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Algebra II - Problem Set 2

(1) Let R be a commutative ring and let S be a R-algebra. Let M be a left S-module. A left S-module is naturally a left R-module.

(a) Write down the natural R-module structure on M.
Let $f : R \to S$, which satisfies $f(1) = 1$ and $f(R) \in Z(S)$, be the ring homomorphism associated with the R-algebra structure on S. Since M is a left S-module, we have an action

$$\begin{aligned} S \times M &\to M \\ (s, m) &\mapsto sm \end{aligned}$$

Then we can define the R action on M giving it a left R-module structure by

$$\begin{aligned} R \times M &\to M \\ (r, m) &\mapsto f(r)m \end{aligned}$$

(b) Write down the natural R-module structure on S.
The action is given by

$$\begin{aligned} R \times S &\to S \\ (r, s) &\mapsto f(r)s \end{aligned}$$

where we are using the ring multiplication in S.

(c) Assume that S is finitely generated as an R-module. Prove that M is a finitely generated left S-module if and only if it is finitely generated as an R-module.

Proof. Since S is a finitely generated R-module, there is a subset $\{s_i\}_{i=1}^n \subset S$ such that for any $s \in S$ there exist $r_i \in R$ so

$$s = \sum_{i=1}^{n} f(r_i)s_i.$$

Assume that M is finitely generated as an R-module. Then there is a subset $\{m_i\}_{i=1}^k \subset M$ such that for any $m \in M$ we have $r_i \in R$ so

$$m = \sum_{i=1}^{k} f(r_i)m_i.$$

Since each $f(r_i) \in S$, we see that the same subset $\{m_i\}_{i=1}^k$ generates M as a left S-module. Conversely, assume that M is finitely generated as an S-module. Then there is a subset $\{m_i\}_{i=1}^k$ such that for all $m \in M$ there are $s_i' \in S$ so

$$m = \sum_{i=1}^{k} s_i' m_i.$$

Let $m \in M$. Find $s_i' \in S$ that satisfy the above formula. For each s_i', there are $r_{ij} \in R$ for $j = 1, \dots, n$ such that

$$s_i' = \sum_{j=1}^{n} f(r_{ij}) s_j.$$

Using associativity of the S-action on M, we can write m as

$$m = \sum_{i=1}^{k} \left(\sum_{j=1}^{n} f(r_{ij}) s_j \right) m_i = \sum_{i=1}^{k} \sum_{j=1}^{n} f(r_{ij})(s_j m_i).$$

This shows that $\{s_j m_i\}_{i=1,j=1}^{k,n} \subset M$ is a finite generating set for M as an R-module. $\square$

(2) Let R be a ring. In this problem, "module" means "left module." Let the free object functor $\mathcal{F}\colon \underline{\text{Sets}} \to \underline{R\text{-modules}}$ be defined on objects by sending a set X to the R-module $R^{\oplus X}$. Let the forgetful functor $\mathcal{G}\colon \underline{R\text{-modules}} \to \underline{\text{Sets}}$ be defined by sending a R-module to its underlying set.

(a) Write down what the free object functor $\mathcal{F}$ does to morphisms

Let X and Y be sets, and $f \in \operatorname{Hom}(X, Y)$. Apply the universal property of free R-modules to $R^{\oplus X}$ and the map $i_Y \circ f$ (here i_X and i_Y are the inclusions defined in class), and define $\mathcal{F}f$ to be the unique R-module morphism which makes the diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ R^{\oplus X} & \overset{\mathcal{F}f}{\dashrightarrow} & R^{\oplus Y} \end{array}$$

Functoriality of $\mathcal{F}$ follows essentially from the universal property of free R-modules. For instance, $\mathcal{F}(\mathrm{id}_X)$ is the unique morphism such that $\mathcal{F}(\mathrm{id}_X) \circ i_X = i_X$. Since $\mathrm{id}_{R^{\oplus X}}$ is also an R-module morphism satisfying this property, we must have $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{R^{\oplus X}}$ by uniqueness. To check that $\mathcal{F}$ preserves composition, let X, Y, Z be sets and $f \in \operatorname{Hom}(X, Y), g \in \operatorname{Hom}(Y, Z)$. Then $\mathcal{F}(g \circ f)$ is the unique morphism satisfying

$$\mathcal{F}(g \circ f) \circ i_X = i_Z \circ g \circ f.$$

Since

$$\mathcal{F}(g) \circ \mathcal{F}(f) \circ i_X = \mathcal{F}(g) \circ i_Y \circ f = i_Z \circ g \circ f,$$

uniqueness implies that $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

(b) Write down what the forgetful functor $\mathcal{G}$ does to morphisms. Given R-modules V and W and a morphism $g \in \operatorname{Hom}(V, W)$, define $\mathcal{G}(g) = g$ as just the map of sets (forgetting R-linearity). Then it is easy to observe that $\mathcal{G}$ is a functor.

(c) Look up the definition of a pair of adjoint functors. Prove that $\mathcal{F}$ and $\mathcal{G}$ are a pair of adjoint functors, and be sure to say which one is the left (or right) adjoint.

My reference for the definition of adjoint functors is Tom Leinster's book *Basic Category Theory*, available on the arXiv.

We will show that $\mathcal{F}$ is **left adjoint** to $\mathcal{G}$. In particular, we must show that for any set X and R-module V

$$\operatorname{Hom}(\mathcal{F}(X), V) \cong \operatorname{Hom}(X, \mathcal{G}(V))$$

naturally, meaning there is a specified bijection above for each X and V which satisfies a naturality axiom. Given a set X and R-module V, the correspondence is denoted in both directions by an over bar:

$$(g\colon \mathcal{F}(X) \to V) \mapsto (\overline{g} : X \to \mathcal{G})$$

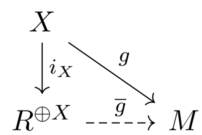
and vice versa. The naturality axioms are

$$\overline{(\mathcal{F}(X) \xrightarrow{g} V \xrightarrow{q} W)} = (X \xrightarrow{\overline{g}} \mathcal{G}(V) \xrightarrow{\mathcal{G}(q)} \mathcal{G}(W)),$$

and

$$\overline{(X \xrightarrow{p} Y \xrightarrow{f} \mathcal{G}(V))} = (\mathcal{F}(X) \xrightarrow{\mathcal{F}(p)} \mathcal{F}(Y) \xrightarrow{\overline{f}} V).$$

Proof. Let X be a set, M an R-module. Given $f \in \mathrm{Hom}(R^{\oplus X}, M)$, define $\overline{f}\colon X \to M$ by $\overline{f} = f \circ i_X$. Then $\overline{f} \in \mathrm{Hom}(X, M)$ since it is a map of sets. On the other hand, given $g \in \mathrm{Hom}(X, M)$, define $\overline{g} \in \mathrm{Hom}(R^{\oplus X}, M)$ by appeal to the universal property of free R-modules. That is, let $\overline{g}$ be the unique morphism of R-modules that makes the diagram commute:



Now we must check that the naturality axioms are satisfied. Consider a sequence of maps

$$R^{\oplus X} \xrightarrow{\;g\;} M \xrightarrow{\;q\;} M'$$

where X is a set, M and M' are R-modules. From the above definitions,

$$\overline{q \circ g} = q \circ g \circ i_X = q \circ \overline{g} = \mathcal{G}(q) \circ \overline{g},$$

which verifies the first naturality condition.
Finally, consider a sequence of maps

$$X \xrightarrow{\;p\;} Y \xrightarrow{\;f\;} \mathcal{G}(M)$$

where X, Y are sets and M is an R-module. Then $\overline{f}\colon R^{\oplus Y} \to M$ is the unique morphism satisfying $f = \overline{f} \circ i_Y$. Similarly, $\overline{f \circ p}\colon R^{\oplus X} \to M$ is the unique morphism satisfying $f \circ p = \overline{f \circ p} \circ i_X$. Also, $\mathcal{F}(p)$ is the unique morphism from $R^{\oplus X}$ to $R^{\oplus Y}$ for which

$$\mathcal{F}(p) \circ i_X = i_Y \circ p.$$

Using the above compositions, we have

$$\overline{f} \circ \mathcal{F}(p) \circ i_X = \overline{f} \circ i_Y \circ p = f \circ p,$$

Since $\overline{f \circ p}$ is the unique morphism satisfying this composition, we have that

$$\overline{f \circ p} = \overline{f} \circ \mathcal{F}(p).$$

Having verified the naturality axioms, we see that the free object functor is left adjoint to the forgetful functor. □