Renormalization methods for higher order differential equations

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Abstract

We adapt methodology of statistical mechanics and quantum field theory to approximate solutions to an arbitrary order ordinary differential equation boundary value problem by a second-order equation. In particular, we study equations involving the derivative of a double-well potential such as \( u - u^3 \) or \( -u + 2u^3 \). Using momentum (Fourier) space variables we average over short length scales and demonstrate that the higher order derivatives can be neglected within the first cumulant approximation, once length is properly renormalized, yielding an approximation to solutions of the higher order equation from the second order. The results are confirmed using numerical computations. Additional numerics confirm that the main role of the higher order derivatives is in rescaling the length.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Higher order ordinary and partial differential equations (DEs) arise in the context of some important issues in materials science and statistical mechanics. For example, in the phase field approach to free boundaries, retaining wave modes up through \( k^2 \), in the Fourier expansion, leads to the usual second-order phase field equations while retaining higher wave modes, leads to higher order equations. It was shown [4, 8] in some early papers that higher order equations could be useful in understanding the detailed features of an interface such as anisotropy.
For instance, in order to retain the 6-fold anisotropy, one needs to retain terms up to $k^6$, leading to sixth-order DEs, since the anisotropy is largely ‘averaged out’ in the second-order DEs [7].

Second-order DEs with a term arising from the derivative of a double-well have been studied extensively in terms of existence, uniqueness and monotonicity. In higher orders few rigorous results are available such as [21] in which existence of traveling waves was established. In general, the addition of the higher order terms can dramatically change the nature of the solution. Even if they are small they could represent a singular perturbation. Nevertheless under some conditions, it was shown that this was not the case for small coefficients. Fourth and sixth-order equations have been studied in other contexts in [9, 22, 29, 30, 32, 35, 36].

Overall, theoretical methods for higher order DEs are not nearly as well-developed as they are for the second-order equations. Computations that are routine for second-order equations are also more difficult. Hence, it would be useful to develop tools to approximate solutions to such higher order DEs with those of second order.

The general ideas of renormalization group (RG) methods, which are closely related to multi-scaling methods, have revolutionized our understanding of critical phenomena and other areas of statistical mechanics, as well as quantum field theory [1, 10, 11, 16]. In a philosophical sense there are a number of issues involving DEs that are related. One example is the blow-up or decay of solutions to DEs [3, 13, 17, 18, 26, 33]. Here, one has an ‘asymptotic self-similarity’ that is reminiscent of the divergence of key thermodynamic quantities in statistical mechanics. Other recent applications of RG involve dynamical systems [27], difference systems [28], boundary flows [23] and other related problems [2, 12, 31].

In this paper we present a formal derivation, using RG and scaling methods, that shows higher order DEs can be approximated by suitably rescaled second-order equations. This formal derivation is supported by numerical computations that quantify the difference between an accurate solution to the sixth order computed directly, and the approximate solution derived using RG and scaling arguments. These comparisons are presented in section 5.

We develop an idea of using momentum (Fourier) space RG in order to approximate solutions of a DE that is higher order than second and involves a nonlinear term. The nonlinear term is the derivative of a double-well potential, so that solutions of the second order exhibit transition layers. The strategy is to write the solution of the DE as the minimizer of a free energy which is then expressed in Fourier space. We first truncate the Fourier space integrals to eliminate the very short and long space scales, and then express the nonlinear terms in terms of the first cumulant (see, e.g., [10]). Analyzing these terms further we can rewrite the free energy in the same form as the original with rescaled Fourier variables. By adjusting the truncation, we can obtain coefficients that modify the nonlinearity. Transforming these equations by rescaling, we can regain the original nonlinear term but with arbitrarily small coefficients for the higher order derivative terms. Finally, the space scale that has been modified in this process must be renormalized. We have shown that this can be done in two ways. One is that, since we know that the rescaled solution to the second order is an approximate solution, we can minimize over functions of the same type. This is just a minimization over a single parameter (which represents the length scale in the second-order DE), so that we need only to solve a polynomial to determine the scale. The other perspective is by using the surface tension as an empirical quantity that rescales the length.

The key results of our RG analysis are (i) solutions of the higher order DE’s are close (in $L^2$ and $L^\infty$) to those of the second-order DE with rescaled length; (ii) solutions of the sixth-order equation do not change significantly (as measured by the $L^2$ and $L^\infty$ norms) as the coefficient of the higher order derivatives varies, provided space is rescaled appropriately.
The methodology can be illustrated with the prototype equation
\[ c_6 \phi^{(6)} + c_2 \phi^{(2)} + \phi - \phi^3 = 0 \]  
with \( c_6, c_2 \in \mathbb{R}^+ \) (and not necessarily small) subject to boundary conditions
\[ \phi^{(k)} (\pm \infty) = 0 \quad \text{for} \quad k = 1, 2, 3 \text{ and } \phi (0) = 0. \]  
(2)

One can consider the effect of changing, say \( c_6 = 10 \) to \( c_6 = 0 \) (and appropriately dropping the boundary conditions \( \phi^{(k)} (\pm \infty) = 0 \) for \( k = 2, 3 \)). In most nonlinear ordinary differential equations (ODEs) and partial differential equations (PDEs) the highest order derivative is of crucial importance, and there would be a profound difference between the sixth-order and second-order equations. Even if the coefficient of the highest order is small, the perturbation from zero to a small value, \( \epsilon \), is generally singular and there is no guarantee that the difference in solutions will be small. Thus, it is surprising to see that the solutions of (1), (2) are close to those of the second order, particularly for large values of \( c_6 \).

The formal derivations are complemented by numerical computations (section 5). We compare the solutions of the ODEs obtained from the numerical solution and RG approximation scheme. For the numerical solutions a multi-dimensional version of Newton’s method is implemented. We find that, as the sixth-order coefficient is varied from \( c_6 = 0 \) to \( c_6 = 10 \), not only do the two solutions (numerical and approximate) of (1), (2) stay qualitatively similar, as both exhibit single transition layer solutions, but also they are very close in the \( L^2 \) and \( L^\infty \) norms. We also show, in this section, that the effect of the sixth-order coefficient is weak in the sense that, as \( c_6 \) is increased to large values, the true (numerically computed) solutions do not change appreciably. For example, solutions to (1), (2) for \( c_6 = 8 \) and \( c_6 = 10 \) differ by 0.002 in the \( L^2 \) norm, and 0.006 in the \( L^\infty \) norm. In fact, further numerical experiments show that the main effect of changing the sixth-order coefficient is just rescaling the solution. In short, the renormalization analysis that we perform indicates that the main contribution of the higher order derivatives involves rescaling the length.

The rest of this paper is organized as follows. In section 2 we define the \( 2N \)th order DE and the associated functional that solutions must minimize. In section 3, we describe our methodology for the first stage of renormalization, namely rewriting the functional in Fourier space and truncating in order to eliminate the small length scales. We analyze a key integral, and show how the DE can be modified as a result. In section 4 we perform the second stage in the RG process, namely, rescaling the length. Finally, in section 5 we perform numerical computations on the sixth-order equations and compare with our approximate solutions. In the conclusion (section 6) we summarize the results, and discuss additional research problems. The appendix contains much of the RG calculation and proofs of some technical lemmas.

2. Differential equations and statistical mechanics

2.1. Minimizers/differential equations

We consider, for \( N \in \mathbb{N} \), the \( 2N \)th order DE
\[ \sum_{j=1}^{N} c_j \phi^{(2j)} - 2a\phi - 4b\phi^3 = 0, \]  
(3)

with the boundary conditions
\[ \phi^{(j)} (\pm \infty) = 0 \quad \text{for} \quad j = 1, \ldots, N. \]  
(4)

Solutions to (3), (4) can be cast as a minimization problem. Let \( \phi \in C^{2N} (\mathbb{R}) \cap L^\infty (\mathbb{R}) \) and satisfy
(i) boundary conditions $(4)$ and $\int_{\mathbb{R}} |\phi^{(j)}|^2 < \infty$,
(ii) $\text{sgn}(ab) = -1$ and $\int_{\mathbb{R}} |a|\phi^2 + |b|\phi^4 < \infty$.

We consider potentials for which solutions, $\phi(x)$, approach $\pm \sqrt{-a/b}$ or 0 as $x \to \pm \infty$.

Define
\[
L[\phi] := \int_{\mathbb{R}} \sum_{j=1}^{N} \left( (-1)^{j+1} \frac{C_{2j}}{2} [\phi^{(j)}]^2 \right) + a\phi^2 + b\phi^4.
\] (5)

The following lemma is standard.

**Lemma 1.** Let $\phi \in C^N(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and satisfy conditions (i) and (ii). If $\phi$ is a minimizer of (5) then it satisfies (3).

Next we convert the sums into Fourier space by defining
\[
\hat{\phi}(k) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi(x) e^{-ikx} \, dx, \quad \phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{\phi}(k)e^{ikx} \, dk.
\] (6)

**Lemma 2.** The functional $L[\phi]$ can be written as:
\[
L[\phi] = \hat{L}[\hat{\phi}] := \int_{-\infty}^{\infty} dk \sum_{j=1}^{N} \left( (-1)^{j+1} \frac{C_{2j}}{2} k^{2j} \right) \hat{\phi}(k)^2
\]
\[
+ \frac{b}{2\pi} \int_{-\infty}^{\infty} dk_1 \ldots \int_{-\infty}^{\infty} dk_4 \hat{\phi}(k_1) \ldots \hat{\phi}(k_4) \delta(k_1 + \cdots + k_4).
\] (7)

**Proof.** See appendix A.

2.2. Renormalization group method/statistical mechanics

Having noted that solutions to (1), (2) are minimizers of (7), we use RG ideas to obtain an approximation to (7) that involve only terms associated with the second derivatives. The main ansatz is that there is a parallel between minimizing the Landau free energy and the concept of the partition function in statistical mechanics. The minimizers of the free energy describe the behavior of the system (as solutions of a DE). The partition function $Z$ (defined more precisely below) which is obtained by taking the Landau free energy $\hat{L}[\hat{\phi}]$ as a weight, and summing over all possible $\hat{\phi}$ in the Fourier space, also contains all essential information with regards to the physical system at hand. Hence, we can use RG methods to eliminate the short length scales and write $Z$ in terms of sum of an effective free energy. The minimizers of this new free energy, $\hat{L}^{\text{new}}$, should also well describe the behavior of the same system. Indeed, it will be seen that $\hat{L}^{\text{new}}$ has the same form as the original but with new parameters. By making particular choices, the parameters can be adjusted to obtain a new functional in which the coefficient of the higher (than two) order derivatives are much smaller than that of the second-order derivative. Iterating this procedure, one can make the coefficients arbitrarily small. Using the formal asymptotics and existing rigorous results [21] in the case of the sixth-order DEs (as an example), these can be set to zero, leading to an approximation of the solution to the higher order equation by that of the second order.

We start with (7) and first truncate the variables $k, k_i$ at some large $\Lambda$ corresponding to a small length scale. In statistical mechanics this could be an atomic length for example. Thus
we write
\[
\hat{L}[\hat{\phi}] = \int_{-\Lambda}^{\Lambda} dk \left\{ \sum_{j=1}^{N} \left( (-1)^{j+1} \frac{C_j k^{2j}}{2} + a \right) |\hat{\phi}(k)|^2 \right\} + \frac{b}{2\pi} \int_{-\Lambda}^{\Lambda} dk_1 \ldots \int_{-\Lambda}^{\Lambda} dk_4 \{ \hat{\phi}(k_1) \hat{\phi}(k_2) \hat{\phi}(k_3) \hat{\phi}(k_4) \delta (k_1 + \cdots + k_4) \}. \tag{8}
\]
Note that all integrals are over both positive and negative \(k\) since they arise from Fourier transforms.

Using this notation, we can define the partition function over the momentum space by the magnitude of such integrals (we omit the absolute value symbols below)
\[
Z = \left| \left| \int \prod_{0 < |k| < \Lambda} d\hat{\phi}(k) e^{-\hat{L}[\hat{\phi}(k)]} \right| \right|. \tag{9}
\]
Note that if \(k\) were discrete then we would have a large product of such integrals. Since we are considering a continuum of \(k\) values, we regard the integrals over \(k\) in the functional integral sense \[15\].

Given any \(l > 0\), we define
\[
\hat{\phi}_>(k) := \hat{\phi}(k) \quad \text{if} \quad \frac{\Lambda}{l} < |k| < \Lambda, \quad \hat{\phi}_< (k) := 0 \quad \text{if} \quad \frac{\Lambda}{l} < |k| < \Lambda.
\]
\[
:= 0 \quad \text{if} \quad 0 < |k| < \frac{\Lambda}{l} \quad \hat{\phi}(k) \quad \text{if} \quad 0 < |k| < \frac{\Lambda}{l}. \tag{10}
\]

Now, by the coarse-graining procedure, one can show (see appendix C) that the partition function can be (approximately) written as
\[
Z = \left| C \int \prod_{0 < |k| < \Lambda/l} d\hat{\phi}(k) e^{-\hat{L}_{\text{new}}[\hat{\phi}(k)]} \right|. \tag{11}
\]
where \(\hat{L}_{\text{new}}\) is given below.

**Result 1.** The functional \(\hat{L}\) in (8) can be effectively replaced by
\[
\hat{L}_{\text{new}}[\hat{\phi}] := \int_{-\Lambda/l}^{\Lambda/l} dk \left\{ \beta(k) + \frac{3bC}{\pi} \right\} |\hat{\phi}_<(k)|^2 + \frac{b}{2\pi} \int_{-\Lambda/l}^{\Lambda/l} dk_1 \ldots \int_{-\Lambda/l}^{\Lambda/l} dk_4 \hat{\phi}_<(k_1) \ldots \hat{\phi}_<(k_4) \delta (k_1 + \cdots + k_4) \tag{12}
\]
with
\[
\beta(k) := \sum_{j=1}^{N} (-1)^{j+1} \frac{C_j k^{2j}}{2} + a, \quad C(\Lambda, l) := \int_{\Lambda/l}^{\Lambda} \frac{1}{\beta(k)} dk. \tag{13}
\]

The derivation of result 1, involving field theoretic methods of RG, is presented in appendix C.

Thus we obtain an effective free energy \(\hat{L}_{\text{new}}\) in a form close to (8), except that the \(dk\) integration does not extend to \(\Lambda\), but only to \(\Lambda/l\). By rescaling \(k\) we can write (12) in a form that is identical to the original.

### 2.3. Renormalization and justification in this context

(The subsequent analysis can be read independently of this subsection.) The success of RG in statistical mechanics \[14, 34\] can be attributed to the idea that one can repeatedly average over smaller length scales provided one rescales the interactions in a way that yields a fixed point in the transformation. This fixed point must correspond to the exponent that characterizes the
divergence of the thermodynamic quantity near the critical point where the system exhibits a loss of scale due to the infinite correlation length.

A basic idea is that RG leads to successful calculation of critical exponents in statistical mechanics due to the principle of universality. This principle states that the exponents are ‘universal’ such that the changes in the details of interactions cannot alter the exponents. For example, altering the interactions from nearest to next-nearest neighbor interactions in a lattice model produces not change in the exponents. However, dimension of the space or varying the symmetry group of the spins can alter the exponents.

This universality arises from the fact that the correlation length (roughly speaking, the maximum distance at which spins influence one another). Consequently, the process of averaging over short length scales should not alter the basic properties of the system provided one adjusts, or renormalizes the interaction strengths to compensate for the reduction in size. Thus, the heart of the RG process and universality is the feature that one is interested in length scales that are much larger than the scales on which the averaging is taking place.

In the present framework one can distinguish the universal and non-universal quantities/features. What is universal in our setting is the form of the DE (as being a byproduct of the free energy). As a result of this, the general features of the solutions to DE are also universal. However, the coefficients of the DE are non-universal and depend on the particular microscopic information which cannot be obtained through RG. Moreover, one can implement an averaging process over small length scales (in Fourier space as described below), provided one renormalizes the length scale in order to preserve the surface tension. In other words, a transition layer is associated with a surface tension (see section 4) which is the crucial quantity in terms of the physical characteristics of the system (e.g., solution of the equation, the stability of the interface, etc). There is a class of solutions that will have the same surface tension. Hence, if we perform an averaging process in which the higher order derivatives are eliminated, then we can obtain the same behavior in terms of these universal quantities provided we renormalize and endow the new, simpler system with the same surface tension.

There are several ways of performing the averaging and rescaling process [10]. Real space RG involves averaging the spins in a geometric region that can be used to render a smaller replica of the original system. Momentum or Fourier space RG involves transforming the free energy into Fourier space, and then integrating out, in the partition function, the fields corresponding to larger wavelengths or shorter physical lengths in the problem. This latter approach is closest to the methodology we pursue for our problem. The idea is that if we have an integral over $dk$ on the range $[0, \Lambda]$ then we would like to divide this integral into $[0, \Lambda/l]$ and $[\Lambda/l, \Lambda]$ for some $l > 0$ to be determined. We then integrate over the larger $k$ portion, namely $[\Lambda/l, \Lambda]$, leaving the free energy as an integral over just the smaller $k$ part, namely $[0, \Lambda/l]$.

Historically, the justification of RG methodology involved proximity of the system to a critical temperature at which point correlation length diverges. In the original Landau energy, temperature appears in the coefficient of the quadratic term of the functional $(a\phi^2 \sim (T - T_c)\phi^2)$. Near the critical point this coefficient diminishes toward zero. Although we start with a coefficient (of $\phi^2$) which has a fixed value, one can show by simple scaling of the fields followed by changing the length scale, as done explicitly in section 3, that the functional can be cast into a form where this coefficient is small (in fact the coefficient can be made to assume arbitrary values). Upon performing this rescaling, we can regard the system as close to the critical point (since the coefficient of the $\phi^2$ term will be small), and hence all the RG arguments are justified. Of course, once the averaging process is over, we need to renormalize in order to restore the functional to its original form with new parameters. This
process, however does not destroy the essential information contained in the functional which will exhibit itself in the minimizers, i.e., the solution of the original problem.

In general, the justification of this process depends on the particular system. For a statistical mechanics system consisting of spins $(\pm 1)$ on a triangular lattice, for example, it involves the idea that taking a ‘majority rules’ procedure to replace the three spins with just one, thereby reducing the size of the system by three. Of course, if one does not simultaneously adjust the interaction strengths, one will obtain a trivial result. The ansatz here is that the length scale of the smallest triangle is very small compared with the correlation length, so that averaging cannot alter a fundamental structure of the divergent quantity. Creswick et al [10] describe a number of mathematical problems (fractals, random walk, etc) in which the justification of the transformations is based upon exact or approximate self-similarity. In the case of a random walk of $N \gg 1$ steps, one can divide the walk into smaller subwalks of $N/l$ steps that are averaged out and rescaled so that the distribution has the same form as the original, but with different parameters. In this way, one recovers the classical central limit theorems.

The philosophy of RG has been adapted to DEs in several works primarily in calculating some key features. For example in Goldenfeld et al [17], and Caginalp [5] RG was used to calculate the nonclassical decay exponent of DEs. The former considered the Barenblatt equation of porous media flow. The latter considered the heat equation with a nonlinearity. These were along the lines of real-space RG with the justification of ‘asymptotic self-similarity.’ In particular, if one has an equation in which there is decay of solutions for large $x$ (space) and $t$ (time), then one does not have exact self-similarity as in the fractals (for any finite $x$ and $t$), but rather one approaches self-similarity as $x$ and $t$ approach $\infty$ within an appropriate scaling. DEs that exhibit a ‘blow-up’ of solutions [5] at a particular point have a similar self-similarity, but as $x$ approaches the blow-up point rather than infinity. In many cases the results of the RG analysis of these equations have been confirmed by exact calculations and theorems [6, 25].

The RG philosophy can also be used to derive simpler or alternative equations while preserving the essential physical information. For example, in [19], a set of ‘amplitude equations’ were derived from the phase field crystal equation that work on a much larger length scale thereby significantly reducing the computational time. Another example of the same spirit was given in [24] where RG was used to derive Navier–Stokes equations from the Boltzman equation. These examples all show that RG approach can be used in a broader context and has far reaching application areas outside the realm of equilibrium statistical mechanics.

The DEs problems we consider in this paper have four main features: (i) they arise as critical points of a free energy functional, (ii) they have large order derivatives, (iii) solutions exhibit a transition layer, (iii) one can define a ‘surface free energy’ related to the transition layer and the free energy functional. Like the decay and blow-up problems, the transition layer in which the solution moves from $-1$ to $+1$ can be viewed as asymptotically self-similar. Hence, RG ideas can be implemented with the same justification that led to successful calculations for the DEs discussed above. In particular, small length scales and higher order derivatives should not have a significant impact on the transition layer except through rescaling length. A key property of DEs with transition layers is surface tension, which is a bona fide physical quantity in the phase field equations, but can be defined more generally in a purely mathematical formulation in terms of $L^2$ norms of solutions, for example. Studies involving higher order phase field equations [7] indicate that these higher order derivatives serve to modify the surface tension, but otherwise have a smaller influence than the second-order derivative. Thus, the ansatz is that we can utilize the higher order derivatives to match the empirical surface tension which is an intrinsic quantity of the physical system.
3. Renormalization of the free energy

The original free energy (8) has been transformed using the momentum space RG procedure into (12). Our goal now is to rescale the length and the field in order that the form of the free energy is exactly the same as the original with the new parameters substituted for the old. The length will be scaled twice, to restore the integration limits and the form of the nonlinearity.

With $C$ defined by (13), the new parameters are given by
\[
a \rightarrow a + \frac{3bC}{\pi} = a^\text{new}, \quad b \rightarrow b = b^\text{new}.
\]
In order to obtain the same form for the free energy, with the new parameters, we need to rescale the integrals by setting $k' = lk$ and write
\[
\int_0^\infty dk := \int_{-\Lambda/l}^{\Lambda/l} dk = \int_{-\Lambda}^{\Lambda} dk'/l
\]
and substitute into (12) to yield
\[
\hat{\mathcal{L}}^\text{new} [\hat{\phi}] := \int_{-\Lambda}^{\Lambda} dk \left\{ \beta(k) + \frac{3bC}{\pi} \right\} |\phi(k)|^2 + b \int_{-\Lambda}^{\Lambda} dk \phi(k_1) \phi(k_2) \delta (k_1 + k_2)
\]
\[
= \int_{-\Lambda}^{\Lambda} \frac{1}{l} dk' \left\{ \beta \left( \frac{k'}{l} \right) + \frac{3bC}{\pi} \right\} |\phi(k'/l)|^2 + b \int_{-\Lambda}^{\Lambda} \frac{1}{l} dk'_1 \phi(k'_1) \phi(k'_2) \delta \left( k'_1 + \ldots + k'_4 \right).
\]

The delta function satisfies
\[
\delta(k'/l) = \frac{1}{l} \delta(k').
\]

For some $z$ in $\mathbb{R}^+$ to be determined, we let
\[
\hat{\phi}(k'/l) := \hat{\phi}_1 (k') z.
\]
Using again $k = k'/l$ we have in place of (16) the free energy
\[
\hat{\mathcal{L}}^\text{new} [\phi] = \int_{-\Lambda}^{\Lambda} \frac{1}{l} dk' \left\{ \sum_{j=1}^{N} \left( -1 \right)^{j+1} \frac{c_{2j}}{2} \left( \frac{k'}{l} \right)^{2j} + (a^\text{new}) \right\} |\hat{\phi}_1 (k)|^2 + b \int_{-\Lambda}^{\Lambda} \frac{1}{l} dk'_1 \phi(k'_1) \phi(k'_2) \delta \left( k'_1 + \ldots + k'_4 \right).
\]

Basic Scaling. We set $z = l$ so that the integrand (coarse-grained free energy density) has the same form as the original density up to a factor of $l$. Hence,
\[
c_{2j}' = c_{2j}/l^{2j}, \quad (a^\text{new})' = a^\text{new}, \quad b' = b.
\]
With the primed parameters, we can write the free energy, (19), in the form
\[
\hat{\mathcal{L}}^\text{new} [\phi] = l \int_{-\Lambda}^{\Lambda} dk' \left\{ \sum_{j=1}^{N} \left( -1 \right)^{j+1} \frac{c_{2j}'}{2} \left( k' \right)^{2j} + a_l^\text{new} \right\} |\hat{\phi}_1 (k)|^2
\]
\[
+ \frac{l b}{2\pi} \int_{-\Lambda}^{\Lambda} dk'_1 \ldots dk'_4 \phi(k'_1) \phi(k'_2) \delta (k'_1 + \ldots + k'_4).
\]
This corresponds to the DE
\[
\sum_{j=1}^{N} c_j \phi_j^{(2j)} - 2a^{\text{new}} \phi_1 - 4b\phi_1^3 = 0. \tag{22}
\]

Note that we can rescale the space variable with \( \tilde{x} = lx \) and \( \tilde{\phi} (\tilde{x}) = \phi_l (x) \) and write
\[
\sum_{j=1}^{N} c_j \phi_j^{(2j)} - 2 \left( a + \frac{3bC}{\pi} \right) \phi - 4b\phi^3 = 0. \tag{23}
\]

We will show below that, under appropriate conditions, \( C \) can be chosen arbitrarily in \((0, \infty)\) by choosing \( \Lambda \) and \( l \) appropriately, so that \( a + 3bC/\pi \) is arbitrarily small. Letting \( A := -2 (a + 3bC/\pi) \) and \( B := 4b \), we use the simple scaling described below to rewrite (23) in a form in which the coefficient of the terms beyond the second order are small if \( A \) is small.

### 3.1. Analysis of \( C \)

Let \( N \) be odd and \( c_{2N} > 0 \). Recall the definitions of \( \beta(k) \), \( C(\Lambda, l) \) in (13) and let \( \varepsilon := l^{-1} \) where \( a < 0 \), and note that the leading term, \( c_{2N}k^{2N}/2 \) is positive. Thus, the polynomial \( \beta(k) \) has a real and positive root. We define \( k_+ \) as the largest real root of \( \beta(k) = 0 \). Hence we can write
\[
\beta(k) = (k^2 - k_+^2)^p Q(k^2) = (k - k_+)^p (k + k_+)^p Q(k^2) \quad \text{for } p \in \{1, \ldots, N\},
\]

where \( Q \) is a polynomial of order \( 2N - p \) and \( Q (k_+^2) > c > 0 \). Thus, the integral in \( C(\Lambda, l) \) has the dominant term
\[
\int_{\varepsilon \Lambda} \frac{dk}{(k - k_+)^p}
\]

which diverges as \( \Lambda \varepsilon \to k_+ \) for \( p \geq 1 \).

Note that for any \( \Lambda \in \mathbb{R}^+ \) we have \( C(\Lambda, 1) = 0 \). Also, by first choosing a large fixed \( \Lambda \), and then adjusting \( \varepsilon \) sufficiently close to \( k_+ / \Lambda \), we can make \( C(\Lambda, \varepsilon^{-1}) \) arbitrarily large. Hence, for any \( q \in \mathbb{R}^+ \) we can find a pair \( (\Lambda_q, \varepsilon_q) \) with \( \Lambda_q \gg 1 \) and \( 0 < \varepsilon < 1 \) such that \( C(\Lambda_q, \varepsilon_q) = q \). We summarize our results below using either of the two conditions.

**Condition A.** Let \( N \) be odd, and assume \( c_{2N} > 0 \), \( a < 0 \).

**Condition B.** Let \( N \) be odd, and let \( \beta(k) \), defined by (13), have at least one positive root.

**Lemma 3.** Under condition A or B, for sufficiently large \( \Lambda \), given any positive real number \( \delta \), we can find \( \varepsilon \) depending on \( q \) and \( \Lambda \) such that
\[
C(\Lambda, \varepsilon^{-1}) = \delta. \tag{24}
\]

For \( a > 0 \) we can utilize a less restrictive condition. Since \( C(\Lambda, \varepsilon^{-1}) \) can be made arbitrarily small (note \( C(\Lambda, 1) = 0 \), if we can show that for some \( \varepsilon \) and \( \Lambda \) one has that
\[
a + \frac{3bC}{\pi} = 0 \tag{25}
\]

then by continuity we can choose \((\Lambda, \varepsilon)\) such that \( a + 3bC/\pi \) attains any value between 0 and \( a \).

**Condition C.** Let \( N \) be odd. For \( a > 0 \) and \( b < 0 \) we require the existence of \((\Lambda, \varepsilon)\) such that
\[
\infty > C(\Lambda, \varepsilon^{-1}) = \int_{\varepsilon \Lambda}^{\Lambda} \frac{dk}{\beta(k)} \geq \frac{\pi}{3} \left| \frac{a}{b} \right|. \tag{26}
\]
When condition C is satisfied, one can find \((\Lambda, \varepsilon)\) such that \(A := a + 3bC/\pi\) is arbitrarily small.

In the RG analysis, we assume that condition A, B or C hold as indicated below.

**Example.** Let \(c_6 > 0\) and \(c_2 > 0\) with all other \(c_i = 0\). Set \(a = 1/2\) and \(b = -1/2\) so that the equation to be considered is
\[
c_6\phi^{(6)} + c_2\phi^{(2)} - \phi + 2\phi^3 = 0. \tag{27}
\]

Since we can make \(\varepsilon\) arbitrarily small, and \(\Lambda\) arbitrarily large, it will suffice to show
\[
\int_0^\infty \frac{2dk}{c_6k^6 + c_2k^2 + a} \geq \frac{\pi}{3} \equiv 1.0472. \tag{28}
\]
One can verify numerically that condition on C will be satisfied, for example, if \((c_6, c_2)\) satisfy any of the conditions: \(c_6 \leq 40, c_2 \leq 1\), or \(c_6 \leq 4, c_2 \leq 4.5\), or \(c_6 \leq 0.1, c_2 \leq 7\).

### 3.2. Rescaling of the field

Here, we complete the renormalization process by scaling the field and the length so that the original form of the DE is preserved. The equation
\[
\sum_{j=1}^N c_2 j \phi^{(2j)} + A\phi - B\phi^3 = 0 \tag{29}
\]
can be written as
\[
\sum_{j=2}^N c_2 j A^{-1} \psi^{(2j)} + c_2 \psi^{(2)} + \psi(1 - \psi^2) = 0, \tag{30}
\]
with \(\psi(x) := B^{1/2} A^{-1/2} \phi(x/A^{1/2})\). The RG analysis showed that the original equation (3) can be replaced by another in which the coefficients \((a, b)\) are transformed into \((a + 3bC/\pi, b)\) with a change in length scale. The analysis above showed that with a suitable choice of \((\Lambda, \varepsilon)\) one can adjust \(C\) so that \(-2(a + 3bC/\pi)\) is arbitrarily small provided \(a\) and \(b\) are of different signs.

Now using the simple scaling with \(\delta := A := -2(a + 3bC/\pi)\) and \(B := 1\) in (10) we have the equation
\[
\sum_{j=2}^N c_2 j \delta^{j-1} \psi^{(2j)} + c_2 \psi^{(2)} + \psi(1 - \psi^2) = 0. \tag{31}
\]

Thus, we have derived the following.

**Result 2.** Solutions of the DE (3) can be approximated by those of (31) for arbitrarily small \(\delta\), subject to the same boundary conditions (4) for some choice of length scale.

Hence, the coefficients of all terms beyond the second order can be made arbitrarily small. Thus, the RG transformations are used to obtain a small coefficient for the \(\phi\) term, which can then be transformed into small coefficients for the higher (than two) order terms in the derivatives.
4. Rescaling the length

Recapitulating our results thus far, the original equation (3) for \( a \leq 0 \) and \( b \geq 0 \) can be transformed via the simple scaling above to one of the prototype equations (32) and (33) below. Moreover, the RG scheme above shows that solutions to the equations (for \( N \) odd)

\[
\sum_{j=1}^{N} c_2 j \phi^{(2j)} + \phi - \phi^3 = 0, \quad \text{subject to conditions A or B} \tag{32}
\]

and

\[
\sum_{j=1}^{N} c_2 j \phi^{(2j)} - \phi + 2\phi^3 = 0, \quad \text{subject to condition C} \tag{33}
\]

with suitable boundary conditions, can be approximated by the solutions of the second-order equation. However, the transformations have altered the spatial scale. The next step in our analysis is to set the length scale. This can be done in two ways discussed below.

4.1. Minimization of the functional integral to set length scale

The original DE, by construction, is the minimizer of the functional \( L \). Our approximate solution class consists of all functions which solve the second-order DE with an altered spatial scale. So, the best approximation among all candidates is the one with a stretching factor that minimizes the functional \( L \).

Example 1. We consider first prototype equation for the case \( c_6 > 0 \), \( c_2 > 0 \) (and all other \( c_i = 0 \)) and \( a = -1/2 \) and \( b = 1/4 \), so that the potential is \(|\phi^2 - 1|^2/4\) yielding the equation

\[
c_6 \phi^{(6)} + c_2 \phi^{(2)} + \phi - \phi^3 = 0. \tag{34}
\]

Since \( \tanh \left( x/(2^{1/2}c_2^{1/2}) \right) \) is an exact solution for the second-order DE in this problem, the RG procedure suggests that an approximate solution has the form

\[
\psi_\eta(s) = \tanh \left( \frac{s}{2^{1/2} \eta} \right) \tag{35}
\]

for some value \( \eta > 0 \). We minimize the functional

\[
L[\psi_\eta] = \int_{\mathbb{R}} \left\{ \frac{c_6}{2} \left[ \psi_\eta^{(3)} \right]^2 + \frac{c_2}{2} \left[ \psi_\eta^{(1)} \right]^2 + \frac{1}{4} \left[ \psi_\eta^2 - 1 \right]^2 \right\} \tag{36}
\]

over all \( \psi_\eta \). Differentiating with respect to \( \eta \) we obtain the condition for the minimum as

\[
0.667 \eta^6 - 0.667 c_2 \eta^4 - 1.904 c_6 = 0. \tag{37}
\]

This has a unique positive root \( \eta_* \) for all positive \( c_2 \), and \( c_6 \). The approximate solution is then given by \( \psi_{\eta_*}(s) \). For example, if \( c_6 := 10 \), \( c_2 := 1 \) one has \( \eta_* = 1.852 \), while \( c_6 = 1 \) and \( c_2 = 1 \) implies \( \eta_* = 1.357 \). Numerical results below will show that this yields a good approximation to the computed sixth-order solution as measured by the \( L^2 \) norm.

Given this approximate solution, \( \psi_{\eta_*} \), to the sixth-order equation, one can also compute the surface tension \([7], \sigma\), which is a crucial quantity in phase field models. Roughly, speaking it is the difference per unit area of the free energy with the transition, minus the average of the free energies of the constant states (e.g., \( \phi = \pm 1 \)). For the free energies we consider, one has (see [7])

\[
\tilde{\sigma}(\eta, c_2, c_4, \ldots, c_N) := \sum_{j=1}^{N} j c_2 j \left\| \psi_\eta^{(j)} \right\|_2^2 = 3 c_6 \left\| \psi_\eta^{(3)} \right\|_2^2 + c_2 \left\| \psi_\eta^{(1)} \right\|_2^2. \tag{38}
\]
The quantity \( \tilde{\sigma} \) would be identical to the theoretical surface tension if \( \psi_\eta(x) \) were replaced by the true solution to the full 2Nth order DE. For \( c_0 = 10, c_2 = 1, \sigma \) is approximated by \( \tilde{\sigma} = 1.231 \) (compared with the true value of 1.176). For \( c_0 = 1, c_2 = 1, \sigma \) is approximated by 1.046 compared with the true value of 1.02. For \( c_0 = 5 \) one has \( \tilde{\sigma} = 1.173 \) compared with the true value of 1.116. By ‘true value’ we refer to the value of \( \sigma \) obtained by using the numerically computed solution of the sixth-order equation (discussed below).

**Example 2.** We implement the same procedure on a different prototype for potentials with \( a > 0 \) and \( b < 0 \), namely, \(-\frac{1}{2} (\phi'' - \frac{1}{2})^2 + \frac{1}{8}\), yielding the sixth-order equation

\[
c_6 \phi^{(6)} + c_2 \phi^{(2)} - \phi + 2\phi^3 = 0
\]

subject to the same Neumann conditions. The second-order equation now has a solution that is qualitatively different from the first prototype. It is given by \( \Psi(x) := \psi_1(x) \) where

\[
\psi_\eta(x) := \frac{1}{\cosh(x/\eta)} .
\]

We use again the general solution and determine the minimizing value of \( \eta \). Defining \( m_1 \) and \( m_3 \) below, and setting analogously, \( m_0 := \int_\mathbb{R} \frac{1}{2} [\psi_\eta^2 - \psi_\eta^4] \) we compute using the new \( \psi_\eta \) defined above, \( m_0 = 1/3, m_1 = 2/3 \) and \( m_3 = 2.95 \). One has similarly,

\[
5c_6 (2.95) \left( \frac{1}{\eta} \right)^6 + 2 \frac{2}{3} c_2 \left( \frac{1}{\eta} \right)^2 - \frac{2}{3} = 0 .
\]

For example, for \( c_6 = 10 \) and \( c_2 = 1 \) we have the solution \( \eta_\ast = 2.531 \). For \( c_6 = 5 \) and \( c_2 = 1 \) we have \( \eta_\ast = 2.271 \) and for \( c_6 = 1 \) and \( c_2 = 1 \) we have \( \eta_\ast = 1.786 \).

Using these values, we can approximate the surface tension, \( \sigma \approx \tilde{\sigma} \), using (38) together with \( \psi_\eta, \eta \) with the results \( \tilde{\sigma} = 1.114 \) (\( c_6 = 10 \)), \( \tilde{\sigma} = 1.023 \) (\( c_6 = 5 \)) and \( \tilde{\sigma} = 0.861 \) (\( c_6 = 1 \)). These compare with the true values (i.e., those obtained from the numerically computed solution of the sixth-order equation) of 1.07, 0.97 and 0.82 respectively, so that the estimate is 4\% to 5\% higher than the true value. Hence, this methodology allows one to compute the surface tension of the sixth-order model to this level of accuracy with information from the second-order equation alone.

The conclusion for the general 2Nth order equation is summarized below.

**Scaling by minimizing functional.** Consider the general 2Nth order equations (32) (or (33)) (for \( N \) odd) \( \psi(x) \) being the solution of the second-order equation. Let

\[
m_0 := \int_{-\infty}^{\infty} [a \psi^2 + b \psi^4] ds , \quad m_j := \| \psi^{(j)} \|_2^2 , \quad j = 1, 2, \ldots
\]

\[
\psi_\eta(x) := \psi(x/\eta) , \quad \mathcal{L}[\psi_\eta] = \sum_{j=1}^{N} (-1)^{j+1} c_{2j} (\frac{1}{\eta})^{2j} m_j + \eta m_0 .
\]

The derivative of this functional satisfies

\[
0 = \frac{d\mathcal{L}[\psi_\eta]}{d\eta} , \quad \text{or}, \quad \sum_{j=1}^{N} (-1)^{j+1} (2j - 1) c_{2j} m_j (\frac{1}{\eta})^{2j} = 2m_0 .
\]

For \( N \) odd, this will have a positive solution for all \( c_{2j} \) and \( m_j \) (\( j = 1, \ldots, N \)). Then, \( \eta_* \) is defined as the unique global minimizer, i.e., the value of \( \eta \) for which \( \mathcal{L}[\psi_\eta] \) is the minimum among finitely many roots.

**Result 3.** Let \( \eta_* \) be defined as the minimizing stretching factor for the functional \( \mathcal{L} \). Then the function \( \psi_{\eta_*}(x) \) is an approximation to the solution of (32) (or (33)).
While we have focused on two equations that have exact solutions for the second-order equation, the process is the same for any potential with similar qualitative solutions. The general theory asserts the existence of a solution with these boundary conditions, so that a numerical solution would be obtained in place of $\psi_\eta$ to calculate the $m_i$.

4.2. Using empirical surface tension to set length scale

As an alternative to minimizing the functional integral to obtain the value of $\eta^*$, we now use an idea that is based on statistical mechanics instead. We regard the surface tension (38) as an empirical macroscopic quantity that the solution of the 2Nth order DE must satisfy. Let the true solutions of (32) (or (33)) be denoted by $\phi(x; c_2, \ldots, c_N)$ and recall that the exact solutions to the second-order equation are given by $\phi(x; c_2, 0, \ldots, 0) = \psi_\eta(x) := \tanh[x/(2^{1/2}\eta)]$ (or $\psi_\eta(x) := 1/cosh[x/(\eta)]$). As before, we can compute

$$\|\psi_\eta^{(j)}(s)\|_2^2 = \frac{m_j}{\eta^{j-1}}; \quad m_j := \int_{-\infty}^{\infty} \left\{ \frac{d^j\psi(x)}{ds^j} \right\}^2 ds. \quad (40)$$

Then, we can rewrite $\tilde{\sigma}$ as

$$\tilde{\sigma}(\eta, c_2, \ldots, c_N) = \sum_{j=1}^{N} (-1)^{j+1} j c_2 \frac{m_j}{\eta^{j-1}}. \quad (41)$$

In phase field models and other physical applications, $\tilde{\sigma}$ defined by (38) represents the (theoretically computed) surface tension. In order to renormalize the length scale we set $\tilde{\sigma} := \sigma_*$, where $\sigma_*$ is the empirical value of the surface tension. Now let $c_2, \ldots, c_N$ be fixed, and suppose that we assume the solution to (32) (or (33)) is of the form of $\psi_\eta(s)$ for some $\eta$. We then propose that, among all possible functions $\psi_\eta(s)$, the better approximate solution is the one for which the surface tension $\tilde{\sigma}$ coincides with $\sigma_*$. This is to say that we are looking for a positive solution, $\eta_*$, of

$$\sigma_* = \sum_{j=1}^{N} (-1)^{j+1} j c_2 \frac{m_j}{\eta_*^{j-1}}, \quad \text{or}$$

$$Nc_2m_N = \sigma_* \eta_*^{2N-1} - \sum_{j=1}^{N-1} (-1)^{j+1} j c_2 m_j \eta_*^{2(N-j)}. \quad (42)$$

The procedure described above can be summarized as fixing the surface tension based on empirical considerations. Denoting this by $\sigma_*$, we solve for $\eta_*$ in (42) and use this value in $\psi_\eta(s)$ as the approximation to the solution to the 2Nth degree equation (3), (4). This addresses the question: if one knows the surface tension of the 2Nth equation, how well is the solution profile approximated by the solution to the second-order DE with the same surface tension? What is left is then to find a reasonable value for $\sigma_*$ (aside from physical measurement). We will consider two ways.

Scaling (surface tension) 1. Use the surface tension obtained from the second-order equation as $\sigma_*$.

Scaling (surface tension) 2. Use the true surface tension that is obtained from the numerical solution (see section 5) of the 2Nth order DE.

Example 3. We again consider the sixth-order first prototype equation for which we will perform numerical computations in the subsequent section:

$$c_0\phi^{(6)} + c_2\phi^{(2)} + \phi - \phi^3 = 0. \quad (43)$$
Let \( c_2 := 1 \) for which the surface tension for the second-order equation is given by (38) as 
\[
\sigma_5 = \frac{3m_3}{\eta^3} + c_2 m_1.
\]
In order to retain the same surface tension in the sixth-order equation, we set the value \( \sigma_6 := m_1 \). We also compute \( m_3 = \frac{3m_3}{\eta^3} \geq 0.539 \). Given any value, \( \sigma_6 \), for surface tension, if the solution profile for this sixth-order equation were given exactly by \( \psi(s, \eta) \) and had surface tension \( \tilde{\sigma}(\eta_1, c_6) \) then we would have
\[
\sigma_6 = \tilde{\sigma}(\eta_1, c_6) = \frac{3c_6 m_3}{\eta^3} + c_2 m_1.
\]
In order to retain the same surface tension as in the second-order equation, we set \( \sigma_6 := \sigma_1 = 0.943 \). Thus, \( \eta_1 \), can be defined as the unique (real) positive solution to the fifth degree polynomial
\[
0.943\eta^5 - 0.943\eta^4 - 3 (0.539) c_6 = 0.
\]
One can readily see that (45) has a unique real solution, \( \eta_1 \), which solves the equation (43) and the approximate solution (46) for these values of \( c_6 \). Once, the solution is computed, we can use the computed surface tension for the sixth-order equation as \( \sigma_6 \) in (44) to obtain a better \( \eta_1 \) value (see section 5).

**Example 4.** Let \( N = 5 \) with the \( c_1 = 1 \) in (32) and set \( \sigma_5 := m_1 = 0.943 \), so (42) is then (upon computing the \( \sigma_5 \))
\[
(0.943) \eta^5 - (0.943) \eta^4 + 2 (0.377) \eta^3 - 3 (0.539) \eta^2 + 4 (1.51) \eta^1 - 5 (6.857) = 0,
\]
with unique real solution \( \eta_1 = 1.559 \). The approximation to the 10th order equation is then given by \( \Psi_\eta(x) = \tanh\left(\frac{x}{\sqrt{2}\eta_1}\right) \).

**Example 5.** We consider the second prototype, namely equation (39). For \( c_6 := 0 \) and \( c_2 := 1 \) one has the exact solution \( \Psi(x) := \Psi_\eta(x) \) where \( \Psi_\eta(x) := 1 / \cosh (x/\eta) \) as before. Once again we have \( \Psi_\eta^{1,2,3}(x) := m_j \) and the \( m_j \) are given as in (40). Then we have \( m_1 = 2/3 \) and \( m_3 = 2.95 \) so that
\[
\tilde{\sigma}(\eta_1, c_2, c_6) = 3c_6 \|\psi_\eta^{(3)}\|_2^2 + c_2 \|\psi_\eta^{(1)}\|_2^2
\]
\[
= 3c_6 \frac{m_3}{\eta^5} + c_2 \frac{m_1}{\eta^3}.
\]
Using the same idea as above, we define \( \sigma_6 := m_1 = 2/3 \) so \( \eta_1 \) is the real root of
\[
2 \frac{2}{3} \eta^5 - 2 \frac{2}{3} \eta^4 - 3c_6 (2.95) = 0.
\]
For example, if \( c_6 = 10 \) then \( \eta_1 = 2.89 \). Then, the approximate solution is \( \psi_\eta(x) = 1 / \cosh \left( \frac{x}{\eta_1} \right) \) which solves the equation \( \eta_1^2 \psi_\eta^{(2)} = -\psi + 2 \psi^3 = 0 \). We can also use the computed (for the sixth-order equation) surface tension \( \sigma_6 = 1.03 \) and obtain \( \eta_1 = 2.56 \), so that the approximate solution is \( \cosh \left( \frac{x}{2.56} \right) \). These will be compared with the numerically computed solutions below.

**Result 4.** With \( \eta_1 \) defined by either scaling 1 or 2, the function \( \psi_\eta(x) \) is an approximation to the solution of (32) (or (33)).
5. Numerical computations and comparisons

In this section we describe the numerical techniques for the solutions of two prototype (sixth-order) DEs. Both examples are in the form

\[ 0 = c_6 \phi^{(6)} + c_4^{(2)} \phi - W'(\phi), \]

where \( W_1(x) = (1 - x^2)^2/4 \) and \( W_2(x) = (x^2 - x^4)/2 \). Although both forms appear to be similar, they exhibit solutions with quite different behavior due to the differing signs. We solve the above DEs in the interval \([-10, 10] \) with the following respective boundary conditions

\[
I : \phi(\pm 10) = \pm 1, \quad \phi^{(1)}(\pm 10) = 0, \quad \phi^{(2)}(\pm 10) = 0, \\
II : \phi(\pm 10) = 0, \quad \phi^{(1)}(\pm 10) = 0, \quad \phi^{(2)}(\pm 10) = 0. \tag{49}
\]

We use a finite difference scheme in order to solve these high order nonlinear DEs, and discretize the interval \([-10, 10] \) into \( N \) equal pieces and enumerate the points \( x_0 = -10, \ldots, x_N = 10 \). The second, fourth and sixth-order derivatives are numerically approximated by

\[
\frac{d^2 \phi(x_i)}{dx^2} \approx \frac{\phi(x_{i+1}) - 2\phi(x_i) + \phi(x_{i-1})}{\Delta x^2},
\]

\[
\frac{d^4 \phi(x_i)}{dx^4} \approx \frac{\phi(x_{i+2}) - 4\phi(x_{i+1}) + 6\phi(x_i) - 4\phi(x_{i-1}) + \phi(x_{i-2})}{\Delta x^4},
\]

\[
\frac{d^6 \phi(x_i)}{dx^6} \approx \frac{\phi(x_{i+3}) - 6\phi(x_{i+2}) + 15\phi(x_{i+1}) - 20\phi(x_i) + 15\phi(x_{i-1}) - 6\phi(x_{i-2}) + \phi(x_{i-3})}{\Delta x^6}.
\]

respectively.

The discrete system then constitutes a system of nonlinear algebraic equations. If we denote the value of \( \phi \) at \( x_i \) as \( \phi_i \), then the system of equations can be regarded as a nonlinear map \( S : \mathbb{R}^N \rightarrow \mathbb{R}^N \) such that \( S(\phi) = 0 \), where \( \phi \in \mathbb{R}^N \). To solve it we use a multi-dimensional version of Newton’s method. Starting with a reasonable initial solution profile (for instance \( \phi(x) = \tanh x \) for the first type of equation, \( 1 / \cosh(x) \) for the second), one can perform the iteration scheme

\[ \phi^{n+1} = \phi^n - [DS(\phi^n)]^{-1}S(\phi^n), \]

where \( [DS(\phi^n)]^{-1} \) denotes the inverse of the gradient of \( S \). Actually, in order to avoid construction of large inverse matrices and save computational power, we implement an equivalent procedure by solving an auxiliary linear equation, and inserting the temporary solution as the second term in the above equation, i.e.,

\[ DS(\phi^n)V_{\text{temp}}^n = S(\phi^n), \]

\[ \phi^{n+1} = \phi^n - V_{\text{temp}}^n. \]

Once the solution is obtained the surface tension integral is evaluated numerically on the same mesh. The mesh size needs to be chosen judiciously. For mesh sizes \( \Delta x < \frac{1}{30} \), it is observed that surface tension values do not change appreciably. Also, the matrices constructed in the scheme become close to singular when \( \Delta x < \frac{1}{60} \). The codes have been written using the MATLAB software.

5.1. Numerical solutions for the first prototype equation

We consider equation (43) for \( c_6 \geq 0 \), together with (49). We let \( \| \cdot \|_2 \) and \( \| \cdot \|_\infty \) denote the usual \( L^2 \) and \( L^\infty \) norms and denote by \( \phi(c_2, c_6) \) the solutions to with those particular
parameters. The objective is to examine the difference between the $L^2$ and $L^\infty$ norms between solutions to the sixth-order equation and the approximations discussed above, namely, $\psi_{\eta_* (c_6)} (x) = \tanh \left( \frac{x}{2 \eta_* (c_6)} \right)$ with $\eta_*$ determined as described in each of the three procedures. (i) For $c_6 = 10$, the surface tension computed from the sixth order equation is $\sigma = 1.176$. The polynomial equations to be solved for the two approaches that use the surface tension (to set the length scale) are

\[
0.94 (\eta^5 - \eta^4) - 16.2 = 0 \text{ (scaling 1)},
\]

if one uses the surface tension obtained from the second-order equation, and

\[
1.176 \eta^5 - 0.94 \eta^4 - 16.2 = 0 \text{ (scaling 2)},
\]

if one uses the true surface tension. The latter is obtained from solving the sixth-order equation, but we can regard it as a proxy for the empirical surface tension. The results of these two methods in determining $\eta_*$, together with the minimization method described above, are shown in the figure and tables for varying values of $c_6$.

For $c_6 = 10$, the results are depicted in figure 1 and table 1.

As a measure of the differences, we can compute $\| \phi (c_2, 0) - \phi (1, 0) \|_3$ where $\phi (c_2, c_6)$ is the exact computed solution to the DE for a range of values of $c_2$. In

Table 1. The $L^2$ and $L^\infty$ differences between the true solution and approximations for $c_6 = 10$.

<table>
<thead>
<tr>
<th>$c_6 = 10$</th>
<th>Pure</th>
<th>Scaling 1</th>
<th>Scaling 2</th>
<th>Scale by min</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$ value</td>
<td>1</td>
<td>2.02</td>
<td>1.89</td>
<td>1.86</td>
</tr>
<tr>
<td>$L^2$ diff</td>
<td>0.45</td>
<td>0.31</td>
<td>0.25</td>
<td>0.23</td>
</tr>
<tr>
<td>$L^\infty$ diff</td>
<td>0.25</td>
<td>0.12</td>
<td>0.10</td>
<td>0.09</td>
</tr>
<tr>
<td>$ST = 1.176$</td>
<td>0.94</td>
<td>0.94</td>
<td>1.17</td>
<td>1.23</td>
</tr>
</tbody>
</table>
This way we can see the relative effect of changing $c_6$ as opposed to changing $c_2$. For example, $\|\phi (10, 0) - \phi (1, 0)\|_2 \approx 1.63$. One can also note the relative norm difference $\left\{\int_{-\infty}^{\infty} |\phi (1, 0) - \text{sgn} (x)|^2 \, dx \right\}^{1/2} \approx 1.48$ as a measure of the numerical difference between the two similarly behaving functions.

(ii) For $c_6 = 5$, the pertaining polynomial equations for scaling 1 and scaling 2 are

$$0.94(\eta^5 - \eta^4) - 8.1 = 0,$$

$$1.115\eta^5 - 0.94\eta^4 - 8.1 = 0$$

respectively, and the surface tension is 1.115. The results are shown in figure 2 and table 2.

(iii) For $c_6 = 1$, the respective polynomial equations for the two type of scalings are

$$0.94\eta^5 - 0.94\eta^4 - 1.62 = 0,$$

$$1.02\eta^5 - 0.94\eta^4 - 1.62 = 0$$

The results are shown in table 3.
5.2. Numerical solutions for the second prototype equation

Next, we compute solutions to the equation (39) and compare the true (i.e., directly computed solutions) for the sixth-order equation with the exact solution (pure) for the second-order and the (rescaled) approximations.

(iv) For \( c_6 = 10 \) one obtains the true surface tension \( \sigma = 1.07 \) by computing the solution to the sixth-order DE. One also computes \( m_1 = 2/3 \) and \( m_3 = 2.95 \). The polynomial equations to be solved are

\[
0.66(\eta_5^5 - \eta_1^4) - 88.5 = 0,
\]

\[
1.07\eta_2^5 - 0.66\eta_2^4 - 88.5 = 0
\]

for the two scalings, respectively.

The results are displayed in figure 3 and table 4.

(v) For \( c_6 = 5 \) we note that the true surface tension is \( \sigma = 0.97 \) and obtain the pertaining polynomials as

\[
0.66(\eta_5^5 - \eta_1^4) - 88.5/2 = 0,
\]

\[
0.97\eta_2^5 - 0.66\eta_2^4 - 88.5/2 = 0
\]

for the two scalings. The results are displayed in figure 4 and table 5.
Figure 4. The true solution (red) for the full sixth-order (second prototype) equation \((c_6 = 5)\) is displayed together with the solution of the second-order equation (blue) and modified second-order equations with scaling 1 (green), scaling 2 (yellow) and minimization (black).

Table 4. The \(L^2\) and \(L^\infty\) differences between the true solution and approximations for \(c_6 = 10\).

<table>
<thead>
<tr>
<th>(c_6) value</th>
<th>Pure</th>
<th>Scaling 1</th>
<th>Scaling 2</th>
<th>Scale by min</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta) value</td>
<td>1</td>
<td>2.9</td>
<td>2.56</td>
<td>2.53</td>
</tr>
<tr>
<td>(L^2) diff</td>
<td>0.85</td>
<td>0.82</td>
<td>0.61</td>
<td>0.59</td>
</tr>
<tr>
<td>(L^\infty) diff</td>
<td>0.4</td>
<td>0.25</td>
<td>0.19</td>
<td>0.18</td>
</tr>
<tr>
<td>(ST)</td>
<td>1.07</td>
<td>0.66</td>
<td>1.06</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Table 5. The \(L^2\) and \(L^\infty\) differences between the true solution and approximations for \(c_6 = 5\).

<table>
<thead>
<tr>
<th>(c_6) value</th>
<th>Pure</th>
<th>Scaling 1</th>
<th>Scaling 2</th>
<th>Scale by min</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta) value</td>
<td>1</td>
<td>2.56</td>
<td>2.39</td>
<td>2.27</td>
</tr>
<tr>
<td>(L^2) diff</td>
<td>0.71</td>
<td>0.72</td>
<td>0.61</td>
<td>0.52</td>
</tr>
<tr>
<td>(L^\infty) diff</td>
<td>0.35</td>
<td>0.24</td>
<td>0.20</td>
<td>0.17</td>
</tr>
<tr>
<td>(ST)</td>
<td>0.97</td>
<td>0.66</td>
<td>0.97</td>
<td>1.016</td>
</tr>
</tbody>
</table>

(vi) For \(c_6 = 1\) we have the analogous polynomials

\[
0.66(\eta_1^5 - \eta_1^4) - 8.85 = 0, \\
0.82\eta_2^5 - 0.66\eta_2^4 - 8.85 = 0.
\]

The results are displayed in figure 5 and table 6.
5.3. Additional numerical analysis of prototype equations

We examine the accuracy of the RG approximation by presenting additional computations that confirm the ansatz. All computed tables of this section are given in appendix D.

Let $\phi(c_2, c_6)$ be the true solution for the sixth-order equation (i.e., (1), (2)), that we compute directly (without RG). Let $\phi_{\text{app}}(c_2, c_6)$ denote the approximation that we obtain from RG and the scaling via minimization. We investigate the $L^2$ and $L^\infty$ norms of the differences between $\phi(c_2, c_6)$ and $\phi_{\text{app}}(c_2, c_6)$. We have already presented, in the tables and the graphs of the previous subsections, the norm differences and qualitative behavior of $\phi(c_2, c_6)$, $\phi_{\text{app}}(c_2, c_6)$ for $c_6 = 1, 5, 10$. Now, we increase $c_6$ to even higher values (for the first prototype equation) and investigate the difference in norm (with $c_2 := 1$).
Implementing a linear regression of these values \( c_6 \geq 12 \) (table D1) we obtain
\[
\frac{\|\phi(1, c_6) - \phi_{ap}(1, c_6)\|_1}{\|\phi(1, c_6) - 0\|_1} \approx 0.8 + 0.01 c_6 \approx 0.04 + \frac{c_6}{2000}.
\]
By this measure, the error is almost independent of \( c_6 \) for large values of \( c_6 \). This is a surprising result, since higher order derivatives are often regarded as dominant in the behavior of the solution.

The RG analysis that is supported by the numerical computations suggests that exact solutions of the sixth-order equation (i.e., (1), (2) with \( c_2 := 1 \)) are not sensitive to changes in \( c_6 \) when \( c_6 \) is large. We test this idea by computing the solutions directly and examining the quantity 
\[
e(c_6, n, p) := \|\phi(1, c_6) - \phi(1, c_6 + n)\|_p,
\]
as shown in table D2. We see that there is a small difference between the true solutions corresponding to largely differing \( c_6 \) e.g., \( c_6 = 16 \) and \( c_6 = 26 \). The difference decreases as \( c_6 \) increases. A similar observation can be made for the second prototype DE. Table D3 shows that for larger values of \( n \) the differences \( e(c_6, n, p) \) decrease with increasing \( c_6 \) for the second prototype equation.

We can further investigate the nature of the small difference between the solutions of (32) corresponding to different values of \( c_6 \) (with \( c_2 = 1 \) fixed). The RG ansatz suggests that the main difference should be a rescaling of space. In order to test this hypothesis, we can vary the length scale of a solution for a given \( c_6 \) and compare that with those of the equation in which \( c_6 \) is replaced by \( c_6 + n \). In other words, for \( \phi_{\alpha n}(x) := \phi_{\alpha}(\alpha x) \), we compute
\[
\min_{\alpha \in (0,1,0)} \|\phi_{\alpha n} - \phi_{\alpha n+\alpha}\|_p.
\]
If this is small for a broad range of \( c_6 \) (particularly large values of \( c_6 \)), then it provides good evidence for the assertion that varying \( c_6 \) essentially rescales the solution. In appendix D, we list the norm differences for the \( (c_6, n) \) for various \( \alpha \) values. The results summarized in tables D4–D7 confirm the ansatz that rescaling the solutions with the right parameter leads to approximate solutions with errors (measured by the \( L^p \) norms) that are five to ten times smaller than without the scaling. As a measure of the size of this norm difference, we compute the norms of the true solutions as shown below.

We compare them with the ‘relative errors’ where a standard measure of the ‘relative error’ can be given as
\[
re(c_6, n, p) := \min_{\alpha \in (1,1.5)} \frac{\|\phi_{\alpha n} - \phi_{\alpha n+\alpha}\|_p}{\text{avg}(\|\phi_{\alpha n}\|_p, \|\phi_{\alpha n+\alpha}\|_p)}.
\]
Computing this expression for \( c_6 = 4 \), we obtain
\[
re = \frac{\min_{\alpha \in (1,1.5)} \|\phi_{\alpha n} - \phi_{4+1}\|_1}{\text{avg}(\|\phi_4\|_1, \|\phi_5\|_1)} = 0.036/77.55 \approx 5 \cdot 10^{-4},
\]
\[
re = \frac{\min_{\alpha \in (1,1.5)} \|\phi_{\alpha n} - \phi_{4+1}\|_2}{\text{avg}(\|\phi_4\|_2, \|\phi_5\|_2)} = 0.01/8.75 \approx 0.001.
\]
Hence, the relative error in the rescaled solution is very small.

In a similar way we can form tables for the second prototype equation as shown in tables D8–D11. Again, we observe that not only are the norm differences of the solutions at large \( c_6 \) values small, but also, solutions corresponding to different \( c_6 \) coefficients, are extremely close to one another upon changing length-scale. Summarizing these computations we have the following:

**Result 5.** For both prototype equations, the computations show that the relative error arising from a significant change in the coefficient of the sixth-order derivative leads to a very small change in the solution as measured by \( L^1, L^2 \) and \( L^\infty \) differences.
6. Conclusion

Motivated by statistical mechanics and quantum field theory, we have presented a formal RG methodology that can be used to approximate solutions of a class of higher (than two) order differential equations (DEs) of boundary value type by second order. The central ideas involve writing the DE as a minimizer of a functional which is then treated as a Hamiltonian in statistical mechanics. By writing the free energy arising from this functional, one can average over the large Fourier modes, i.e., the short length scales. Going through this process, one can render the coefficients of the higher order derivatives arbitrarily small, so that the equation can be approximated by one of second order. The length scale, however, has been modified in the process. Thus, the second step in the RG procedure is to rescale length. We showed that this can be done in two ways. One of these involves determining the length scale by minimizing the functional. This is easily done since there is only one parameter, and it amounts to solving a polynomial.

An alternative procedure involves using the surface tension as an empirical constant that can be used to rescale the equations. One can use the surface tension obtained from the second order in order to set the scale (scaling 1) or utilize the surface tension of the sixth-order solution (scaling 2). The latter is a proxy for using the measured surface tension in obtaining the solution.

The surface tension that is an inherent feature of equations exhibiting a transition layer is particularly important in phase field models (see references in [7])). We find that the surface tension of the computed solution to the sixth order differs from the approximation (using minimization) by 2%–5% in the equations studied. This range holds even when the coefficient of the sixth-order term is as large as 10. Mathematically, the surface tension can be defined in terms of Sobolev norms such as $\|\phi'\|_{L^2}$.

With either approach for rescaling length, the solutions provide an approximation that shows a qualitative agreement (in terms of exhibiting a similar transition layer), and good quantitative agreement in terms of the $L^2$ and $L^\infty$ norms. The minimization approach provides slightly better accuracy.

The RG analysis suggests that the chief role of the higher derivatives in a sixth-order equation involves rescaling length. Using computed solutions to the two basic prototypes of sixth-order equations, we have shown that the main effect of varying the coefficient of the sixth-order derivative is to rescale the length scale.

The overall philosophy of our approach entails examining the underlying physics for one possible derivation of a DE. Using approximation methods particular to that application, we obtain simplified equations. In many physics and engineering problems one has a system of many equations. In situations of high symmetry one has a system of ODEs which can be written as a single higher order ODE. Thus our methodology is a first step in significantly simplifying large systems.

Numerical computations on DEs of high order pose many challenges such as stability. While our numerical examples have focused on sixth-order equations, the same RG methodology can be expected to provide a good approximation for DEs of much higher order whose solutions are much more difficult to compute using typical methods.

In many problems involving transition layers, one can utilize a local coordinate system in which one coordinate is orthogonal to the interface. Thus, while our methods have been applied to ordinary DEs, there is a good reason to believe that it will be an integral part of a study of partial DEs in which there are transition layers, for example. Moreover, whenever one has a problem in which there is a large space scale (e.g., the transition) and complex behavior (e.g., the intricacies of interfacial pattern) one should be able to use this approach, since the complex
behavior is governed properties of the larger scale (e.g., surface tension) that are preserved during this RG process. Clearly a next step in this direction of research would be to find a method for successive approximations that converge to the true solution in a particular norm. The cumulant expansion can be carried out to higher terms, thereby eliminating the error in this approximation. Another direction is to obtain rigorous error bounds for the approximations obtained from the RG approach.

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Appendix A. Interaction terms in Fourier space

We define the Fourier transform

\[ \hat{\phi}(k) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi(x) e^{-ikx} dx, \quad \phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{\phi}(k) e^{ikx} dk \]

and calculate the interaction terms in the Hamiltonian. One way to do this is by considering the discrete version and taking the continuum limit as discussed in [4, 7]. We can also compute these quantities directly in the continuum:

\[ A_1 := \int_{-\infty}^{\infty} dx \partial_x^2 \phi^2 = \int_{-\infty}^{\infty} dx (2\pi)^{-1} \left( \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \hat{\phi}(q) e^{-iqx} dq \right)^2 \]

\[ = -\int_{-\infty}^{\infty} dq \hat{\phi}^* (q) \hat{\phi}(-q) q (-q) = \int_{-\infty}^{\infty} dq |\hat{\phi}(q)|^2 q^2. \tag{A.2} \]

\[ A_4 := \int_{-\infty}^{\infty} dx [\phi(x)]^4 = (2\pi)^{-2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \hat{\phi}(q_1) e^{-iq_1x} dq_1 \ldots \int_{-\infty}^{\infty} \hat{\phi}(q_4) e^{-iq_4x} dq_4 \]

\[ = (2\pi)^{-2} \int_{-\infty}^{\infty} dq_1 \ldots \int_{-\infty}^{\infty} dq_4 \int_{-\infty}^{\infty} dx e^{-i(q_1 + \ldots + q_4)(\hat{\phi}(q_1) \ldots \hat{\phi}(q_4))} \]

\[ = (2\pi)^{-1} \int_{-\infty}^{\infty} dq_1 \ldots \int_{-\infty}^{\infty} dq_4 \delta(q_1 + \ldots + q_4) \hat{\phi}(q_1) \ldots \hat{\phi}(q_4) \tag{A.3} \]

\[ A_3 := \int_{-\infty}^{\infty} dx [\phi_{,x}^4 (x)]^2 = \int_{-\infty}^{\infty} dx \left[ (2\pi)^{-1/2} \frac{\partial^3}{\partial x^3} \int_{-\infty}^{\infty} dq \hat{\phi} (q) e^{-iqx} \right]^2 \]

\[ = \int_{-\infty}^{\infty} dx \left[ (2\pi)^{-1/2} \frac{\partial^3}{\partial x^3} \int_{-\infty}^{\infty} dq \hat{\phi} (q) e^{-iqx} \right]^2 \]

\[ = -\int_{-\infty}^{\infty} dq \delta(\delta + \hat{\delta}) \hat{\phi} (q) \hat{\phi} (-\hat{q}) q^3 (-q)^3 \int_{-\infty}^{\infty} dq |\hat{\phi}(q)|^2 q^6. \tag{A.4} \]

Appendix B. The cumulant expansion

The cumulant expansion (see p 260 of [16])

\[ \langle e^{-V} \rangle_0 = \left( 1 - V + \frac{1}{2} V^2 + \ldots \right) = 1 - \langle V \rangle_0 + \frac{1}{2} \langle V^2 \rangle_0 + \ldots \]
\[ \log(\langle e^{-V'} \rangle_0) = \log \left( 1 - \langle V \rangle_0 + \frac{1}{2} \langle (V^2) \rangle_0 + \cdots \right) \]
\[
\approx - \langle V \rangle_0 + \frac{1}{2} \left[ \langle (V^2) \rangle_0 - \langle (V^2) \rangle_0^2 \right] + \cdots \quad (B.1)
\]
Hence, one has the approximation
\[ \langle e^{-V'} \rangle_0 \approx e^{-\langle V \rangle_0} \frac{1}{4} \left( \langle (V^2) \rangle_0 - \langle (V^2) \rangle_0^2 \right) \]
so that the first cumulant approximation is
\[ \langle e^{-V'} \rangle_0 \approx e^{-\langle V \rangle_0} \cdot \quad (B.2) \]

**Appendix C. Derivation of result 1**

The partition function is defined by
\[ Z := \int_0<|k|<\Lambda \prod d\tilde{\phi}(k) e^{-\tilde{L}[\phi(k)]}. \]
We note some identities related to complex integration before we delve into actual computations. Let \( z \) denote the complex conjugate of \( z \), and write
\[ \int dz \bar{dz} := \int dz \wedge d\bar{z} = \int dz (x + iy) d(x - iy) = \int dx \wedge dx - 2i \int dx \wedge dy + \int dy \wedge dy = -2i \int dx \wedge dy. \quad (C.1) \]
Here we are using the standard ideas of area integration in complex variables (see, for example, [20]). For any fixed \( k \), we can define the integral \( \int d\tilde{\phi}(k) \) analogously since the integrals will occur in complex conjugate pairs as defined by \( (C.1) \). With \( dz \) denoting \( d\tilde{\phi}(k) \), the product integral, and noting that the definition implies \( \tilde{\phi}(k) \) and \( \phi(-k) \) are complex conjugates we write \( \int d\tilde{\phi}(k) d\phi(-k) = \int dz d\bar{z} \). Also note that the integrals in the definition of \( Z \) involve pairs \((k, -k)\) yielding a grouping in the form of \( (C.1) \). Thus, we have the identity
\[ \left| \prod_{0<|k|<\Lambda} d\tilde{\phi}(k) \right| = \left| \prod_{0<|k|<\Lambda} d\tilde{\phi}(k) d\phi(-k) \right| = \left| \prod_{0<|k|<\Lambda} 2 \int d(Re \tilde{\phi}(k)) d(Im \tilde{\phi}(k)) \right|, \quad (C.2) \]
and note that the factor of two is a constant factor that will not be relevant to our results though we retain it.

For subsequent calculations we may write the integrals in \( (C.2) \) more concisely as
\[ \int \prod_{0<|k|<\Lambda} d\tilde{\phi}(k) = \prod_{0<k<\Lambda} 2 \int d(Re \tilde{\phi}(k)) d(Im \tilde{\phi}(k)) \]
\[
= \prod_{0<k<\Lambda/2} 2 \int d(Re \tilde{\phi}_+(k)) d(Im \tilde{\phi}_+(k)) \prod_{\Lambda/2<k<\Lambda} 2 \int d(Re \tilde{\phi}_-(k)) d(Im \tilde{\phi}_-(k)) \]
\[
= \int \prod_{0<k<\Lambda/2} 2 d\tilde{\phi}_+(k) \int \prod_{\Lambda/2<k<\Lambda} 2 d\tilde{\phi}_-(k) =: \int \Pi^+ d\tilde{\phi}(k) \int \Pi^- d\tilde{\phi}(k). \quad (C.3) \]
Recall the definitions (10) and abbreviate the second line of (8), the quartic nonlinearity, by $V$ (see below). To simplify the notation, let

$$\int_{-\Lambda}^{\Lambda} dk := \int_{|k| < \frac{\Lambda}{2}} dk$$

and use it to rewrite (C.6) as

$$e^{-\beta_{\text{old}} F} := \int \prod_{0 < |k| < \Lambda} d\phi(k) e^{-\hat{\mathcal{L}}[\hat{\phi}(k)]}$$

Using (C.3) we write (omitting the absolute values)

$$Z = e^{-\beta_{\text{old}} F} := \int \prod_{0 < |k| < \Lambda} d\phi(k) e^{-\hat{\mathcal{L}}[\hat{\phi}(k)]} = \int \Pi^< d\phi(k) e^{-\int^< dk \beta(k) \phi(k)^2} \int \Pi^> d\phi(k) e^{-\int^> dk \beta(k) \phi(k)^2} e^{-V[\phi(k)]}.$$

We would like to perform the integration over the large wavelengths, namely $\int^< dk$, rendering the exp $\{-V[\phi(k)]\}$, namely the $\phi^2$ terms into a form that is essentially an average over the $\phi^2$ terms. By doing this integration we are averaging over short length scales. We define the average

$$\langle A[\phi(k)] \rangle := \frac{\int \Pi^> d\phi(k) e^{-\int^> dk \beta(k) \phi(k)^2} A[\phi(k)]}{\int \Pi^> d\phi(k) e^{-\int^> dk \beta(k) \phi(k)^2}}$$

and use it to rewrite (C.6) as

$$e^{-\beta_{\text{old}} F} = \int \Pi^< d\phi(k) e^{-\int^< dk \beta(k) \phi(k)^2} \langle e^{-V[\phi(k)]} \rangle_\text{(num)} \langle 1 \rangle_\text{num}.$$

We use the first cumulant expansion (see appendix B) and approximate

$$\langle e^{-V[\phi(k)]} \rangle_\text{(num)} \approx e^{-V[\phi(k)]},$$

so that (C.8) can be written, within this approximation, as

$$e^{-\beta_{\text{old}} F} = \int \Pi^< d\phi(k) e^{-\int^< dk \beta(k) \phi(k)^2} e^{-V[\phi(k)]} \langle 1 \rangle_\text{num} = \int \Pi^< d\phi(k) e^{-\int^< dk \beta(k) \phi(k)^2} \int \Pi^> d\phi(k) e^{-\int^> dk \beta(k) \phi(k)^2} e^{-V[\phi(k)]} \langle 1 \rangle_\text{num}.$$

Note that $\langle V[\phi(k)] \rangle_\text{num}$ does not depend on $\int \Pi^> d\phi(k)$ integrals since they have already been integrated out. Now we can view (C.10) as a partition function in terms of a new free energy, and write

$$e^{-\beta_{\text{old}} F} = \int \prod_{0 < |k| < \Lambda} d\phi(k) e^{-\hat{\mathcal{L}}_{\text{new}}[\hat{\phi}(k)]}$$

where

$$\hat{\mathcal{L}}_{\text{new}} = \int_{0 < |k| < \frac{\Lambda}{2}} dk \int_{\frac{\Lambda}{2} < |k| < \Lambda} dk \beta(k) \phi(k)^2 + \langle V[\phi(k)] \rangle_\text{num}.$$
Our goal now is to evaluate \( \langle V [\phi(k)] \rangle \), and rescale variables so that we can obtain a free energy in the same form as the original, namely, (8). First we compute

\[
\langle V [\phi(k)] \rangle = \frac{\int \Pi^* \, d\phi(k) \, e^{-\int \delta k \phi(k)^2} V [\phi(k)]}{\int \Pi^* \, d\phi(k) \, e^{-\int \delta k \phi(k)^2}}
\]

(C.12)

\[
= B \int \Pi^* \, d\phi(k) \, e^{-\int \delta k \phi(k)^2} \frac{b}{2\pi} \int_{-\Lambda}^\Lambda \, dk_1 \ldots
\]

(C.13)

and let \( B := \int \Pi^* \, d\phi(k) \, e^{-\int \delta k \phi(k)^2} \). We write (C.13) by splitting up the \( \int_{-\Lambda}^\Lambda \, dk_1 \ldots \int_{-\Lambda}^\Lambda \, dk_4 \) integrals in terms of the long and short wave modes:

\[
\int \, dk_i = \int^\infty \, dk_i + \int^\infty \, dk_i,
\]

resulting in 16 terms. Using (10) we also write

\[
\phi(k_i) = \phi_c(k_i) + \phi_s(k_i).
\]

One can then write (C.13) as

\[
B \langle V [\phi(k)] \rangle = \frac{b}{2\pi} \int \Pi^* \, d\phi(k) \, e^{-\int \delta k \phi(k)^2} \{ I_1 + 4I_2 + 6I_3 + 4I_4 + I_5 \}
\]

(C.14)

\[
I_1 := \int^\infty \, dk_1 \int^\infty \, dk_2 \int^\infty \, dk_3 \int^\infty \, dk_4 \phi_c(k_1) \phi_c(k_2) \phi_c(k_3) \phi_c(k_4) \delta (k_1 + \cdots + k_4)
\]

\[
I_2 := \int^\infty \, dk_1 \int^\infty \, dk_2 \int^\infty \, dk_3 \int^\infty \, dk_4 \phi_c(k_1) \phi_c(k_2) \phi_s(k_3) \phi_s(k_4) \delta (k_1 + \cdots + k_4)
\]

\[
I_3 := \int^\infty \, dk_1 \int^\infty \, dk_2 \int^\infty \, dk_3 \int^\infty \, dk_4 \phi_s(k_1) \phi_s(k_2) \phi_s(k_3) \phi_s(k_4) \delta (k_1 + \cdots + k_4)
\]

\[
I_4 := \int^\infty \, dk_1 \int^\infty \, dk_2 \int^\infty \, dk_3 \int^\infty \, dk_4 \phi_s(k_1) \phi_c(k_2) \phi_s(k_3) \phi_s(k_4) \delta (k_1 + \cdots + k_4)
\]

\[
I_5 := \int^\infty \, dk_1 \int^\infty \, dk_2 \int^\infty \, dk_3 \int^\infty \, dk_4 \phi_s(k_1) \phi_s(k_2) \phi_s(k_3) \phi_c(k_4) \delta (k_1 + \cdots + k_4)
\]

(C.15)

**Lemma 4.** \( Q_2 = Q_4 = 0 \).

**Proof.** We prove \( Q_2 = 0 \); the proof of \( Q_4 = 0 \) is similar. We compute (see (C.3) for the definition of \( \int \Pi^* \, d\phi(k) )

\[
\frac{2\pi}{b} Q_2 = \int \Pi^* \, d\phi(k) \, e^{-\int \delta k \phi(k)^2} \, \int^\infty \, dk_1 \int^\infty \, dk_2 \int^\infty \, dk_3 \int^\infty \, dk_4 \, \phi_c(k_1) \phi_c(k_2) \phi_s(k_3) \phi_s(k_4) \delta (k_1 + \cdots + k_4)
\]

Now use the fact that \( \phi_c(k_1) \) is independent of the \( \int^\infty \, dk_i \) integrals and evenness of \( \beta \) to write this as

\[
\frac{2\pi}{b} Q_2 = \int^\infty \, dk_1 \int^\infty \, dk_2 \int^\infty \, dk_3 \int^\infty \, dk_4 \, \phi_c(k_1) \phi_c(k_2) \phi_c(k_3) \phi_c(k_4) \delta (k_1 + \cdots + k_4)
\]

\[
= \int \Pi^* \, d\phi(k) \, e^{-\int \delta k \phi(k)^2} \phi_c(k_4) \delta (k_1 + \cdots + k_4)
\]

26
From the definition we write

\[ Q_1 = \frac{\hbar b}{2\pi} \int dk_1 \ldots dk_4 \phi_{\phi} (k_1) \ldots \phi_{\phi} (k_4) \delta (k_1 + \cdots + k_4). \]

**Proof.** We compute

\[
\begin{align*}
\frac{2\pi}{b} Q_1 := & \int dk_1 \ldots dk_4 \Pi \phi_{\phi} (k) \phi_{\phi} (k) e^{-\int d^4k|\phi(k)|^2} \\
& \cdot \phi_{\phi} (k_1) \ldots \phi_{\phi} (k_4) \delta (k_1 + \cdots + k_4) \\
= & \int dk_1 \ldots dk_4 \phi_{\phi} (k_1) \ldots \phi_{\phi} (k_4) \delta (k_1 + \cdots + k_4) \\
& \cdot \int \Pi \phi_{\phi} (k) e^{-\int d^4k|\phi(k)|^2} \\
= & B \int dk_1 \ldots dk_4 \phi_{\phi} (k_1) \ldots \phi_{\phi} (k_4) \delta (k_1 + \cdots + k_4).
\end{align*}
\]

Note that \( B \) also decouples, but we will not need to evaluate it. Note that the factors of \( B \) cancel in the exponent for \( \langle V [\phi(k)] \rangle \) in (C.14).

**Lemma 5.** \( Q_1 = \frac{\hbar b}{2\pi} \int dk_1 \ldots dk_4 \phi_{\phi} (k_1) \ldots \phi_{\phi} (k_4) \delta (k_1 + \cdots + k_4). \)

**Proof.** From the definition we write

\[
\begin{align*}
\frac{2\pi}{b} Q_3 := & \int \Pi \phi_{\phi} (k) e^{-\int d^4k|\phi(k)|^2} \\
& \cdot \phi_{\phi} (k_1) \phi_{\phi} (k_2) \phi_{\phi} (k_3) \phi_{\phi} (k_4) \delta (k_1 + \cdots + k_4) \\
= & \int \Pi \phi_{\phi} (k) e^{-\int d^4k|\phi(k)|^2} \phi_{\phi} (k_3) \phi_{\phi} (k_4)
\end{align*}
\]

and let

\[
T_3 := \int \Pi \phi_{\phi} (k) e^{-\int d^4k|\phi(k)|^2} \phi_{\phi} (k_3) \phi_{\phi} (k_4).
\]
We now divide the evaluation of (C.20) into three parts. In the sum over $k_3$ and $k_4$ we evaluate it for (i) $k_3 \neq k_4$, (ii) $k_3 = k_4$, and (iii) $k_3 = -k_4$.

Case (i) $k_3 \neq k_4$: The integrals in (C.21) decouple, and noting that we have pairs of integrals including

$$\int d\phi (k_3) \, d\phi (-k_3) \, e^{-2\beta |\phi (k_3)|^2} \phi_+ (k_3)$$

$$= \int dz \, d\bar{z} \, e^{-2\beta |z|^2} = 2 \int dx \, dy \, e^{-2\beta (x^2 + y^2)} (x + iy) = 0$$

(identical to the calculation of (C.17). Hence $T_3 = 0$ and so these terms vanish.

Case (ii) $k_3 = k_4$: We evaluate $T_3$ as

$$T_3 := \int \Pi_2^{\mathbf{k} \neq \pm \mathbf{k}_1} d\phi (k) \, e^{-\int_{\sum_{k_2} \, d\beta (k)|\phi (k)|^2} T_{31}$$

$$T_{31} := \int d\phi (k_3) \, d\phi (-k_3) \, e^{-2\beta (k_3)|\phi (k_3)|^2} \phi (k_3)^2$$

$$= \int d\bar{z} \, d\bar{z} \, e^{-2\beta (\bar{z})|\bar{z}|^2} \bar{z}^2 \, (\text{converting to polar coordinates})$$

$$= 2 \int_0^\infty r \, dr \, \int_0^{2\pi} d\theta \, e^{-2\beta (r) r^2} \, r^2 \, e^{2i\theta} = 0,$$

since the $\theta$ integral vanishes.

Case (iii) $k_3 = -k_4$: In this case $T_3$ is expressed as

$$T_3 = \int \Pi_2^{\mathbf{k} \neq \pm \mathbf{k}_1} d\phi (k) \, e^{-\int_{\sum_{k_2} \, d\beta (k)|\phi (k)|^2} T_{32}$$

$$T_{32} := \int d\phi (k_3) \, d\phi (-k_3) \, e^{-2\beta (k_3)|\phi (k_3)|^2} \phi (k_3) \phi (-k_3). \quad (C.22)$$

We perform the complex area integration for $T_{32}$ in polar coordinates:

$$T_{32} = \int d\bar{z} \, d\bar{z} \, e^{-2\beta (\bar{z})|\bar{z}|^2} |\bar{z}|^2$$

$$= 2 \int_0^\infty r \, dr \int_0^{2\pi} d\theta \, e^{-2\beta (r) r^2} \, r^2$$

$$= \frac{4\pi}{\beta^2} \int_0^\infty (\beta^{1/2} r)^3 \, e^{-2(\beta^{1/2} r)^2} \, d(\beta^{1/2} r)$$

$$= \frac{\pi}{2 \beta (k)^2}. \quad (C.23)$$

Now we evaluate the remaining integrals in (C.22):

$$\int d\phi (k) \, d\phi (-k) \, e^{-2\beta (k)|\phi (k)|^2} = 2 \int_0^{2\pi} d\theta \int_0^\infty r \, dr \, e^{-2\beta (k) r^2}$$

$$= 4\pi \frac{1}{-4\beta (k)} \int_0^\infty (-4\beta r) \, e^{-2\beta (r)^2} \, dr = \frac{\pi}{\beta (k)}. \quad (C.24)$$

Using (C.22)–(C.24) to evaluate $T_3$, we obtain the identity

$$T_3 = \left( \prod_{k \neq k_3, k \neq 0} \frac{\pi}{\beta (k)} \right) \frac{\pi}{2 \beta (k_3)^2} = \frac{1}{2} \frac{1}{\beta (k_3)^2} \prod_{k_3 > 0} \frac{\pi}{\beta (k_3)}. \quad (C.25)$$

28
Now recalling the definition of \( Q_5 \) and noting the results of the three cases so that only \( k_3 = -k_4 \) contributes, we see that the \( dk_4 \) integral leads to a delta function and yields

\[
\frac{2\pi}{b} Q_3 = \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \phi_\prec (k_1) \phi_\prec (k_2) \int_{-\infty}^{\infty} dk_3 \int_{-\infty}^{\infty} dk_4 \\
\cdot \delta (k_1 + \cdots + k_4) \int \Pi^\prec \phi (k) e^{-\int^\prec dk \beta (k) \phi^2 (k)} \phi_\prec (k_3) \phi_\prec (k_4)
\]

and observe that the \( dk_4 \) integration introduces a delta function \( \delta (k_3 + k_4) \). Thus, we have

\[
\frac{2\pi}{b} Q_3 = \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \phi_\prec (k_1) \phi_\prec (k_2) \int_{-\infty}^{\infty} dk_3 \int_{-\infty}^{\infty} dk_4 \\
\cdot \delta (k_1 + \cdots + k_4) \frac{1}{2} \beta (k_3) \prod_{k = 0}^{\infty} \frac{\pi}{\beta (k)}
\]

\[
= \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \phi_\prec (k_1) \phi_\prec (k_2) \int_{-\infty}^{\infty} dk_3 \delta (k_1 + k_2) \frac{1}{2} \beta (k_3) \prod_{k = 0}^{\infty} \frac{\pi}{\beta (k)}
\]

\[
= \int_{-\infty}^{\infty} dk_1 \phi_\prec (k_1) \phi_\prec (-k_1) \int_{-\infty}^{\infty} dk_3 \frac{1}{2} \beta (k_3) \prod_{k = 0}^{\infty} \frac{\pi}{\beta (k)}.
\] (C.26)

Recall that \( B := \int \Pi^\prec \phi (k) e^{-\int^\prec dk \beta (k) \phi^2 (k)} \) so that by the calculation of (C.24) we have

\[
B = \prod_{k = 0}^{\infty} \frac{\pi}{\beta (k)},
\] (C.27)

and we can write \( Q_3 \) as

\[
\frac{2\pi}{b} Q_3 = B \int_{-\infty}^{\infty} dk_1 \phi_\prec (k_1) \phi_\prec (-k_1) \int_{-\infty}^{\infty} dk_3 \frac{1}{2} \beta (k_3)
\]

\[
= BC \int_{-\infty}^{\infty} \phi (k_1) |\phi_\prec (k_1)|^2
\]

\[
C := \int_{-\infty}^{\infty} dk_3 \frac{1}{2} \beta (k_3).
\] (C.28)

The calculation of \( C \) is in section 3.1. \( \square \)

**Lemma 7.** \( Q_5 = \text{Const.} \)

**Proof.** We note that

\[
\frac{2\pi}{b} Q_3 = \frac{b}{2\pi} \int_{-\infty}^{\infty} dk_1 \cdots \int_{-\infty}^{\infty} dk_4 \delta (k_1 + \cdots + k_4)
\]

\[
\cdot \int \Pi^\prec \phi (k) e^{-\int^\prec dk \beta (k) \phi^2 (k)} \phi_\prec (k_1) \cdots \phi_\prec (k_4)
\]

\[
= \text{Const}
\] (C.29)

since each of the terms integrates out. \( \square \)

**Lemma 8.** One has the result

\[
\langle V [\phi (k)] \rangle_\prec = \frac{1}{B} \{ Q_1 + 4Q_2 + 6Q_3 + 4Q_4 + Q_5 \}
\]

\[
= \frac{b}{2\pi} \int_{-\infty}^{\infty} dk_1 \cdots \int_{-\infty}^{\infty} dk_4 \phi_\prec (k_1) \cdots \phi_\prec (k_4) \delta (k_1 + \cdots + k_4)
\]

\[
+ 6C \frac{b}{2\pi} \int_{-\infty}^{\infty} \phi_\prec (k_1) |\phi_\prec (k_1)|^2 + Q_5
\] (C.30)

where \( Q_5 \) is a constant (independent of \( \phi \)).
Recall \((C.10)\), namely
\[
e^{-\beta s F} = \int \Pi^c \phi(k) e^{-\int^c d\beta(k)\phi(k)} e^{-\int^c d\beta(k)\phi(k)} e^{-\int^c d\beta(k)\phi(k)} e^{-\int^c d\beta(k)\phi(k)} e^{-\int^c d\beta(k)\phi(k)} e^{-\int^c d\beta(k)\phi(k)} e^{-\int^c d\beta(k)\phi(k)}.
\]

We note that \(e^{-\int^c d\beta(k)\phi(k)}\) has already been integrated over \(\int^c\) and so does not depend on \(|k| > \Lambda/l\). Hence we can integrate the term
\[
\int \Pi^c \phi(k) e^{-\int^c d\beta(k)\phi(k)} =: B
\]
by itself. Using \((C.30)\) in \((C.10)\) then we can write \((C.10)\) as
\[
e^{-\beta s F} = B \int \Pi^c \phi(k) e^{-\int^c d\beta(k)\phi(k)} \cdot \exp \left[ - \left\{ 6C \frac{b}{2\pi} \int^c dk_1 |\phi_{<}(k_1)|^2 + \frac{b}{2\pi} \int^c dk_1 \ldots dk_4 |\phi_{<}(k_1) \ldots \phi_{<}(k_4)| \delta(k_1 + \ldots + k_4) + Q_3 \right\} \right]
\]

We can combine the \(|\phi_{<}(k_1)|^2\) terms in the exponential and write the partition function in the form of \((C.11)\), namely,
\[
e^{-\beta s F} = B e^{-Q_3} \int \prod_{0<|k|<\Lambda/l} e^{-\hat{\mathcal{L}}_{\text{sea}}(\phi(k))}.
\]

**Appendix D. Tables**

Here we list the tables used in section 5.3

**Table D1.** The \(L^2\) and \(L^\infty\) differences between the true solution and approximations for \(c_b = 10, \ldots, 35\).

<table>
<thead>
<tr>
<th>(c_b)</th>
<th>10</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>20</th>
<th>23</th>
<th>25</th>
<th>27</th>
<th>30</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta)</td>
<td>1.856</td>
<td>1.906</td>
<td>1.965</td>
<td>2.020</td>
<td>2.056</td>
<td>2.098</td>
<td>2.123</td>
<td>2.149</td>
<td>2.182</td>
<td>2.238</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>1.176</td>
<td>1.194</td>
<td>1.217</td>
<td>1.237</td>
<td>1.246</td>
<td>1.266</td>
<td>1.277</td>
<td>1.286</td>
<td>1.300</td>
<td>1.317</td>
</tr>
<tr>
<td>(L^2)</td>
<td>0.232</td>
<td>0.241</td>
<td>0.250</td>
<td>0.257</td>
<td>0.264</td>
<td>0.270</td>
<td>0.273</td>
<td>0.276</td>
<td>0.28</td>
<td>0.289</td>
</tr>
<tr>
<td>(L^\infty)</td>
<td>0.94</td>
<td>0.954</td>
<td>0.097</td>
<td>0.10</td>
<td>0.10</td>
<td>0.103</td>
<td>0.102</td>
<td>0.103</td>
<td>0.104</td>
<td>0.105</td>
</tr>
</tbody>
</table>

**Table D2.** List of \(e(c_b, n, \infty)\) values for the first prototype equation.

<table>
<thead>
<tr>
<th>(c_b)</th>
<th>(n = 2)</th>
<th>(n = 4)</th>
<th>(n = 6)</th>
<th>(n = 8)</th>
<th>(n = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c_b = 2)</td>
<td>0.042</td>
<td>0.06</td>
<td>0.09</td>
<td>0.1</td>
<td>0.11</td>
</tr>
<tr>
<td>(c_b = 4)</td>
<td>0.026</td>
<td>0.045</td>
<td>0.06</td>
<td>0.07</td>
<td>0.08</td>
</tr>
<tr>
<td>(c_b = 6)</td>
<td>0.019</td>
<td>0.035</td>
<td>0.047</td>
<td>0.058</td>
<td>0.069</td>
</tr>
<tr>
<td>(c_b = 8)</td>
<td>0.015</td>
<td>0.028</td>
<td>0.039</td>
<td>0.050</td>
<td>0.061</td>
</tr>
<tr>
<td>(c_b = 10)</td>
<td>0.013</td>
<td>0.024</td>
<td>0.035</td>
<td>0.046</td>
<td>0.051</td>
</tr>
<tr>
<td>(c_b = 12)</td>
<td>0.011</td>
<td>0.022</td>
<td>0.033</td>
<td>0.038</td>
<td>0.044</td>
</tr>
<tr>
<td>(c_b = 16)</td>
<td>0.012</td>
<td>0.017</td>
<td>0.024</td>
<td>0.031</td>
<td>0.037</td>
</tr>
</tbody>
</table>
Table D3. List of $e(c_6, n, 2)$ values for the second prototype equation.

<table>
<thead>
<tr>
<th>$c_6$</th>
<th>$n = 2$</th>
<th>$n = 4$</th>
<th>$n = 6$</th>
<th>$n = 8$</th>
<th>$n = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.084</td>
<td>0.14</td>
<td>0.18</td>
<td>0.21</td>
<td>0.24</td>
</tr>
<tr>
<td>4</td>
<td>0.055</td>
<td>0.097</td>
<td>0.13</td>
<td>0.16</td>
<td>0.18</td>
</tr>
<tr>
<td>6</td>
<td>0.042</td>
<td>0.076</td>
<td>0.101</td>
<td>0.130</td>
<td>0.151</td>
</tr>
<tr>
<td>8</td>
<td>0.034</td>
<td>0.045</td>
<td>0.088</td>
<td>0.110</td>
<td>0.131</td>
</tr>
<tr>
<td>10</td>
<td>0.029</td>
<td>0.054</td>
<td>0.076</td>
<td>0.097</td>
<td>0.114</td>
</tr>
<tr>
<td>12</td>
<td>0.025</td>
<td>0.036</td>
<td>0.056</td>
<td>0.086</td>
<td>0.103</td>
</tr>
<tr>
<td>16</td>
<td>0.022</td>
<td>0.034</td>
<td>0.055</td>
<td>0.071</td>
<td>0.085</td>
</tr>
</tbody>
</table>

Table D4. $\min_{n \in (1, 1.5)} \| \phi_{4,n} - \phi_{4+n} \|_{L^p}$ is reached is for $\alpha = \alpha_{\min} = 1.05$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>1.01</th>
<th>1.02</th>
<th>1.03</th>
<th>1.04</th>
<th>1.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{min} | \phi_4 - \phi_{4+n} |_{L^p}$</td>
<td>0.103</td>
<td>0.068</td>
<td>0.038</td>
<td>0.036</td>
<td>0.056</td>
<td>0.086</td>
</tr>
<tr>
<td>$\text{min} | \phi_4 - \phi_{4+n} |_{L^p}$</td>
<td>0.031</td>
<td>0.020</td>
<td>0.011</td>
<td>0.01</td>
<td>0.018</td>
<td>0.028</td>
</tr>
<tr>
<td>$\phi_4 - \phi_{4+n} |_{L^p}$</td>
<td>0.017</td>
<td>0.011</td>
<td>0.006</td>
<td>0.004</td>
<td>0.009</td>
<td>0.014</td>
</tr>
</tbody>
</table>

Table D5. $\min_{n \in (1, 1.5)} \| \phi_{8,n} - \phi_{8+n} \|_{L^p}$ is reached is for $\alpha = \alpha_{\min} = 1.03$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>1.01</th>
<th>1.02</th>
<th>1.03</th>
<th>1.04</th>
<th>1.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{min} | \phi_8 - \phi_{8+n} |_{L^p}$</td>
<td>0.124</td>
<td>0.083</td>
<td>0.044</td>
<td>0.022</td>
<td>0.051</td>
<td>0.087</td>
</tr>
<tr>
<td>$\text{min} | \phi_8 - \phi_{8+n} |_{L^p}$</td>
<td>0.034</td>
<td>0.022</td>
<td>0.011</td>
<td>0.006</td>
<td>0.015</td>
<td>0.026</td>
</tr>
<tr>
<td>$\phi_8 - \phi_{8+n} |_{L^p}$</td>
<td>0.015</td>
<td>0.009</td>
<td>0.005</td>
<td>0.002</td>
<td>0.006</td>
<td>0.011</td>
</tr>
</tbody>
</table>

Table D6. $\min_{n \in (1, 1.5)} \| \phi_{12,n} - \phi_{12+n} \|_{L^p}$ is reached is for $\alpha = \alpha_{\min} = 1.04$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>1.01</th>
<th>1.03</th>
<th>1.04</th>
<th>1.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{min} | \phi_{12} - \phi_{12+n} |_{L^p}$</td>
<td>0.18</td>
<td>0.137</td>
<td>0.054</td>
<td>0.032</td>
<td>0.053</td>
</tr>
<tr>
<td>$\text{min} | \phi_{12} - \phi_{12+n} |_{L^p}$</td>
<td>0.048</td>
<td>0.036</td>
<td>0.013</td>
<td>0.007</td>
<td>0.015</td>
</tr>
<tr>
<td>$\phi_{12} - \phi_{12+n} |_{L^p}$</td>
<td>0.022</td>
<td>0.017</td>
<td>0.006</td>
<td>0.002</td>
<td>0.007</td>
</tr>
</tbody>
</table>

Table D7. $\min_{n \in (1, 1.5)} \| \phi_{20,n} - \phi_{20+n} \|_{L^p}$ is reached is for $\alpha = \alpha_{\min} = 1.05$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>1.02</th>
<th>1.04</th>
<th>1.05</th>
<th>1.06</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{min} | \phi_{20} - \phi_{20+n} |_{L^p}$</td>
<td>0.324</td>
<td>0.199</td>
<td>0.089</td>
<td>0.079</td>
<td>0.098</td>
</tr>
<tr>
<td>$\text{min} | \phi_{20} - \phi_{20+n} |_{L^p}$</td>
<td>0.072</td>
<td>0.044</td>
<td>0.020</td>
<td>0.017</td>
<td>0.025</td>
</tr>
<tr>
<td>$\phi_{20} - \phi_{20+n} |_{L^p}$</td>
<td>0.029</td>
<td>0.019</td>
<td>0.009</td>
<td>0.006</td>
<td>0.010</td>
</tr>
</tbody>
</table>
Table D8. \[ \min_{u \in (1,1.5)} \| \Phi_{u,n} - \Phi_{4.1,n} \|_{L^p} \] is reached is for \( \alpha = \alpha_{\min} = 1.03. \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>1</th>
<th>1.01</th>
<th>1.02</th>
<th>1.03</th>
<th>1.04</th>
<th>1.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \Phi_{u} - \Phi_{4.1,u} ]</td>
<td>1</td>
<td>0.172</td>
<td>0.112</td>
<td>0.064</td>
<td>0.025</td>
<td>0.071</td>
</tr>
<tr>
<td>[ \Phi_{u} - \Phi_{4.1,u} ]</td>
<td>2</td>
<td>0.049</td>
<td>0.033</td>
<td>0.017</td>
<td>0.007</td>
<td>0.018</td>
</tr>
<tr>
<td>[ \Phi_{u} - \Phi_{4.1,u} ] ( \infty )</td>
<td>0.022</td>
<td>0.015</td>
<td>0.007</td>
<td>0.002</td>
<td>0.007</td>
<td>0.013</td>
</tr>
</tbody>
</table>

Table D9. \[ \min_{u \in (1,1.5)} \| \Phi_{u,n} - \Phi_{8.1,n} \|_{L^p} \] is reached is for \( \alpha = \alpha_{\min} = 1.03. \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>1</th>
<th>1.01</th>
<th>1.02</th>
<th>1.03</th>
<th>1.04</th>
<th>1.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \Phi_{u} - \Phi_{8.1,u} ]</td>
<td>1</td>
<td>0.203</td>
<td>0.141</td>
<td>0.082</td>
<td>0.022</td>
<td>0.071</td>
</tr>
<tr>
<td>[ \Phi_{u} - \Phi_{8.1,u} ]</td>
<td>2</td>
<td>0.055</td>
<td>0.038</td>
<td>0.020</td>
<td>0.007</td>
<td>0.017</td>
</tr>
<tr>
<td>[ \Phi_{u} - \Phi_{8.1,u} ] ( \infty )</td>
<td>0.023</td>
<td>0.016</td>
<td>0.008</td>
<td>0.002</td>
<td>0.006</td>
<td>0.013</td>
</tr>
</tbody>
</table>

Table D10. \[ \min_{u \in (1,1.5)} \| \Phi_{12,n} - \Phi_{12.1,n} \|_{L^p} \] is reached is for \( \alpha = \alpha_{\min} = 1.04. \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>1</th>
<th>1.01</th>
<th>1.02</th>
<th>1.03</th>
<th>1.04</th>
<th>1.05</th>
<th>1.06</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \Phi_{12} - \Phi_{12.4,u} ]</td>
<td>1</td>
<td>0.294</td>
<td>0.229</td>
<td>0.163</td>
<td>0.100</td>
<td>0.046</td>
<td>0.083</td>
</tr>
<tr>
<td>[ \Phi_{12} - \Phi_{12.4,u} ]</td>
<td>2</td>
<td>0.076</td>
<td>0.058</td>
<td>0.040</td>
<td>0.0023</td>
<td>0.009</td>
<td>0.017</td>
</tr>
<tr>
<td>[ \Phi_{12} - \Phi_{12.4,u} ] ( \infty )</td>
<td>0.030</td>
<td>0.023</td>
<td>0.016</td>
<td>0.008</td>
<td>0.003</td>
<td>0.005</td>
<td>0.012</td>
</tr>
</tbody>
</table>

Table D11. \[ \min_{u \in (1,1.5)} \| \Phi_{20,n} - \Phi_{20.1,n} \|_{L^p} \] is reached is for \( \alpha = \alpha_{\min} = 1.05. \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>1</th>
<th>1.02</th>
<th>1.04</th>
<th>1.05</th>
<th>1.06</th>
<th>1.07</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \Phi_{20} - \Phi_{20.8,u} ]</td>
<td>1</td>
<td>0.381</td>
<td>0.237</td>
<td>0.096</td>
<td>0.043</td>
<td>0.094</td>
</tr>
<tr>
<td>[ \Phi_{20} - \Phi_{20.8,u} ]</td>
<td>2</td>
<td>0.095</td>
<td>0.058</td>
<td>0.021</td>
<td>0.009</td>
<td>0.020</td>
</tr>
<tr>
<td>[ \Phi_{20} - \Phi_{20.8,u} ] ( \infty )</td>
<td>0.036</td>
<td>0.022</td>
<td>0.008</td>
<td>0.003</td>
<td>0.006</td>
<td>0.013</td>
</tr>
</tbody>
</table>

References


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