ALGEBRA 2 – MIDTERM REVIEW – UNIVERSITY OF PITTSBURGH, FALL 2019

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Here are the midterm instructions that you will see on the exam: *You do not need to prove what you claim in detail, but you should give justification in all cases. The most common way to do this is to clearly state a theorem from the course, and then explain why this theorem implies something. If you cannot remember a theorem precisely, try to state clearly the consequence of the theorem that you want to use.*

Additional note: The final topic in the list below ("The Nullstellensatz") will only appear on the midterm in that you may be expected to know definitions and what major theorems say. The only exception is the topic of radical ideals, which is fair game to appear in more detail.

1. INITIAL TOPICS

- The notion of a ring and a ring homomorphism
- Modules over a ring, and their homomorphisms
- Example of $\mathbb{Z}$, and its special property (initial object in the category of (unital) rings)
- Recall of well-ordering principle, induction
- Injections, surjections, and isomorphisms of modules
- Sums and products of modules, and their universal properties
- Free modules

2. TOPICS RELATED TO THE STRUCTURE THEOREM FOR FINITELY GENERATED MODULES OVER A PID

- Abelian groups are $\mathbb{Z}$-modules
- Finitely generated modules
- Isomorphism of categories between "k-vector spaces with a k-linear endomorphism" and "$k[x]$-modules"
- Commutative domains and PIDs
- The structure theorem for finitely generated modules over a PID
- Canonical forms of matrices, minimal and characteristic polynomials; and their relationship with this structure theorem
- UFDs; PIDs are UFDs
- Recall of ideals, prime ideals, and maximal ideals
- Example: Euclidean domains are PIDs
- Noetherian modules
- Torsion elements of modules, torsion modules, and torsion-free modules
- Theorem on the structure of submodules of a finitely generated free module over a PID (main part of the argument for the structure theorem for finitely generated modules over a PID)
- The annihilator ideal of a module

3. INTRODUCTION TO COMMUTATIVE ALGEBRA, ESPECIALLY FINITENESS NOTIONS OF COMMUTATIVE RINGS

- Chinese remainder theorem
- Products of rings
- Rings of fractions, also known as localizations; fraction fields
- The universal property of localization homomorphisms

- Local rings (definition)
- Krull dimension
- Behavior of ideals under contraction and extension along a localization homomorphism
- Algebras over a commutative ring: finite generation, finite presentation
- Noetherian rings and the Hilbert basis theorem
- Gauss's lemma, and the behavior of the UFD property by taking polynomial algebras
- Recall of algebraicity of field extensions
- Integrality of elements and integrality of R-algebras
- Integral closure
- Relation of integrality to other finiteness properties

4. Number rings and Dedekind domains

- Number rings, rings of integers of number fields, and the definition of Dedekind domains
- Properties of Dedekind domains
- Behavior of integral closure under localization
- DVRs, and various characterizations of DVRs
- Equivalent definition of Dedekind domains in terms of DVRs
- Fractional ideals; over Dedekind domains, they are invertible
- The class group of a Dedekind domain (definition)
- Rings of integers of quadratic number fields
- Example of failure of unique factorization in rings of integers of quadratic number fields

5. Introductory algebraic geometry: The Nullstellensatz

- The coordinate ring of affine space
- Affine algebraic sets: zero sets of polynomials in affine space
- The ideal of polynomial functions on affine space that vanish on an algebraic subset
- Radical ideals and radicals of ideals
- Coordinate rings of affine algebraic sets
- Morphisms of affine algebraic subsets
- The relationship between morphisms of affine algebraic subsets and homomorphisms
- The Nullstellensatz and weak Nullstellensatz
- Noether's normalization lemma
- The going up and going down theorems