NUMERICAL COMPUTATION OF FIRST-CROSSING BOUNDARY PROBLEM

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Abstract.

1. Introduction

In this paper, we study the inverse first-crossing problem. This problem originates from the Merton's structural model [7] for credit risk management, derived as follows.

Consider a company whose asset value and debt at time $t \geq 0$ are denoted by $A(t)$ and $D(t)$ respectively. Assume the following:

(1) $D(0) \leq A(0)$ *and the company is in* **default** *at a time* $t > 0$ *if* $A(t) < D(t)$.

(2) $\{A(t)\}$ *is a log-normal process before the first time at which the company defaults.*

In mathematical finance, it is convenient to use the **default index** $X(t)$ and the **barrier function** $b(t)$ defined by

$$X(t) := \log \frac{V(t)}{V(0)}, \quad b(t) := \log \frac{D(t)}{V(0)}.$$

Then the assumption (2) is translated as

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB^t \qquad \forall\, t < \tau, \tag{1.1}$$

where $\tau = \inf\{s \geq 0 \mid X(s) \leq b(s)\}$ is the default time, B^t the standard Brownion Motion, μ the growth rate, and σ the volatility.

Of importance are the following two problems:

(1) **The first-crossing problem:** *Given a barrier function* $b(t)$*, find the survival probability* $p(t)$ *that a company does not default before* t.

$$p(t) := \text{Prob}\{\tau > t\}. \tag{1.2}$$

(2) **The inverse first-crossing problem:** *Given a survival probability function* $p(t)$*, find a barrier function* $b(t)$*, such that (1.2) holds.*

In our paper "Inverse First-Crossing Boundary Problem", we have formulating the inverse first-crossing problem into a free boundary PDE problem and we proved that there exists a unique viscosity weak solution to this PDE. Furthermore we estimate the free boundary when t is small and derived the integral equations for the purpose of numerical computation. In this paper we will apply those integral equations to compute the default boundary.

To compute the boundary, Avellaneda and Zhu [4] applied the finite difference scheme to the PDE, Zucca, Sacerdote and Peskir [10] applied the secant to a integral equation and I.Iscoe and A.Kreinin [6] use the Monte-Carlo approach. We will introduce their work in this paper compare all the schemes.

2. A Numerical Scheme

For the simplicity when calculating the free boundary, we set $\mu = 0$ and $\sigma = 1$.

2.1. **Our Numerical Scheme.** We use the integral equations

$$\Gamma(b(t), t) = \int_0^t \Gamma(b(t) - b(\tau), t - \tau)\, dq(\tau), \tag{2.1}$$

$$q'(t) = \Gamma_x(b(t), t) - \int_0^t \Gamma_x(b(t) - b(\tau), t - \tau)\, dq(\tau), \tag{2.2}$$

derived in § 5 of "Inverse First-Crossing Boundary Problem" to compute the free boundary.

Suppose $b(\tau)$ is known for all $\tau \in [0, t)$. Set

$$Q(x, t) = \int_0^t \Gamma(x - b(\tau), t - \tau)\, dq(\tau).$$

We want to solve $b = b(t)$ from the equation

$$\Gamma(b, t) - Q(b, t) = 0.$$

Note from (2.2) that

$$\frac{\partial}{\partial x}\Big\{\Gamma(x, t) - Q(x, t)\Big\}\Big|_{x=b} = q'(t).$$

Thus, if we use Newton's method, a new approximation b^{new} can be obtained from the old approximation b^{old} by

$$b^{new} = \hat{b}^{old} + \frac{Q(b^{old}, t) - \Gamma(b^{old}, t)}{q'(t)}. \tag{2.3}$$

From this, we can implement a numerical scheme as follows. Let $\{t_i\}_{i=0}^N$ be the mesh points where $0 = t_0 < t_1 < t_2 < \cdots$. Denote by b_i the approximate value of $b(t_i)$. For

$n \geq 1$, suppose $b_0, \cdots, b_{n-1}$ have been calculated. We define b_n by iteration

$$\begin{cases} b^{(0)} &= \frac{\sqrt{t_n}}{\sqrt{t_{n-1}}} b_{n-1}, \\ b^{(k+1)} &= b^{(k)} + \dfrac{Q(b^{(k)}, t_n) - \frac{1}{\sqrt{2\pi t_n}} e^{-(b^{(k)})^2/(2t_n)}}{q'(t_n)}, \quad k = 0, 1, \cdots, K. \\ b_n &= b^{(K)}, \end{cases}$$

where the initial guess $b^{(0)}$ is obtained by the fact that $\frac{b(t)}{\sqrt{2t}}$ changes slowly in the $\log t$ scale. The iteration ends until

$$\left| b^{(k+1)} - b^{(k)} \right| = \left| \frac{Q(b^{(k)}, t_n) - \frac{1}{\sqrt{2\pi t_n}} e^{-(b^{(k)})^2/(2t_n)}}{q'(t_n)} \right| < \epsilon$$

and ε is tolerance, say 10^{-5}. Since Q is an integral, a quadrature rule is needed for its numerical estimation. To take care of the singularity, we write it as

$$\begin{aligned} Q(x, t_n) &= \frac{1}{\sqrt{2\pi}} \int_0^{t_n} e^{-(x-b(\tau))^2/2(t_n-\tau)} \frac{q'(\tau) d\tau}{\sqrt{t_n - \tau}} \\ &= \sqrt{\frac{2}{\pi}} \int_0^{t_n} q'(\tau) e^{-(x-b(\tau))^2/[2(t_n-\tau)]} d\left(-\sqrt{t_n - \tau}\right), \end{aligned}$$

when $q'(t)$ is bounded. The left-point rule can be written as:

$$Q(x, t_n) = \sqrt{\frac{2}{\pi}} \sum_{i=0}^{n-1} q'(t_i) \exp\Big(- \frac{(x - b_i)^2}{2(t_n - t_i)} \Big) \Big\{ \sqrt{t_n - t_i} - \sqrt{t_n - t_{i+1}} \Big\}$$

Similarly the trapezoid rule is

$$\begin{aligned} Q(x, t_n) \;&= \sqrt{\tfrac{2}{\pi}} \Big(q'(t_n) \tfrac{\sqrt{t_n - t_{n-1}}}{2} + q'(0) \exp\Big(-\tfrac{x^2}{2t_n} \Big) \tfrac{\sqrt{t_n} - \sqrt{t_n - t_1}}{2} \\ &\quad + \textstyle\sum_{i=1}^{n-1} q'(t_i) \exp\Big(-\tfrac{(x-b_i)^2}{2(t_n - t_i)} \Big) \tfrac{\sqrt{t_n - t_{i-1}} - \sqrt{t_n - t_{i+1}}}{2} \Big) \end{aligned}$$

Nevertheless, due to the singularity of the integral, higher order quadrature rules (e.g., the Simpson's rule) are not recommended. Indeed, as we shall see, the trapezoid rule is quite satisfied at least for small t. The complexity of the scheme is $O(N^2)$ where N is the total number of mesh points.

There are some other numerical treatments for the calculation of default boundary $b(t)$. Avellaneda and Zhu ([4]) used the finite difference method to (**??**). Zucca, Sacerdote and Peskir ([9])used the secant method to (**??**). Iscoe and Kreinin treated the problem as a conditional probability problem. In readers's convenience, we provide a little bit more of their contributions.

2.2. **Integral Equation by Peskir.** Peskir derived a sequence of integral equations

$$t^{n/2}H_n\left(\frac{b(t)}{\sqrt{t}}\right)-\int_0^t (t-s)^{n/2}H_n\left(\frac{b(t)-b(s)}{\sqrt{t-s}}\right)\dot{q}(s)ds=0, n=-1,0,1... \tag{2.4}$$

where $H_{-1}(x)=\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and $H_n(x)=\int_x^\infty H_{n-1}(z)dz$ for $n\geq 0$. In particular, when $n=-1$ and $n=0$, they become

$$\frac{1}{\sqrt{2\pi t}}e^{-b^2(t)/2t}-\int_0^t \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-(b(t)-b(s))^2/2(t-s)}\dot{q}(s)ds=0, n=-1, \tag{2.5}$$

$$\int_{\frac{b(t)}{\sqrt{t}}}^\infty \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz-\int_0^t\left(\int_{\frac{b(t)-b(s)}{\sqrt{t-s}}}^\infty \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz\right)\dot{q}(s)ds=0, n=0. \tag{2.6}$$

One observe that (2.5) is what we used in calculating the boundary. Instead of using (2.5), Peskir and Zucca used (2.6) to calculate the boundary. They discretize (2.5) by the scheme:

$$\begin{cases} \int_{b(t_i)/t_i}^\infty \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz=\sum_{j=1}^i \int_{\frac{b(t_i)-b(t_j)}{\sqrt{t_i-t_j}}}^\infty \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz\,\dot{q}(t_j)\,h & i=2,...n,\\ \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz=\frac{1}{2}\,\dot{q}(t_1)\,h & i=1.\end{cases} \tag{2.7}$$

(2.7) yields a non-linear system of n equations with n unknowns $b(t_1),...b(t_n)$. The secant method can used to solve it.

We point the difference of this scheme with our scheme. In our approximation, we use the Newton iteration since we derived the derivative of (2.5). Whereas for Peskir and Zucca's scheme, they used the secant method.

2.3. **Avellaneda-Zhu's Scheme.** The spatial translated density function $f(y,t)=u(y+b(t),t)$ satisfies

$$\begin{cases} f_t(y,t)=b'(t)f_y(,y,t)+\frac{1}{2}f(y,t)_{yy} & \text{for } y>0,\ t>0,\\ f(0,t)=0 & \text{for } y=0,\ t>0,\\ f(y,0)=\delta_0(y-b(0)) & \text{for } y>0,\ t=0,\\ \frac{1}{2}f_y(0,t)=q'(t) & \text{for} y=0,\ t>0.\end{cases} \tag{2.8}$$

Here $\delta_0(\cdot)$ is a Dirac Measure concentrated at 0. It must be regularized consistently with the boundary condition $f(0,t)=0$ for all $t\geq 0$. Indeed, to take care of the singularity, Avellaneda and Zhu used the idea of 'initial layer'. For a chosen small t_0, they replace the solution to $[0,t_0]$ by an explicit solution to the first passage problem. A numerical simulation carries on after t_0. Based on Avellaneda and Zhu's idea of using initial layer and finite difference scheme, we implement their scheme as the following.

1. Analytic Small Time Approximation

For any $\alpha > 0$ and $\beta \in \mathbb{R}$, (2.8) admits an exact solution.

$$\left\{ \begin{array}{l} \bar{f} = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \left(1 - e^{-2\alpha(x+\alpha+\beta t)/t}\right), \\ \bar{b} = -\alpha - \beta t. \end{array} \right. \tag{2.9}$$

Then the corresponding default probability and its derivative are given by:

$$\begin{aligned} \bar{q}(t;\alpha,\beta) &= 1 - \int_{b(t)}^{\infty} \bar{f}(y,t)dy = N\left(\frac{-\alpha-\beta t}{\sqrt{t}}\right) + e^{-2\alpha\beta} N\left(\frac{-\alpha+\beta t}{\sqrt{t}}\right), \\ \dot{\bar{q}}(t;\alpha,\beta) &= \frac{\alpha}{t\sqrt{2\pi t}} e^{-(\alpha+\beta t)^2/2t}. \end{aligned}$$

For a given t_0, the parameter α and β are chosen such that

$$\left\{ \begin{array}{l} q(t_0) = \bar{q}(t_0;\alpha,\beta), \\ \dot{q}(t_0) = \dot{\bar{q}}(t_0;\alpha,\beta). \end{array} \right. \tag{2.10}$$

For example, when $q(t) = 0.1t$ and $t_0 = 0.1$, solution to (2.10) is $\alpha = 0.4672$ and $\beta = 4.3575$.

2. Numerical Simulation

Apply the finite difference scheme to the first equation of (2.8) with the initial condition at $t = t_0$ given by (2.9). Define $y_i = ih$ (i=1,2,...,M), $t_n = t_0 + n\Delta t$ and let f_i^n represent the numerical approximation to $f(y_i, t_n)$ and $b'_{n+\frac{1}{2}}$ to $b'(t_{n+\frac{1}{2}})$. Their initial values are taken to be $f_i^0 = \bar{f}(y_i, 0)$ and $b'_0 = -\beta$. The boundary conditions are $f_0^n = 0$ and $f_{M+1}^n = 0$. A Crank-Nicholson scheme reads as

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = b'_{n+\frac{1}{2}} \frac{f_{i+\frac{1}{2}}^{n+\frac{1}{2}} - f_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{h} + \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{4h^2} + \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{4h^2}. \tag{2.11}$$

$f_{i+\frac{1}{2}}^{n+\frac{1}{2}}$ is the numerical approximation of $f(y + \frac{\Delta y}{2}, t + \frac{\Delta t}{2})$ by both Taylor expansion and upwind scheme:

$$\begin{aligned} f_{i+\frac{1}{2}}^{n+\frac{1}{2}} &= f_{i+1}^n + \frac{1}{2}(\lambda^{n-\frac{1}{2}}\Delta t - \Delta y)(f_y)_{i+1}^n + \frac{1}{4}(f_{yy})_{i+1}^n \Delta t, \quad \text{if} \lambda^{n-\frac{1}{2}} \geq 0, \\ f_{i+\frac{1}{2}}^{n+\frac{1}{2}} &= f_i^n + \frac{1}{2}(\lambda^{n-\frac{1}{2}}\Delta t + \Delta y)(f_y)_i^n + \frac{1}{4}(f_{yy})_i^n \Delta t, \quad \text{if} \lambda^{n-\frac{1}{2}} < 0, \end{aligned}$$

where $(f_y)_i^n$ and $(f_{yy})_i^n$, are estimated by standard central difference and second order one-sided difference approximations for the boundary points

$$\begin{aligned} (f_y)_i^n &= \frac{f_{i+1}^n - f_{i-1}^n}{2h}, \quad (f_{yy})_i^n = \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{h^2}, \quad i = 1, ..., M-1. \\ (f_y)_0^n &= \frac{-3f_0^n + 4f_1^n - f_2^n}{2h}, \quad (f_{yy})_0^n = \frac{2f_0^n - 5f_1^n + 4f_2^n - f_3^n}{h^2} \\ (f_y)_M^n &= \frac{-3f_M^n + 4f_{M-1}^n - f_3^{M-2}}{2h}, \quad (f_{yy})_M^n = \frac{2f_M^n - 5f_{M-1}^n + 4f_{M-2}^n - f_{M-3}^n}{h^2}. \end{aligned}$$

Suppose $f^1,...,f^n$, b^n and $\lambda^{n-\frac{1}{2}}$ have been calculated, then so is $f_{i+\frac{1}{2}}^{n+\frac{1}{2}}$. (2.11) gives a system of $M-1$ linear equations with M unknown, $f_i^{n+1}(2 \le i \le M)$ and $b'_{n+\frac{1}{2}}$. To determine these M unknowns, another equation is required. Then the last equation in (2.8) when $t = t_{n+1}$: $\frac{1}{2}f_y(0,t_{n+1}) = q'(t_{n+1})$ is used, which can be rewritten by difference approximation as:

$$\frac{f_1^{n+1} - f_0^{n+1}}{2h} = q'(t_{n+1}). \tag{2.12}$$

(2.11) and (2.12) gives a system of M linear equations with M unknowns. Solve the system of equations, we get the solution $f^{n+1}$to the PDE and then update the boundary by

$$\begin{aligned} b^{n+1} &= b^n + b'_{n+\frac{1}{2}}\Delta t \\ b^0 &= -\alpha - \beta t_0. \end{aligned}$$

2.3.1. *An Alternative Explicit Scheme.* In light of Avellanda and Zhu's scheme, where they impose an explicit finite difference scheme, Crank-Nicholson scheme, we discretize the PDE in (2.8)as

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = b'_{n+1}\frac{f_i^{n+1} - f_{i-1}^{n+1}}{h} + \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{2h^2}. \tag{2.13}$$

We corporate (2.12)with (2.13) as $i = 1$. Namely, solving f_1^{n+1} from (2.13) and substituting the result into (2.12) gives an equation involving b'_{n+1}. where solution is

$$b'_{n+1} = \frac{2hq'(t_{n+1}) - f_1^n + (\Delta t/2h^2)(f_0^n - 2f_1^n + f_2^n)}{2\Delta t q'(t_{n+1})}. \tag{2.14}$$

Using (2.14) we can estimate the solution and update the boundary by

$$\begin{aligned} f_j^{n+1} &= \frac{f_j^n - \frac{\Delta t}{h}\lambda^{n+1} f_{j-1}^{n+1} + (\Delta t/2h^2)(f_{j-1}^n - 2f_j^n + f_{j+1}^n)}{1 - \frac{\Delta t}{h}\lambda^{n+1}}, \\ b^{n+1} &= b^n + \lambda^{n+\frac{1}{2}}\Delta t. \end{aligned}$$

The CFL stability condition requires that to be stable, we take $\Delta t = \frac{1}{2}h^2$.

2.4. **Conditional default probability.** Iscoe and Kreinin use a different approach to calculate the boundary. Instead of using the partial differential equations or integral equations to estimate the boundary, they used the theory of probability. They reduced the problem of estimating the default boundary to a sequential estimation of the quantities of the conditional default distributions. Instead of considering the continuous process, they consider a discrete-time, mean zero process, $S_n, n = 0, 1, 2, ..., S_0 = 0$, having a finite variance $\sigma_n^2 = t_n$ at time t_n. It can be normalized by taking the value η_n at time t_n by

$$\eta_n = \frac{S_n}{\sigma_n}, \quad n = 1, 2, \ldots; \quad \eta_0 = 0,$$

which satisfies the relation $E\eta_n = 0$, $E\eta_n^2 = 1$ for $n \geq 1$. Then the default time τ can be formulated as $\tau = min_{n\geq 1}\{n : \eta_n < b_n/\sigma_n\}$. Denote $Q(n) := Prob\{\tau \leq t_n\}$, $\pi_n = Prob\{\tau = t_n\}$ and $\hat{Q}_n = Prob\{\tau = t_n|\tau \geq t_n\}$. Iscoe and Kreinin proved that the boundary $\{b_k\}_{k=1}^N$, the probability π_n and $\hat{Q}_n$ satisfy the following equations when $n = 1, 2, ...N$

$$\begin{aligned} \pi_n &= Prob\left\{\bigcap_{k=1}^{n-1}\{\eta_k \geq \frac{b_k}{\sigma_n}\}, \eta_n < \frac{b_n}{\sigma_n}\right\}, \\ \pi_n &= Q(n) - Q(n-1), \\ \hat{Q}_n &= \frac{\pi_n}{1 - Q(n-1)}. \end{aligned}$$

Based on this result, they estimated the boundary b_n as follow.

1. Estimation of b_1. Based on $Q(1) = Prob\{\eta_1 \leq b_1\}$, one can calculate that $b_1 = F_1^{-1}(Q(1))$, where F_1 denoting the the cdf of the random variable η_1.
2. Compute the conditional probabilities $\hat{Q}_n$ based on (**??**) and (**??**).
3. Suppose that the default boundary b_k has already been computed for $k = 1, ..., n-1$. To compute b_n, generate a large number, $M >> 1$, of i.i.d sample paths $\eta(m) = (\eta_1(m), \eta_2(m), ..., \eta_n(m))$, $(m = 1, 2, ..., M)$, and retain only those vectors $\eta(m)$ that satisfy the inequality

$$\eta_k(m) \geq \frac{b_k}{\sigma_k}, \quad k = 1, 2, ..., n-1. \tag{2.15}$$

4. Let $F_n(x)$ denote the conditional empirical cdf of the random variable η_n under the condition (2.15), b_n is then the quantile of the distribution F_n corresponding to the probability.

3. Numerical Simulation

3.1. **Linear boundary.** If default probability function and its' density function are given by (2.10) and (2.10) then the boundary is a straight line

$$b(t) = -\alpha - \beta t, \quad \alpha > 0, \quad t > 0.$$

Here is the picture of the linear boundary for all four schemes with $T = 1$ and $t_0 = 0.5$. To have a better view, we plot four schemes separately.

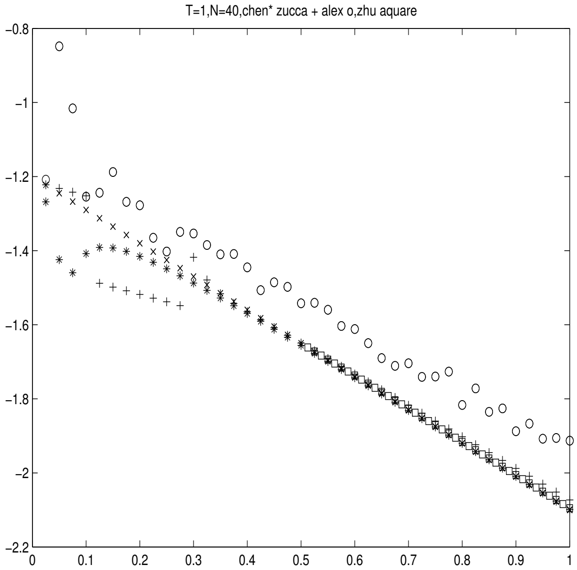

FIGURE 1. Linear Boundary (Our scheme $*$, Peskir and Zucca $+$, Alex o, Zhu $\square$, original line $\times$)

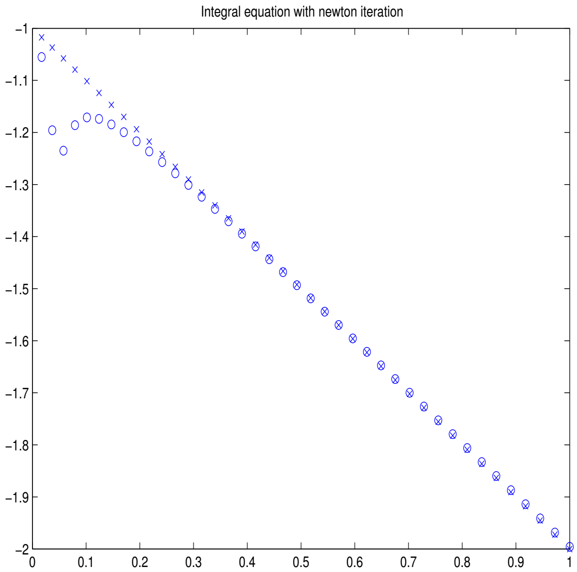

FIGURE 2. Linear Boundary with N=40 (Our scheme o, original line x)

3.2. **Our scheme vs Zhu and Avellanda's.** Use Zhu and Avellanda's finite difference method, we can get not only the free boundary, but also the solution of the PDE. Meanwhile we get more information from this scheme than all the other three schemes. However they used the idea of 'initial layer' so that the boundary estimated at least starting from t_0. Here is an example for $q(t) = 0.1$, with $T = 1$, $t_0 = 0.1$ (Table (1), (2) and (3)).

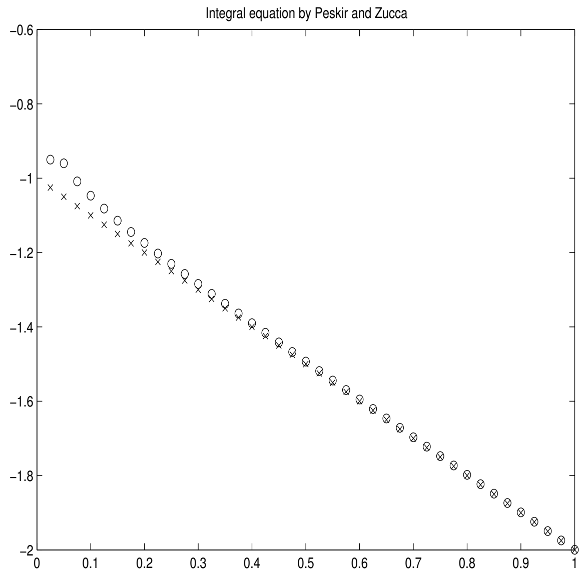

FIGURE 3. Linear Boundary with N=40 (Peskir and Zucca o, original line x)

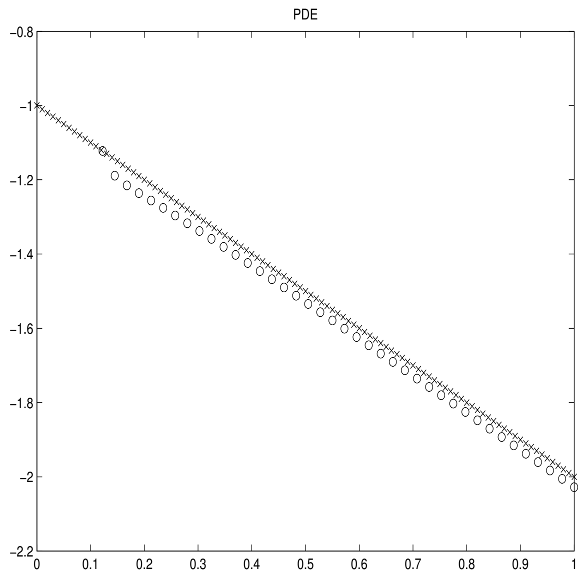

FIGURE 4. Linear Boundary with N=40 $t_0 = 0.1$(Zhu o, original line x)

3.3. **Our scheme vs Peskir and Zucca's.** Use the scheme by Peskir and Zucca, the computation result is very closed to ours. In fact the integral equations used by Peskir, Zucca an us are both from the sequence of equations refpeskir3). The difference is that for the first point from initial point, we use the estimation from section 4, however Peskir and Zucca used (**??**). Both schemes need to solve the nonlinear equations. We used the newton iteration and they used the secant method. Here we list the results by both

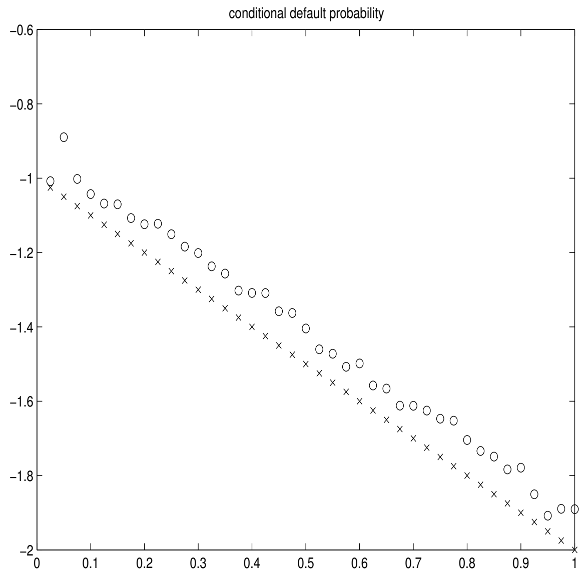

Figure 5. Linear Boundary with N=40 m=10000 (alex o, original line x)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	-1.844521822	N/A	N/A	0.219
20	-1.840072931	0.004448891	N/A	0.375
40	-1.840788211	-0.00071528	-6.2198	1.312
80	-1.840643468	0.000144742	-4.9417	4.875
160	-1.840320038	0.000323431	0.4475	16.797
320	-1.840093945	0.000226093	1.4305	61.641
640	-1.839965963	0.000127982	1.7666	222.328
1280	-1.839898223	6.77405E-05	1.8893	765.703
2560	-1.839863301	3.49215E-05	1.9398	2818.6

Table 1. Default Boundary at $T = 1$ with $q(t) = 0.1t$ (Our scheme)

schemes with the different default probability functions with both left-point rule and the trapezoid rule (inside the parenthesis) .

1. $q(x) = t$ and $T = 0.01$ (table (4) and (5))
2. $q(t) = \sqrt{t}$ (table (6) and (7))

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	-2.7206	N/A	N/A	0.016
20	-2.2248	0.4958	N/A	0.031
40	-2.0275	0.1973	2.5123	0.031
80	-1.9389	0.0886	2.2294	0.032
160	-1.897	0.0419	2.1126	0.078
320	-1.8743	0.0227	1.844	0.172
640	-1.8612	0.0131	1.7343	0.343
1280	-1.8533	0.0079	1.6647	0.844
2560	-1.8484	0.0049	1.6092	1.844
5120	-1.8453	0.0031	1.5635	5.281
10240	-1.8433	0.002	1.5266	15.453
20480	-1.8419	0.0014	1.4976	52.86
40960	-1.841	0.0009	1.4754	204.156

TABLE 2. Default Boundary at $T = 1$ with $q(t) = 0.1t$ $t_0 = 0.1$(explicit)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	-2.2406 N/A	N/A	0.016	
20	-2.0411	0.1995	N/A	0.015
40	-1.9395	0.1016	1.9635	0.016
80	-1.889	0.0505	2.0082	0.078
160	-1.8638	0.0252	2.0126	0.141
320	-1.8513	0.0125	2.0096	0.484
640	-1.8451	0.0062	2.0064	2.719
1280	-1.842	0.0031	2.0046	18.89
2560	-1.8404	0.0016	2.0037	93.875
5120	-1.8396	0.0008	2.0031	437.78
10240	-1.8393	0.0003	2.0026	3547.6
20480	-1.8391	0.0002	2.0021	16237

TABLE 3. Default Boundary at $T = 1$ with $q(t) = 0.1t$ $t_0 = 0.1$(Zhu)

We make some adjustment for $Q(x,t)$ and $s^{(k+1)}$ when we do the calculation use our scheme. Indeed

$$\begin{aligned} Q(x,t) &= \int_0^t \frac{1}{\sqrt{2\pi(t-\tau)}} e^{-\frac{(x-s(\tau))^2}{2(t-\tau)}} dq(\tau) \\ &= \int_0^t \frac{1}{\sqrt{2\pi(t-\tau)}} e^{-\frac{(x-s(\tau))^2}{2(t-\tau)}} \frac{1}{2\sqrt{\tau}} d\tau \\ &= \int_0^t \frac{1}{\sqrt{8\pi}} e^{-\frac{(x-s(\tau))^2}{2(t-\tau)}} d\left(\arcsin(\frac{2\tau-t}{t})\right), \end{aligned}$$

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	-0.292244994 (-0.290506149)	N/A N/A	N/A N/A	0.172 (0.188)
20	-0.291252915 (-0.290379889)	0.000992078 (0.00012626)	N/A N/A	0.328 (0.312)
40	-0.290776096 (-0.290338991)	0.00047682 (0.000040897)	2.08 (3.09)	0.813 (1.141)
80	-0.290544143 (-0.290325559)	0.000231953 (0.0000134329)	2.06 (3.05)	2.797 (2.766)
160	-0.290430342 (-0.290320983)	0.000113801 (0.0000045751)	2.04 (2.94)	10.39 (10.375)
320	-0.290374157 (-0.290319459)	0.00005.61846 (0.00000152423)	2.03 (3.00)	40.797 (41.281)
640	-0.290346301 (-0.290318946)	0.0000278562 (0.0000005127)	2.02 (2.97)	163.922 (162.125)
1280	-0.290332456 (-0.290318744)	0.00001.3845 (0.000000202118)	2.01 (2.54)	328.016 (324.562)
2560	-0.290325555 (-0.290318704)	0.00000690107 (0.0000000406215)	2.01 (4.98)	1312 (1293.718)

TABLE 4. Default Boundary at $T = 0.01$ with $q(t) = t$

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	0.2868	N/A	N/A	0.109
20	0.2886	0.0018	N/A	0.234
40	0.2895	0.0009	2.08	0.875
80	0.2899	0.0004	2.05	3.719
160	0.2902	0.0003	1.7125	16.938

TABLE 5. Default Boundary at $T = 0.01$ with $q(t) = t$ (by Peskir & Zucca's scheme)

and

$$s^{(k+1)} = s^{(k)} + 2\sqrt{t_n}Q(s^{(k)}) - \frac{1}{\pi}e^{-\frac{(s^{(k)})^2}{2tn}}.$$

When we use Peskir and Zucca's Scheme, we make some adjustment too. The reason is that $\dot{q}(t) = \frac{1}{2\sqrt{t}} \to \infty$ as $t \to 0$. Note that (2.4) can be written as:

$$t^{n/2}H_n\left(\frac{b(t)}{\sqrt{t}}\right) - \int_0^t (t-s)^{n/2}H_n\left(\frac{b(t)-b(s)}{\sqrt{t-s}}\right) dq(s) = 0,$$

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	-0.211223506 (-0.210002815)	N/A N/A	N/A N/A	0 (0.015)
20	-0.211379633 (-0.209950142)	0.001343874 (0.0000526728)	N/A N/A	0 (0)
40	-0.210683349 (0.209933543)	0.000696284 (0.0000165989)	1.93 (3.17)	0.01 (0)
80	-0.210316997 (-0.20992631)	0.000366352 (0.00000723327)	1.90 (2.30)	0.03 (0.063)
160	-0.210025907 (-0.209921348)	0.000191589 (0.00000334088)	1.91 (2.17)	0.04 (0.094)
320	-0.210025907 (-0.209921348)	0.0000995 (0.00000162083)	1.93 (2.06)	0.14 (0.328)
640	-0.209974456 (-0.20992052)	0.0000514514 (0.000000828147)	1.93 (1.96)	0.27 (0.687)
1280	-0.209947909 (-0.209920182)	0.0000265463 (0.000000338589)	1.94 (2.45)	1.03 (2.75)
2560	-0.209934247 (-0.209920013)	0.000013662 (0.000000168437)	1.94 (2.01)	4.27 (10.93)
5120	-0.209927228 (-0.20991993)	0.00000701974 (0.0000000832255)	1.95 (2.02)	17.48 (46.735)

TABLE 6. Default Boundary at $T = 0.01$ with $q(t) = \sqrt{t}$

(2.7) becomes

$$\int_{b(t_i)/t_i}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \quad = \quad \sum_{j=1}^{i} \int_{\frac{b(t_i)-b(t_j)}{\sqrt{t_i-t_j}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \ (q(t_j) - q(t_{j-1})),$$

and (??) becomes

$$\int_{b(t_1)/t_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{2} q(t_1).$$

3. $q(t) = 1 - e^{-t}$ (table (8) and (9))
4. $q(t) = e^{-1/2t}$ (table (10) and (11))

We make some adjustment for $Q(x,t)$ and $s^{(k+1)}$ when we do the calculation. Since $q'(t) = \frac{1}{t^2} e^{-1/2t}$,

$$\begin{aligned} Q(x,t) &= \int_0^t \frac{1}{\tau^2 \sqrt{2\pi(t-\tau)}} \exp(-\frac{(x-s(\tau))^2}{2(t-\tau)} - \frac{1}{2\tau}) d\tau \\ &= \int_0^t \frac{1}{\sqrt{8\pi}} \exp(-\frac{(x-s(\tau))^2}{2(t-\tau)} - \frac{1}{2\tau}) d\left(-\frac{\sqrt{t-\tau}}{t\tau} + \frac{1}{2t\sqrt{t}} \ln\left|\frac{\sqrt{t-\tau}-\sqrt{t}}{\sqrt{t-\tau}+\sqrt{t}}\right|\right), \end{aligned}$$

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	0.210516993	N/A	N/A	0.11
20	0.210594081	7.70883E-05	N/A	0.234
40	0.210492503	-0.000101578	-0.758909882	1.078
80	0.210361389	-0.000131115	0.774724046	4.391
160	0.210245607	-0.000115782	1.132430959	17.625
320	0.210154449	-9.11576E-05	1.270126385	69.312

Table 7. Default Boundary at $T = 0.01$ with $q(t) = \sqrt{t}$ (by Peskir & Zucca's scheme)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	-0.292542813 (-0.290815038)	N/A N/A	N/A N/A	0.141 (0.109)
20	-0.291557587 (-0.290689858)	0.000985226 (0.00012518)	N/A N/A	0.344 (0.235)
40	-0.291083888 (-0.290649382)	0.000473698 (4.04763E-05)	2.08 (3.09)	1.015 (0.828)
80	-0.29085341 (-0.290636105)	0.000230478 (1.32773E-05)	2.06 (3.05)	3.016 (2.23)
160	-0.290740315 (-0.290631595)	0.000113094 (4.50935E-06)	2.06 (2.94)	11.312 (7.70)
320	-0.290684473 (-0.290630094)	5.58419E-05 (1.50113E-06)	2.03 (3.00)	42.75 (2.99)
640	-0.290656799 (-0.290629499)	2.76748E-05 (5.95518E-07)	2.02 (2.52)	104.797 (77.4)
1280	-0.290643022 (-0.290629392)	1.37766E-05 (1.06181E-07)	2.015 (5.61)	342.734 (236.922)
2560	-0.290636162 (-0.290629352)	6.8598E-06 (4.08239E-08)	2.01 (2.60)	1339.532 (956.281)

Table 8. Default Boundary at $T = 0.01$ with $q(t) = 1 - e^{-t}$

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	0.2871	N/A	N/A	0.094
20	0.2889	0.0018	N/A	0.218
40	0.2898	0.0009	2.08	0.86
80	0.2902	0.0004	2.05	3.703
160	0.2905	0.0003	1.81	16.578

Table 9. Default Boundary at $T = 0.01$ with $q(t) = 1 - e^{-t}$ (by Peskir & Zucca's scheme)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	-0.533022491	N/A	N/A	0.016
20	-0.435398286	0.097624205	N/A	0.031
40	-0.392856847	0.042541439	2.30	0.063
80	-0.356655692	0.036201155	1.18	0.234
160	-0.330071718	0.026583975	1.36	0.687
320	-0.310972696	0.019099022	1.39	2.438
640	-0.297412263	0.013560433	1.41	8.75
1280	-0.287778027	0.009634236	1.48	30.812

TABLE 10. Default Boundary at $T = 0.01$ with $q(t) = e^{-1/2t}$

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	0.276	N/A	N/A	0.094
20	0.2714	-0.0046	N/A	0.25
40	0.2679	-0.0035	1.3019	0.906

TABLE 11. Default Boundary at $T = 1$ with $q(t) = e^{-1/2t}$ (by Peskir & Zucca's scheme)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	-0.2908	N/A	N/A	0.282
20	-0.2878	0.003	N/A	0.796
40	-0.2858	0.002	1.43	2.61
80	-0.2843	0.0015	1.41	8.203
160	-0.2832	0.0011	1.41	25.625
320	-0.2825	0.0007	1.41	93.328
640	-0.282	0.0005	1.41	280.422
1280	-0.2816	0.0004	1.41	997.625

TABLE 12. Default Boundary at $T = 0.01$ with $q(t) = e^{-\frac{1}{2t^{1/2}}}$

and

$$s^{(k+1)} = s^{(k)} + 2t^2 Q(s^{(k)})/e^{-1/2t} - \sqrt{\frac{2t_n^3}{\pi}} e^{-1/2t_n} e^{-\frac{(s^{(k)})^2}{2tn}}$$

For the scheme by Peskir and Zucca, due to the same reason as in the second case when $q(t) = \sqrt{t}$, we use the same adjustment.

5. $q(t) = e^{-\frac{1}{2t^{1/2}}}$ (table (12) and (13))

N (Mesh point)	Free Boundary $s^N(t)$	Difference $s^N(t) - s^{N/4}(t)$	Rate $\frac{s^N(t)-s^{N/4}(t)}{s^{N/4}(t)-s^{N/16}(t)}$	Time used for calculation
10	0.276	N/A	N/A	0.078
20	0.2784	0.0024	N/A	0.219

TABLE 13. Default Boundary at $T = 1$ with $q(t) = e^{-\frac{1}{2t^{1/2}}}$ (by Peskir & Zucca's scheme)

Iscoe and Kreinin's scheme is different from all the other three. The other scheme used either the PDE or the integral equations. However this used neither. It just based on the theory of probability and used the simulation to calculate the boundary. The advantage of this scheme is that it is not like all the other schemes that we have discuss the existence of solutions to the equations. However it takes so much time to do the simulations and the result is a little less accuracy.

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