for an American put option,* Applied Mathematical Finance **5** (1998), 107–116.
[21] D. Lamberton & M. Mikou, *The critical price for the American put in an exponential Lévy model,* Finance Stoch. **12** (2008), 561–581.
[22] L. W. MacMillan, *Analytic approximation for the American put option,* Advances in Future and Option Research, **1 A** (1986), 119–139.
[23] H. P. Jr. McKean, *Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics,* Industrial Management Review, **6** (1965), 32–39.
[24] R. Merton, Continuous-time Finance, Blackwell, 1992.
[25] P. Van Moerbeke, *On optimal stopping and free boundary problems,* Arch. Rational Mech. Anal. **60** (1976), 101–148.
[26] R. Stamicar, D. Ševčovič, & J. Chadam, *The early exercise boundary for the American put near expiry: numerical approximation*, Canad. Appl. Math. Quart., **7** (1999), 427–444.
[27] P. Wilmott, J. Dewynne, & S. Howison, The Mathematics of Financial Derivatives, Cambridge,University Press, New York, 1995.