# ALGEBRA 2 – PROBLEM SET 1 – UNIVERSITY OF PITTSBURGH, FALL 2019

Due on Friday September 6.

**Correction:** As discussed in class on Friday, "unital" conditions need to be added to the definition of rings, their homomorphisms, and also to the definition of module.

(1) (10 points) This question is based on Silvino's question at the end of lecture on Wednesday (Aug 28). Let $R$ be a commutative ring. Let $V$ and $W$ be $R$-modules. We consider the abelian group $\mathrm{Hom}_R(V, W)$, which is the morphisms from $V$ to $W$ in the category of left $R$-modules.
  (a) Show that the $R$-action on $V$ endows $\mathrm{Hom}_R(V, W)$ with the structure of an $R$-module.
  (b) Show that the $R$-action on $W$ endows $\mathrm{Hom}_R(V, W)$ with the structure of an $R$-module.
  (c) Observe that these two $R$-actions above are identical.

(2) (10 points) Let $R$ be commutative, and let $V$ be an $R$-module. Using (1c) and what goes into it, prove that $\mathrm{End}_R(V)$ is an $R$-algebra, where the multiplication operation is composition. Note that this is, from a certain perspective, a multi-part question: you will need to
  - prove that $\mathrm{End}_R(V)$ is a ring, which involves checking the list of ring axioms
  - write down a natural map $R \to \mathrm{End}_R(V)$ and prove that its image is contained in the center of $\mathrm{End}_R(V)$.

Terminology, FYI:

- A category whose Homs are equipped with a structure of abelian groups is called an additive category. This means that the category of left (or right) $R$-modules is an additive category ($R$ is a ring). It also satisfies addition axioms (not discussed here) making it an abelian category.
- Let $R$ be a commutative ring. Then we have observed that $\mathrm{Hom}_R(V, W)$ from (1) has the structure of an $R$-module. This is known as the *internal Hom* in the category of $R$-modules, because the homomorphism set in this category is again an object of the category.
- Let $R, S$ be rings. Let $V$ be an abelian group equipped with the structure of a left $R$-module and right $S$-module. Then $V$ is called a $(R, S)$-*bimodule*.

Not to turn in, but please read for any new content. Of course, you can do them as exercises if you like! Contact me or your classmates if you would like to propose more interesting problems.

(1) Write out a complete proof that $\mathbb{Z}$ is the initial object in the category of rings, filling in the outline given in class. (If this was a problem to turn in, you would be welcome to use mathematical induction as opposed to starting from the definition of $\mathbb{Z}$ given! Other reasonable such assumptions are permitted; just ask me if you have a question as to whether one is OK or not. If it makes the problem trivial, it is probably not OK!)

(2) The *characteristic* of a ring $R$, denoted $\mathrm{char}\, R$, is defined to be
  - the least $n \in \mathbb{Z}_{\geq 1}$ such that $n = 0$ in $R$, if such an $n$ exists; or,
  - the characteristic is said to be 0 if no such $n$ exists.

  (Recall that any $n \in \mathbb{Z}$ gives rise to a a well-defined element of any ring.)
  (a) Let $n \in \mathbb{N}$. Prove that $\mathrm{char}\, R$ divides $n$ if there exists a ring homomorphism $\mathbb{Z}/n\mathbb{Z} \to R$. Prove that the converse is true if $n \in \mathbb{Z}_{\geq 1}$.
  (b) Let $k$ be a field. Prove that $\mathrm{char}\, k$ is either 0 or is a prime number. (Terminology, FYI: any field has a *prime subifeld*. When $p = \mathrm{char}\, k > 0$, it is the image of the canonical homomorphism $\mathbb{F}_p \to k$. When $\mathrm{char}\, k = 0$, it is the image of the canonical homomorphism $\mathbb{Q} \to k$.

(3) Let $k$ be a field and let $R$ be a ring. Given a ring homomorphism $k \to R$, prove that it is injective.

(4) Observe that the statements in assigned problem (1) are not true when $R$ is not commutative (in the sense that there is no natural right or left $R$-module structure on $\mathrm{Hom}_R(V, W)$. Andrew

pointed out this discussion of this fact: https://math.stackexchange.com/questions/637807/why-is-operatornamehomm-n-not-necessarily-an-r-module. Later, we may discuss the following fact, which is the way in which the results of (1) are recovered in the non-commutative setting:

- Let $R, S, T$ be rings. Let $V$ be a $(R, S)$-bimodule and let $W$ be a $(R, T)$-bimodule. Then the abelian group of left $R$-module homomorphisms $\mathrm{Hom}_R(V, W)$ has the natural structure of a $(S, T)$-bimodule.