Math 201a, Topics in Algebra: Moduli Theory of Representations
Brandeis University, Spring 2014

http://people.brandeis.edu/~cwe/201a_2014/index.html

Course Summary

The overall theme of the course is that homological data exert control over the geometry of a moduli space; we will focus on moduli spaces of representations. Each of these topics – representations, moduli spaces, and homological algebra – will be introduced at a pace calibrated to the background and interests of people in the course, which will be surveyed as we begin. I expect the first half-or-so of the course to consist of the introduction of these topics and the exploration some common first examples of homological control of geometry that will appear along the way. The rest of the course will explore concepts introduced in these two papers:

- *The deformation theory of representations of fundamental groups of compact Kähler manifolds* by Goldman and Millson
- *Introduction to A_∞-algebras and modules* by Bernhard Keller

The first will serve to explain how the local structure of a moduli space of representations is controlled by homological data, namely, a differential graded Lie algebra. The second will be used to show how some of the global structure of a moduli space of representations is controlled by homological data, namely, the A_∞-algebra structure of a certain Yoneda algebra.

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Office:	Goldsmith 206
Meeting Location:	Goldsmith 116
First meeting time:	Monday January 13, 12:00PM-12:50PM

Preferences for the weekly meeting schedule and surveys on background will be collected at the first meeting.

Participation

Graduate students interested in attending should register for the course. Each student will undertake a project related to some aspect of the course. Once I have surveyed the people interested in the course, I will post possible topics. The end product will be a talk, presented in the course, and a writeup of the notes for the talk. The topics will very likely be distributed through the course material, so that there will be talks by students interspersed through the course.

Undergraduate students should send me and email and come see me to discuss participating in the course.

Course Description

After introducing and overviewing the course, we will first discuss representation theory, which for this course refers to the finite-dimensional representations of finitely generated groups/algebras, often over a field. We will also introduce quivers as a way of studying the category of representations of a given algebra. As a way of transitioning to moduli theory, we will mention the notions of finite, tame, and wild representation type.

The next topic is moduli theory and the functorial perspective on schemes. We will then construct moduli schemes of representations and observe the natural group action on them. Finally, we will discuss local properties of these moduli schemes, focusing on deformation problems. Background in scheme theory and algebraic groups will be discussed as needed.

Having seen a first example of homological data (a cohomology group) controlling deformations, we will more formally introduce homological algebra: complexes, the derived category, derived functors, etc. The main examples we will use to introduce the subject are group representations (so that group cohomology is the homological data) and more general algebras over a field.

After this, we will explore Goldman and Millson's construction of a deformation problem from a differential graded Lie algebra, introducing the necessary concepts along the way. We will also do an inverse construction of a differential graded Lie algebra out of the deformation problems we are concerned with.

The final topic is control that homological data exerts over the (global) geometry of the moduli space of representations with fixed Jordan-Hölder factors. First we will overview the result of Alastair King (*Moduli of representations of finite-dimensional algebras*) describing certain projective subspaces of the moduli space of representations, along with some explicit examples. Then, we will explain how the techniques described in Keller's papers yield explicit equations for this moduli space in terms of homological data.

Tentative Course Outline

(1) Introduction
(2) Representation theory
  (a) General representation theory background (basic cases, generalizations, Tannakian philosophy and Hopf algebras)
  (b) Quivers and representations of quivers
  (c) Reduction of representation theory to the quiver case
  (d) Discussion of representation type
(3) Schemes, group actions, and moduli theory
  (a) Background on scheme theory as appropriate
  (b) Examples of moduli problems
  (c) Algebraic groups and their actions
    (i) Geometric invariant theory
    (ii) Quotient algebraic stacks
  (d) Construction of moduli spaces of representations
  (e) Deformation theory
    (i) Background on formal schemes as appropriate
    (ii) Deformation problems
    (iii) A first example of homological control of deformations
(4) Homological algebra
  (a) Group cohomology
  (b) Projective and injective modules, resolutions
  (c) The derived category of an abelian category
  (d) Ext and Tor functors
    (i) Global dimension and analogical meaning
    (ii) Examples of computations
(5) Homological control of deformation theory
  (a) Tangent space
  (b) Obstructions and automorphisms
  (c) Introduction to differential graded Lie algebras and differential graded algebras
  (d) Goldman and Millson's construction of a deformation functor from a DGLA
  (e) The DGLA associated to a deformation problem for representations
(6) Homological control of global structure of a moduli space of representations
  (a) Introduction to pseudorepresentations
    (i) Reduction to the case of algebras finite over their center
  (b) Families of representations with fixed semisimplification
    (i) Initial examples
    (ii) Notions of (semi)stability for representations, following King (see link to the paper on the course webpage)
    (iii) Projective spaces of stable representations
  (c) Keller's description of the category of extensions in terms of homological data
    (i) Introduction to A_∞-algebras
    (ii) The category of twisted stalks
    (iii) Families of twisted stalks