ALGEBRA 2 – PROBLEM SET 2 – UNIVERSITY OF PITTSBURGH, FALL 2019

Due on Friday September 13.

(1) Let R be a commutative ring and let S be a R-algebra. Let M be a left S-module. We have discussed that it follows that any left S-module is naturally a (left) R-module.
   (a) (2 points) Write down this natural R-module structure on M. (Just define the R-action; you don't need to write out a full proof that it satisfies the module actions.)
   (b) (2 points) Write down the natural R-module structure on S. (Just define the R-action; you don't need to write out a full proof that it satisfies the module actions.)
   (c) (6 points) Assume that S is finitely generated as an R-module. Prove that M is a finitely generated left S-module if and only if it is finitely generated as an R-module.

(2) (10 points) Let R be a ring. In this problem, "module" means "left module." Let the free object functor $\mathcal{F} : \underline{\text{Sets}} \to \underline{R\text{-modules}}$ be defined on objects by sending a set X to the R-module $R^{\oplus X}$ defined in class. Let the forgetful functor $\mathcal{G} : \underline{R\text{-modules}} \to \underline{\text{Sets}}$ be defined by sending a R-module to its underlying set.
   (a) Write down what the free object functor $\mathcal{F}$ does to morphisms.
   (b) Write down what the forgetful functor $\mathcal{G}$ does to morphisms.
   (c) Look up the definition of a pair of adjoint functors. Prove that $\mathcal{F}$ and $\mathcal{G}$ are a pair of adjoint functors, and be sure to say which one is the left (or right) adjoint.

Not to turn in, but please read for any new content. Of course, you can do them as exercises if you like! Contact me or your classmates if you would like to propose more interesting problems.

(1) Look up the definitions of "category" and "functor" (in Appendix II of Dummit and Foote's book, for example). Convince yourself that you want to work more concretely than this level of generality.
(2) Give an example of a pair of adjoint functors other than the one above.
(3) Think about how the category of $k[x, y]$-modules is the same as the category of k-vector spaces with a ordered pair of *commuting* endomorphisms.
(4) Look up the definition of a "monomorphism" and an "epimorphism" – these are certain kinds of morphisms in a category. Determine whether an injection or surjection of left R-modules fits one or both of these definitions.