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Algebra II - Problem Set 6

2. Prove that if $s_1, \dots, s_n \in S$ are integral over R, then the ring $R[s_1, \dots, s_n]$ is a finitely generated R-module.

Proof. The proof is by induction on n. The case $n = 1$ is via the key proposition on integrality, namely that $s \in S$ is integral over R if and only if $R[s]$ is a finitely generated R-module. Suppose the result is true for some n, and let $s_1, \dots, s_{n+1} \in S$ be integral over R. By the induction hypothesis, $R[s_1, \dots, s_n]$ is finitely generated as an R-module. Since s_{n+1} is integral over R and $R \subset R[s_1, \dots, s_n]$, s_{n+1} is integral over $R[s_1, \dots, s_n]$. Applying the key proposition again, we see that

$$R[s_1, \dots, s_{n+1}] = R[s_1, \dots, s_n][s_{n+1}]$$

is finitely generated as an $R[s_1, \dots, s_n]$ module. Now apply the result from the second homework, stated below, with $S = R[s_1, \dots, s_n]$ and $M = R[s_1, \dots, s_{n+1}]$, to conclude that $R[s_1, \dots, s_{n+1}]$ is finitely generated as an R-module. $\square$

Theorem. *Let R be a commutative ring, S an R-algebra, and M a left R-module. Assume that S is finitely generated as an R-module. Then M is a finitely generated left S-module if and only if it is finitely generated as an R-module.*