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ANALYTICAL AND NUMERICAL APPROXIMATIONS FOR THE EARLY EXERCISE BOUNDARY FOR AMERICAN PUT OPTIONS

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Abstract. Several new analytical and numerical approximations are provided for the location of the early exercise boundary for the American put option. The most complete approximation is in the form of an integro-differential equation for which an iterative scheme can be proven to converge to the unique solution. The current methods are compared to those recently proposed by other authors. This paper summarizes various parts of our joint work [8-10] with L. Jiang (Shanghai), R. Stamicar (Toronto) and W. Zheng (Irvine).

Keywords. American put option, free boundary, numerical and analytical approximations

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1 Statement of the Problem and Previous Results

A European put (call) option gives the purchaser the right, but not the obligation, to sell (buy) a share of an underlying asset at a given price K, the strike, at a given time in the future, T. The payoff at expiry T for a put option can be determined as follows. If the stock price at expiry, $S(T)$, is less than K then the holder can purchase the stock at $S(T)$ and has the right to sell it at K for a payoff of $K - S(T)$. If, on the other hand, $S(T)$ is greater than K the above strategy does not work and if she is holding stock it is preferable to sell it at $S(T)$ in the open market. Thus there is no incentive to exercise the put option and it is worthless in this case. Summarizing, the value of the put option at expiry is

$$p_E(S,T) = \max\ (K - S, 0) \tag{1}$$

The American version of these options allows the holder to exercise them at any time up to expiry, T. Clearly, because the holder has more optionality, the American version is more expensive than the simpler European version. More interestingly from a mathematical viewpoint is the question of determining the early exercise boundary - the asset price at time $t < T, S_f(t)$, below which it is advisable to exercise early.

The Nobel Prize work of Black, Scholes and Merton (see for example [21, Chap. 3]) provides a risk neutral method of pricing these as well as other more exotic options and derivative securities. More specifically, suppose the underlying asset (stock) S follows a geometric Brownian motion [21, pp. 19-25]

$$\frac{dS}{S} = \mu dt + \sigma dW_t \tag{2}$$

where t is the time in years, μ is the average annual rate of return, σ is volatility and W_t is a Wiener process [21, pp. 21-22]. Equation (2) says that the asset price $S(t)$ is a random variable whose return dS/S consist of two parts - a deterministic part μdt and a random part σdW, the latter representing the total of unexpected news and events.

For an American put option on an underlying asset which follows (2), the Black, Scholes, Merton theory gives the price of the option, $p(S,t)$ for $0 \le t \le T$, as the solution of the problem (P) [21, pp. 41-48, 109-114]:

$$p_t + \frac{\sigma^2}{2} S^2 p_{SS} + rSp_S - rp = 0, \quad S_f(t) < S, \ 0 \le t < T \tag{3-a}$$

$$p(S,t) = K - S \text{ on } S = S_f(t), \ 0 \le t < T \tag{3-b}$$

$$p_S(S,t) = -1 \text{ on } S = S_f(t), \ 0 \le t < T \tag{3-c}$$

$$p(S,t) \to 0 \text{ as } S \to \infty \tag{3-d}$$

$$p(S,T) = \max\,(K - S, 0) \tag{3-e}$$

Equation (3a) is a backward "heat" equation which must be solved subject to the final condition (3e). The second term in (3a), the diffusion term, arises from the random term in (2) and the fourth represents discounting with respect to the risk free interest rate r (e.g., the rate of return on a government bond). The big surprise, and one of the main contributions of Black, Scholes and Merton, is that the third term in (3a), the drift term, has the drift rate r rather than μ as in (2). This is the essence of risk neutral pricing of options. In order to spread the risk evenly between the buyer and the seller of the option and to avoid arbitrage (one or other making a positive gain with probability 1), one must use the risk free growth rate r (of a bond) rather that market growth rate μ (of the stock). The boundary condition (3d) says that if the value of the stock is very large, there is very little chance that it will ever get "in the money", $S < S_f(t)$, so that the holder can exercise the option. The early exercise boundary, $S = S_f(t)$ is part of the solution to problem (P) and is determined from (3b, c). Equation (3c) says that at time t, $0 < t < T$, the solution $p(S,t)$ approaches the intrinsic payoff $\max(K - S, 0)$ smoothly; i.e., problem (P) is a parabolic obstacle problem. The necessity for this smooth approach to the boundary is a consequence of the no arbitrage requirement [21, pp. 110-111].

Problem (P) is a free boundary problem for which there is no explicit solution. The simpler European version has no early exercise boundary $S_f(t)$.

The price for a European put, $p_E(S,t)$, satisfies a similar PDE problem in the half space $S > 0$ with conditions (3b, c) replaced by

$$p_E(S,t) = Ke^{-r(T-t)},\ S = 0,\ 0 \leq t < T \tag{4}$$

A logarithmic change of variables reduces this to a constant coefficient problem in $\mathbf{R} \times [0,T)$ which has an explicit solution [21, pp. 48-49] - the so-called Black and Scholes formula.

Because of its added optionality, and in spite of the fact that there is no closed form solution to problem (P), the American version is more widely traded than the European version. Our interest in the problem arose from discussions with traders who wanted precise estimates for the location of the early exercise near expiry (within a day or so of T) because they were holding large positions in options whose assets were "at the money"; i.e., $S(t) \cong K$. In this paper we summarize our work on obtaining precise analytical and numerical estimates for the location of the early exercise boundary for American put options near to, as well as far from, expiry. It is anticipated that these methods can be applied to a wide class of financial derivatives (other payoffs) with free boundaries. In the course of this work we were never able to locate rigorous proofs of the existence, uniqueness and regularity of the solutions to problem (P) nor of the convexity of the early exercise boundary. These appear in [8] and [9] respectively.

During the last 10 to 15 years there has been a great deal of activity in trying to obtain good estimates for the value of the American put option and the location of its early exercise boundary (see, for example [1-7, 12, 13, 15-18]). One of the earliest results on the behavior of the early exercise boundary near expiry shows that this is not a completely trivial problem. More specifically van Moerbeke [20] showed that

$$\frac{dS_f}{dt}\ (t) \rightarrow -\infty \text{ as } t \rightarrow T_-. \tag{5}$$

A more recent result due to Barles et al [2] gives a rigorous estimate for the top order behavior near expiry:

$$S_f(t) \approx K\ \left[1 - \sigma\sqrt{(T-t)|ln(T-t)|}\right],\ t \sim T. \tag{6}$$

In order to provide motivation for the rest of this exposition and to introduce notation for a unified framework for the statement of the results, we take a very short detour to give the equivalent result for the American call option with the underlying asset paying a dividend at the continuous rate D. If $D \neq 0$, this option also has an early exercise boundary $S_{cf}(t)$ for which it can be shown using asymptotic and similarity methods [21, pp. 121-8] that

$$S_{cf}(t) \sim\ \frac{rK}{D}\ \exp\ \left(a\sqrt{\frac{\sigma^2}{2}(T-t)}\right), \tag{7}$$

where a can be determined numerically as the unique solution of a transcendental equation involving error functions. Knowing that the early exercise boundary for the American put starts at the strike at expiry, $S_f(T) = K$, and in view of (5) and (6), the result (7) for the American call might suggest that in terms of the scaled time to expiry $\tau = \frac{\sigma^2}{2}(T-t)$, the behavior of S_f near expiry might take the form.

$$S_f(t) \sim Ke^{a\sqrt{\tau}} \text{ for } \tau = \frac{\sigma^2}{2}(T-t) \to 0. \tag{8}$$

Repeating the analysis for the call one finds, unfortunately, that $a = -\infty$. This suggest that the problem for the put is slightly more complicated, requiring something like $S_f(\tau) \sim Ke^{a(\tau)\sqrt{\tau}}$ with $a(\tau) \to -\infty$ as $\tau \to 0+$.

We conclude this section with a summary of results most relevant for our presentation in terms of the notation and ideas presented above. Specifically, we now write

$$S_f = Ke^{-2\sqrt{\tau}\ \sqrt{\alpha(\tau)}} \text{ where } \tau = \frac{\sigma^2}{2}\ (T-t), \tag{9}$$

(the particular choice of $a(\tau) = -2\sqrt{\alpha(\tau)}$ will become apparent later). The problem then is to find $\alpha(\tau)$ for $\tau \sim 0$ and, if possible, for all $\tau > 0$. The result of Barles et al [2] in equation (6) can be expressed as the first two terms in the expansion of the exponential in (9) with

$$\text{(BBRS)} \quad \alpha(\tau) = -ln\left(\sqrt{T-t}\right) = -ln\sqrt{2\tau/\sigma^2},\ \tau \sim 0. \tag{10}$$

The earlier results of Barone-Adesi & Whaley [4], augmented by MacMillan [17], and summarized in the book by Hull [14, pp. 376-9] can be expressed (after some computations) as the problem of finding $\alpha(\tau)$ from the integral equation

$$\text{(BWM)}\ \sqrt{\pi}\ h(\tau) = \int\limits_{\sqrt{\alpha(\tau)}}^{\infty} e^{-[z-\frac{(k+1)}{2}\sqrt{\tau}]^2}\left\{(1+\eta(\tau))e^{-2\sqrt{\alpha(\tau)}\sqrt{\tau}} - e^{-2z\sqrt{\tau}}\right\}dz \tag{11}$$

where $h(\tau) = 1 - e^{-k\tau}$ and $k = 2r/\sigma^2$ is a dimensionless constant that arises naturally in a scaling argument to be presented later. The more recent results of Kuske & Keller [16] and Bunch & Johnson [6] can also be expressed as the problem of finding $\alpha(\tau)$ from the algebraic equations

$$\text{(KK)}\ \ \alpha\ e^{\alpha} = (9\pi K^2\tau)^{-1/2}, \quad k = 2r/\sigma^2, \tag{12}$$

and

$$\text{(BJ)} \quad \sqrt{\alpha}\ e^{\alpha} - \ \exp(\alpha - (k-1)\sqrt{\tau}\ \sqrt{\alpha}) = b(4k^2\tau)^{-1/2}\ e^{(b-1)(k+1)^2/4} \tag{13}$$

where again $k = 2r/\sigma^2$ and $b = 1 - k^2[(1+k)^2(2+(1+k)^2\tau)]$.

In the next section we investigate the implications of each of these results near expiry and compare them with our own result [10]. In section 3 we look at how well they do for larger times and propose alternate approximations for intermediate times.

2 Behavior Near Expiry

With $S_f(\tau) = ke^{-2\sqrt{\tau}\,\sqrt{\alpha(\tau)}}$ and $k = 2r/\sigma^2$ the problem of determining the behavior of $S_f(\tau)$ when $\tau = \frac{\sigma^2}{2}(T-t) \to 0+$ can be shown to reduce in each of the cases listed in the previous section (equations (11-13)) to finding $\alpha(\tau)$ from

$$\text{(BWM)} \qquad \sqrt{\tau}\,\sqrt{\alpha}\,e^{\alpha} \;\approx\; 1/\sqrt{4\pi k^2}, \tag{14}$$

$$\text{(KK)} \qquad \sqrt{\tau}\,\alpha\,e^{\alpha} \;\approx\; 1/\sqrt{9\pi k^2},\ \text{and} \tag{15}$$

$$\text{(BJ)} \qquad \sqrt{\tau}\,\sqrt{\alpha}\,e^{\alpha} \;\approx\; \left((1-\frac{1}{2}\left(\frac{k}{1+k}\right)^2\Big)/4k^2\right)^{-1/2}. \tag{16}$$

Our own results [10] show that $\alpha(\tau)$ should be obtained from

$$\text{(SSC)} \quad \sqrt{\tau}\,e^{\alpha} \;=\; 1/\sqrt{4\pi k^2}. \tag{17}$$

Solving equation (17) one finds that for $\tau \to 0$

$$\text{(SSC)} \quad \alpha(\tau) \;\sim -ln\left(\sqrt{4\pi k^2\tau}\right) \;=\; -\frac{\xi}{2} \tag{18}$$

where $\xi = ln(4\pi k^2\tau)$. Similarly, equation (14) gives:

$$\frac{1}{2}ln\alpha + \alpha \;\sim\; -ln\left(\sqrt{4\pi k^2\tau}\right).$$

which implies (since $|ln\ \alpha| < |\alpha|$ for $\alpha \to -\infty$) that

$$\text{(BWM)} \quad \alpha \sim -\frac{\xi}{2} - \frac{1}{2}ln\left(-\frac{\xi}{2}\right), \tag{19}$$

that is, has a $ln(|ln|)$ correction to (18). Similarly, starting with equation (15), one obtains

$$\text{(KK)} \quad \alpha \sim -\frac{\xi}{2} + ln\left(\frac{3}{2}\right) - ln\left(-\frac{\xi}{2} + ln\left(\frac{3}{2}\right)\right). \tag{20}$$

Likewise, from equation (16) one obtains a $ln\ ln$ as well as a constant modification to (18). Of course, all of these results agree with Barles et al that, to the first approximation,

$$\alpha(\tau) \sim -ln\left(\sqrt{T-T}\right) \text{ as } T-t \to 0+ \tag{21}$$

The only questions then are which is the correct constant and whether *ln ln* corrections are appropriate. These questions can be answered by applying classical free boundary methods [11, 19] to problem (P).

Once again, with $\tau = \frac{\sigma^2}{2}(T-t)$ and $k = 2r/\sigma^2$, let $x = ln\left(\frac{S}{K}\right)$ and denote the early exercise boundary in the (x,t) coordinates by

$$s(\tau) = ln\left(\frac{S_f}{K}\right); \text{ i.e., } s(\tau) = -2\sqrt{\tau}\ \sqrt{\alpha(\tau)}. \tag{22}$$

In terms of the scaled option price

$$p_{\text{new}} = \begin{cases} 1 - S/K & S < S_f \\ p/K & S > S_f \end{cases} \tag{23}$$

Problem (P) becomes (dropping the subscript):

$$p_\tau - \left\{ p_{xx} + (k-1)p_x - kp \right\} = kH(s(\tau) - x), \tag{24-a}$$

$$p(x,0) = \max\,(1 - e^x, 0), \tag{24-b}$$

where the *rhs* of equation (24a) is the Heaviside function

$$kH(s(\tau) - x) = \begin{cases} k & -\infty < x < s(\tau) \\ 0 & s(\tau) < x < \infty. \end{cases} \tag{25}$$

Of course, the solution $(p(x,\tau), s(\tau))$ of equations (24) still satisfies the free boundary conditions inherited from problem (P):

$$p(s(\tau),\tau) = 1 - e^{s(\tau)} \tag{26-a}$$

$$p_x(s(\tau),\tau) = -e^{s(\tau)} \tag{26-b}$$

for all $0 < \tau < \sigma^2 T/2$. Thus, in the (x,τ) coordinates problem (P) becomes a forward, constant coefficient heat-type equation with the dimensionless constant $k = 2r/\sigma^2$ entering in a natural manner. The fundamental solution of the differential operator on the *lhs* of equation (24a) is

$$\Gamma(x,\tau) = e^{-k\tau}\ F(x + (k+1)\tau, \tau) \tag{27-a}$$

with F given by the heat kernel

$$F(z,\tau) = \frac{1}{2\sqrt{\pi\tau}}\ e^{-z^2/4\tau}. \tag{27-b}$$

The solution of equations (24) can now be written in terms of Γ *and the free boundary* $s(\tau)$:

$$p(x,\tau) = \int_{-\infty}^{\infty} p(y,0)\Gamma(x-y,\tau)dy + k\int_0^{\tau}\int_{-\infty}^{s(u)} \Gamma(x-y,\tau-u)dy\ du. \tag{28}$$

That is, if $s(\tau)$ is known, then (28) is a solution to the problem. The classical methods for free boundary problems involve substituting (26a) or (26b) into equation (28) to obtain a non-linear integral equation for $s(\tau)$ which then is used as above to find $p(x,\tau)$ from (28). However, for the American put problem of interest here, our work shows that this is not an effective approach because, near expiry, the first term (which is the value of the European option) and the second term (which is the premium paid for American optionality) are about the same size. For this reason we take a different approach so that we can better control the terms. Specifically, taking the time derivative, one obtains

$$p_\tau(x,\tau) = \Gamma(x,\tau) + k \int_0^\tau \Gamma(x - s(u), \tau - u)\ \dot{s}(u)\ du. \tag{29}$$

Also, one can show that on the free boundary $p_\tau(s,(\tau),\tau) = 0$. Thus, from (29) one obtains

$$\Gamma(s(\tau),\tau) = -k \int_0^\tau \Gamma(s(\tau) - s(u), \tau - u)\ \dot{s}(u)\ du \tag{30}$$

Now, for small $0 < u < \tau$,

$$\Gamma(s(\tau) - s(u), \tau - u) = F(s(\tau) - s(u), \tau - u)\ [1 + O(\tau)] \tag{31}$$

so that with $\eta = (s(\tau) - s(u))/2\sqrt{\tau - u}$, the *rhs* of (30) can be approximated, for small τ, by

$$rhs\ (30)\ \simeq -k \int_0^{s(\tau)/2\sqrt{\tau}(\to -\infty)} \left[1 - \frac{s(\tau) - s(u)}{2\dot{s}(u)(\tau - u)}\right]^{-1} \frac{e^{-\eta^2}}{\sqrt{\pi}}\ d\eta \tag{32}$$

Moreover, by the convexity of s [9], $0 \le (s(\tau) - s(u))/2\dot{s}(u)(\tau - u) \le \frac{1}{2}$ and $\to \frac{1}{2}$ as $\tau \to 0$ (uniformly in u). Thus rhs (30) $\to k$ as $\tau \to 0$. As a result, from (30) one has for $\tau \simeq 0$ that

$$\Gamma(s(\tau),\tau) = \frac{1}{2\sqrt{\pi\tau}}\ \exp\left\{-\left[\frac{(s(\tau) + (k-1)\tau)^2}{4\tau}\right]\right\}\ e^{-k\tau} \simeq k \tag{33}$$

Solving and ignoring $O(\tau)$ terms one has that for $\tau \simeq 0$

$$s(\tau) \simeq -2\sqrt{\tau}\ \sqrt{-ln(2\sqrt{\pi}\ k\tau^{1/2})} \tag{34}$$

This provides the reason for choosing the form (9) and shows that

$$\alpha(\tau) = -ln(2\sqrt{\pi}k\tau^{1/2}) = -\frac{ln(4\pi k^2\tau)}{2} = -\frac{\xi}{2} \tag{35}$$

as indicated earlier in equation (18). This establishes that among the near expiry approximations (10), (14-17), our choice of $\alpha(\tau)$ (equation (17)) provides the correct choice of the constant term. In the next section we compare approximations (11-13), which as seen lead to *ln ln* corrections, to several alternative intermediate time approximations that follow from (30) using our approach.

3 Intermediate Time Approximations

One could again take equation (30) and, beginning with (35) as the initial approximation for α, use Mathematica to find by iteration an expansion in terms of $1/\xi$ [10]:

$$\alpha = -\frac{\xi}{2} - \frac{1}{\xi} + \frac{1}{2\xi^2} + \frac{17}{3\xi^3} - \frac{51}{4\xi^4} - \frac{1148}{15\xi^5} + \frac{398}{\xi^6} + \cdots . \tag{36}$$

Of course, these are only valid up to $\xi = 0$, i.e., $\tau = 1/4\pi k^2$. In fact, numerical simulations show that approximations of this sort break down earlier, and more importantly, the third (up to $17/3\xi^3$) is better than the second (up to $\frac{1}{2\xi^2}$) only extremely close to expiry. However, a slight modification of this approach can be shown [10] to lead to improved approximations. Because the special choice of $\xi = ln(B\tau)$ with $B = 4\pi k^2$ eliminates the constant term in the expansion for $\alpha(\tau)$, it is possible to translate variables so that the (artificial) singularity (at $\xi = 0$) occurs at larger values of τ. For example, for any $a > 0$, replacing ξ by $\xi - a = ln(4\pi k^2 \tau e^{-a})$, then (36) becomes

$$\alpha(\tau) = -\frac{\xi}{2} - \frac{1}{\xi - a} + \frac{(1+2a)}{2(\xi - a)^2} + \frac{17/3 - a - a^2}{(\xi - a)^3} + \cdots \tag{37}$$

which can now be evaluated up to $\tau < e^a/4\pi k^2$. With $a = 1.93242$, the positive solution of $17/3 - a - a^2 = 0$, the expression

$$\text{(CCS3)} \quad \alpha = -\frac{\xi}{2} - \frac{1}{\xi - a} + \frac{1/2 + a}{(\xi - a)^2}, \; a = 1.93242 \tag{38}$$

is better than (36) in accuracy and interval of validity.

One can also imagine expressing ξ as a function of α. Using (36) (which followed from (30)) one finds [10]

$$-\frac{\xi}{2} = \alpha + ln\left[1 - \frac{1/2}{\alpha + a} - \frac{a/2}{(\alpha + a)^2} + \frac{(1 - a^2)}{2(\alpha + a)^3} + \cdots\right] \text{ as } \alpha \to \infty, \tag{39}$$

Truncating at the third term by taking $a = 1$ one obtains

$$\alpha + ln\left[1 - \frac{1}{2(\alpha + 1)} - \frac{1}{2(1 + \alpha)^2}\right] = -\frac{\xi}{2} = -\frac{1}{2}\, ln(4\pi k^2 \tau), \tag{40}$$

or equivalently,

$$\text{(CCSa)} \quad \sqrt{\tau}\, e^{\alpha} \left[1 - \frac{1}{2(\alpha+1)} - \frac{1}{2(\alpha+1)^2}\right] = 1/\sqrt{4\pi k^2}. \qquad (41)$$

It is interesting to compare this expression to (14-16) where positive powers of α appears on the *lhs* and lead to less accurate *ln ln* corrections to (18).

We can also obtain an implicit approximation that interpolates between the short time behavior (18) and the infinite horizon solution with $S_f(t) \to k/k+1$ (the strike K is taken to be 1) as $t \to -\infty$ due to Merton. Once again, beginning with (30) a series of calculations [10] leads to the following expression to be solved for α:

$$\text{(CCSb)} \quad \sqrt{\tau}\, e^{\alpha}\, \frac{2}{\pi} \int_0^{\sqrt{\alpha}} e^{-z^2} dx \left[\frac{e^{\alpha} + e^{1/\alpha}}{e^{\alpha} + (2k\, ln(\frac{k+1}{k}))e^{1/\alpha}}\right] = 1/\sqrt{4\pi k^2} \quad (42)$$

4 An ODE/IODE Approximation

All of the above provide relatively good local approximations for the location of the early exercise boundary (i.e., given t they provide $S_f(t)$). In this section we provide our most accurate, rigorous global approximation for the entire free boundary. Specifically, we shall obtain a simpler ODE whose solution provides the first approximation to the early exercise boundary. Iterating this through a non-linear integro/ordinary differential equation provides a sequence of improving global approximations which can be proved [8] to converge to the unique solution for the free boundary.

Once again, starting with (30) a series of complicated calculations gives, for all $0 < \tau$, the integro-differential equation

$$\dot{s}(\tau) = \frac{s(\tau)}{2k\tau}\, \Gamma(s(\tau), \tau)\, [1 + m(\tau)] \qquad (43\text{-a})$$

where

$$m(\tau) = k \int_0^{\tau} \left[\frac{s(\tau) - s(u)}{\tau - u}\, \frac{2\tau}{s(\tau)} - 1\right] \frac{\Gamma(s(\tau) - s(u), \tau - u)}{\Gamma(s(\tau), \tau)}\, \dot{s}(u) du. \quad (43\text{-b})$$

and Γ is the explicitly known fundamental solution (27a). These are to be solved with $s(0) = 0$. One could rewrite the problem in terms of α which must then be solved consistent with the near expiry behavior of α proven earlier (18) i.e., $\alpha + \frac{\xi}{2} \to 0$ as $\xi := ln(4\pi k^2 \tau) \to -\infty$. The iteration scheme is [2]:

$$m^0(\tau) = 0 \tag{44-a}$$

$$\frac{d\alpha^{(n)}}{d(ln(4\pi k^2\tau))} = \alpha^{(n)}\left\{\frac{1+m^{(n)}(\tau)}{\sqrt{4\pi k^2\tau}}\exp\left[\frac{k-1}{2}\sqrt{\tau\alpha^{(n)}} -\alpha^{(n)} - \frac{(k+1)^2}{4}\tau\right] - 1\right\} \tag{44-b}$$

$$\alpha^{(n)}(\tau) + \frac{ln(4\pi k^2\tau)}{2} \to 0 \text{ as} \tau \to 0 \tag{44-c}$$

$$s^{(n)}(\tau) = -2\sqrt{\tau}\ \alpha^{(n)}\ (\tau) \tag{44-d}$$

$$m^{(n+1)}(\tau) = k\int_0^\tau \left\{\frac{s^{(n)}(\tau) - s^{(n)}(u)}{\tau - u} \cdot \frac{2\tau}{s^{(n)}(\tau)} - 1\right\} \cdot \cdot k\frac{\Gamma(s^{(n)}(\tau) - s^{(n)}(\tau), \tau - u)}{\Gamma(s^{(n)}(\tau), \tau)} \cdot \dot{s}^{(n)}(u)du \tag{44-e}$$

The first iterate, $\alpha^{(0)}(\tau)$, is a solution of an ODE which can be obtained instantaneously by Mathematica. The higher iterates take about one minute each and by the fifth iterate the process has stabilized. Not only is this an efficient procedure for obtaining accurate estimates of the entire early exercise boundary but we can also prove rigorously [8] that the iterates converge to the unique solution.

5 Conclusions

In this note we have summarized some of our results on the early exercise boundary for American put options. Our rigorous results include:

1. Problem (P) (see equations (3)) has a unique solution given by the expression (28) with $s(\tau)$ a solution of (30) or (43) [8].
2. The transformed boundary $s(\tau)$ is convex ($\ddot{s}(\tau) > 0$ for all $\tau > 0$) and hence the original boundary is concave $\left(\frac{d^2S_f}{dt^2}(t) < 0\right)$ [9].
3. The iteration scheme (44) converges to the unique boundary $s(\tau)$ [8].

We also provide several analytical and numerical approximations for the early exercise boundary that arise from this work. Near expiry we derived (17) which should be contrasted with (10) and (14-16). We also provide several intermediate time estimates (38, 41, 42) which should be contrasted with (11-13). Finally, we provide an efficient and accurate numerical aporoximation for the entire boundary in the form of the iterative scheme (44).

The figure below compares the errors of the various approximations over a wide range of time intervals (ODE is (43a) with $m \equiv 0$, ode1 is the first iterate of (44), etc., BJ is (13), BWM is (11), CCSa is (41) and CCSb is (42)). The "exact" solution to which these are compared is the fifth iterate of (44). In these calculations the strike price is taken to be 1. The cusps in the graphs indicate a change in the sign of the error.

Figure: Error (in $\log_{10}$) of $S_f(t)$ relative to ode5
for various approximations with $r = 0, \sigma = 0.25$ and $K = 1$.

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