

## RESEARCH ARTICLE

### Parameter optimization for differential equations in asset price forecasting

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A system of nonlinear asset flow differential equations (AFDE) gives rise to an inverse problem involving optimization of parameters that characterize an investor population. The optimization procedure is used in conjunction with daily market prices and net asset values to determine the parameters for which the AFDE yield the best fit for the previous  $n$  days. Using these optimal parameters the equations are computed and solved to render a forecast for market prices for the following days. For a number of closed-end funds, the results are statistically closer to the ensuing market prices than the default prediction of random walk. In particular, we perform this optimization by a nonlinear computational algorithm that combines a quasi-Newton weak line search with the BFGS formula. We develop a nonlinear least-square technique with an initial value problem (IVP) approach for arbitrary stream data by focusing on the market price variable  $P$  since any real data for the other three variables  $B$ ,  $\zeta_1$  and  $\zeta_2$  in the dynamical system is not available explicitly. We minimize the sum of exponentially weighted squared differences  $F[\tilde{K}]$  between the true trading prices from day  $i$  to day  $i + n - 1$  and the corresponding computed market prices obtained from the first row vector of the numerical solution  $U$  of the IVP with AFDE for  $i$ th optimal parameter vector where  $\tilde{K}$  is an initial parameter vector. Here, the gradient  $(\nabla F(x))$  is approximated by using the central difference formula and step length  $s$  is determined by the backtracking line search. One of the novel components of the proposed asset flow optimization forecast algorithm is a dynamic initial parameter pool which contains most recently used successful parameters, besides the various fixed parameters from a set of grid points in a hyper-box.

**Keywords:** numerical nonlinear optimization; inverse problem of parameter estimation; asset flow differential equations; financial market dynamics; market return prediction algorithm; data analysis in mathematical finance and economics

**AMS Subject Classification:** 65L09, 90C31, 62P05, 93C15, 91B28, 37N40, 62M20, 90C53, 91B24, 91B42, 91B84

## 1. Introduction

Forecasting of asset market prices is of interest from a practical and theoretical perspective. Methodology in forecasting can be divided broadly into two groups. One of these consists of purely statistical methods, e.g., time series, that strive to uncover a statistically significant pattern in the data. A second involves developing some understanding of the underlying processes and deriving, for example, differential equations. In general, there are some parameters that need to be estimated in order for a prediction to be made. For example, in weather forecasting one could make forecasts of temperatures in a way that is strictly statistical based upon the available data for a particular city and its neighbors. The alternative, however,

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is to utilize physical laws and estimate some parameters statistically whereupon the differential equations can be used to make a forecast. An advantage of the latter is that it greatly reduces the degrees of freedom, which, in a practical sense means that there are many fewer coefficients to be estimated. In some cases, there is a conservation law (e.g., the conservation of mass for weather forecasting) that eliminates a wide spectrum of coefficients. The main disadvantages of the modeling approach versus the purely statistical approach is that the former may not be easily possible until a deeper understanding is attained. A secondary problem is that unfamiliarity with modeling leads some to question whether the conclusions are built into the system that has been derived. On the other hand, in a purely statistical model, it appears to be a clear and fair test of the importance of different effects.

However, the issue of the origin of the model is less relevant when it is possible to perform out-of-sample forecast (as is the case for weather forecasting and stock price forecasting, for example) that can be tested statistically to determine the accuracy of the predictions. Related to this issue is the Akaike [1] criterion which is a measure of the balance between the number of parameters to be estimated and the deviation between the model and the actual data. In other words, dramatically increasing the number of parameters in order to provide a slight improvement in the error does not yield a better model, in general. In the case of out-of-sample forecasting of asset prices, however, the number of parameters to be estimated is purely a practical issue. If one can determine a set of parameters within the computational and time constraints to make an accurate forecast, then it is generally not desirable to reduce the number of parameters considerably in order to obtain a forecast that is almost as good. On the other hand the constraints on time may be sufficiently strong that one may be restricted in the number of parameters and the method of estimation. For example, if one is using the government data released during the trading day in order to optimize parameters and make a forecast, one may have a constraint of one minute. On the other hand, if one is using daily data to make forecasts for the next day, one has the overnight time interval from the close of trading to the open of the next day.

Our approach uses the asset flow differential equations (AFDE) that have been developed by Caginalp and collaborators since 1989 (see Caginalp and Balenovich [11] and references contained therein). This approach has several key ingredients as discussed in the next section. It utilizes a basic supply and demand adjustment equation, but also incorporates the finiteness of assets, rather than the assumption of infinite arbitrage capital of classical finance theory. Furthermore, the supply and demand are determined by a transition rate that is dependent on sentiment. Classically, sentiment should depend only on a discount or premium to valuation. However, recent decades of research work has documented a set of motivations beyond valuation. One of the most significant of these is the price trend, also called momentum. The equations can readily incorporate additional motivational aspects of trading as they are established. In fact, one way of implementing this is to modify the differential equations to difference equations, and then to use statistical methods to evaluate the coefficients. Thus, if one hypothesizes a price trend motivation, the confirmation is the determination of a coefficient that is positive and statistically significant. If the coefficient corresponding to a particular hypothesized behavioral motivation is not actually present, then the coefficient, by definition, will be within the standard error of zero.

The implementation of these differential equations for practical forecasts poses challenging mathematical tasks that are inverse problems. Once a set of parameters characterizing an investor population is specified, the differential equations can be

solved for future times. However, the values of the parameters are not known *a priori*, so that parameters must be determined by optimization using the actual price history up to the present time. We perform this optimization by a nonlinear computational algorithm that combines a quasi-Newton weak line search with the BFGS formula. We use nonlinear least-square technique with initial value problem approach by focusing on the market price variable  $P$  since any real data for the other three variables  $B$ ,  $\zeta_1$  and  $\zeta_2$  in the dynamical system is not available explicitly. Here, the gradient ( $\nabla F(x)$ ) is approximated by using the central difference formula and step length  $s$  is determined by the backtracking line search among several choices in literature (see Nocedal and Wright [29]). We construct a pool of initial parameters  $K_i$  chosen via a set of grid points in a hyper-box. We select an initial parameter vector from the initial parameter pool because the optimization success of quasi-Newton method in the algorithm depends on the initial parameter. Besides the fixed part of various initial parameters, the dynamic part of the pool is updated by adding successful parameters so that we keep a pool of various and most recently used candidate parameters.

In this paper, we utilize a system of differential equations that incorporate the valuation and trend motivations similar to Caginalp and Balenovich [11]. The question of valuation of a particular asset is often difficult to assess unambiguously. Nevertheless, the methodology for developing a valuation model is well established in finance, and a model for valuation can be constructed for a particular asset class. We utilize a particular group of stocks for which the valuation is very clear namely the set of closed-end funds traded on the NYSE. Briefly, a closed-end fund (see Bodie et al. [6]) is formed as investors pool a sum of money for a particular investment, e.g., investing in common stocks in Japan. Shares are allocated to the shareholders, who may then trade their shares on the open market, just like any other stock on the NYSE, and with the same rules. For most closed-end funds, the net asset value per share (NAV) is computed daily or weekly, and reported to the shareholders. Unlike an open-end fund, however, the shares of a closed-end fund cannot be redeemed by the fund except under special circumstances. This, of course, introduces the possibility that the fund may trade (on the NYSE) lower than the NAV (called a discount) or higher (called a premium) than the NAV. The fact that these funds often trade at significant discounts (e.g. 5% to 25%) and sometimes at large premiums (e.g. 50%) has been a puzzle to classical finance, and many papers have addressed these issues (see Anderson and Born for a summary [2]). A number of these papers have focussed on reconciling these discounts to the efficient market hypothesis (EMH) whose centerpiece is the concept that the market trading price should reflect all available public information and should therefore reflect the true value. For example, a fund may trade at a discount due to an inherent tax liability ([2], Chapter 6) due to profits that are unrealized for tax purposes.

Relatively few studies have considered the issue of the changes in the discounts. Even if there is a good reason for the fund to trade at a discount, why should the discount fluctuate significantly? One study focussing on this aspect is Duran and Caginalp [17] which demonstrated that (i) there are many significant changes in the discount on a daily basis, ranging from 2.5% to over 10%, (ii) there is movement in the opposite direction on the days following this significant change, and (iii) there are significant precursors to the significant changes, also in the opposite direction. They found a characteristic overreaction diamond pattern, revealing a symmetry in deviations before and after the significant change. In addition, the magnitude of the reversal increases as the degree of deviation increases. Much of the statistical significance and the patterns disappear when the subtraction of

NAV return is eliminated, suggesting that the frequent changes in fundamentals mask behavioral effects. Furthermore, they (see [16] and [17]) subdivided the data depending on whether the NAV or market price is responsible for the spike in the relative difference. In a majority of spikes, it is the change in market price rather than NAV that is dominant. In general this paper showed that there is a systematic pattern that is consistent with the overall perspective of the asset flow dynamics. In particular these equations suggest that when there is dip below the realistic or fundamental value, it is generally concurrent with a move toward higher cash positions on the part of investors. Coupled with the motivation to buy an asset at a discount, this higher cash position leads quickly to higher prices. The situation is similar in the other direction.

Our goal in this paper is to develop a methodology to optimize the parameters for these differential equations that will make possible out-of-sample predictions. In particular, we utilize the NYSE data for a large set of closed end funds over a large time period comprising 8,415 data points. On any particular day for any closed-end fund, we use the trading price for a recent time period as well as the NAV in order to optimize the parameters of the differential equations. For each set of four parameters, the corresponding differential equations are obtained and solved numerically. We compute the sum of exponentially weighted squared differences between the true trading prices at times  $t_i, \dots, t_{i+n-1}$  and the corresponding computed market price values which are obtained from the first row vector of the numerical solution  $U$  of the IVP for  $i$ th parameter set. The optimization methods are then used to minimize this difference, i.e., the error, along the candidate set of parameters obtained via the quasi-Newton parameter search by selecting suitable step length  $s_k$  to move along search direction  $P_k$ . The parameter minimizing the sum of difference is picked. Using this parameter set, the IVP is solved numerically to render a forecast for the next day. The previously found optimal parameter set is added to the initial parameter pool. The parameter pool is specific to the fund's price behavior. By repeating this procedure one obtains a large set of forecasted values that can be compared with the actual closing prices for the particular closed end fund. A gauge of the accuracy of these estimates can be obtained by comparing the absolute values of the error in each day's forecast with the error in assuming random walk. The latter is the default hypothesis asserted by EMH that the best forecast of tomorrow's price is today's price plus the average return per day (which is negligibly small). The two sets of positive numbers can then be compared statistically to determine if one is smaller than the other with statistical significance.

In literature, many parameter optimization techniques have been studied for various differential equations in specific science and engineering problems or applications by testing with a limited data set rather than arbitrary conditions. For example, unconstrained derivative-free optimization, unconstrained optimization with derivatives, forward sensitivity analysis and adjoint (backward) sensitivity analysis, (see Biegler and Tjoa [5], Bremermann [7], Dennis et al. [14], Dunker [15], Guay and McLean [18], Kiehl [21], Kramer and Leis [22], Lee and Hovland [23], Maly and Petzold [25], Milstein [27], Milstein [28], Serban and Hindmarsh [32] and references contained therein). Derivative-free algorithms employ only function values for the search direction and they may suffer from the stagnation around a relative minimum. Moreover, random directional line search (a coordinate descent method) is an indeterministic method because of the random directions (see [29] for the inefficiency of coordinate descent methods in practice). There is no unique perfect method which works best for all global unconstrained optimization problems. The quasi-Newton method with dynamic initial parameter pool is a feasible

one for our semi-unconstrained optimization problem from financial application.

In different applications we may meet with specific challenging problems. For example,

- Some initial parameters may lead to singularities at  $k = 0$ ,  $B = 0$ ,  $k = 1$ , or  $B = 1$  in the AFDE during parameter optimization process. Our implementation handles this problem.
- We apply nonlinear optimization technique for arbitrary conditions (various initial parameters and variable values, and a large real data set) in a challenging financial application.
- For optimization methods using derivatives in a nonlinear model it is crucial to start the iteration close enough to the potential global minimum to get rid of being caught in a local minimum. It is harder to choose such initial parameters for arbitrary stream data of stock market.
- We believe the space of function curves obtained via real market data is different and richer than that of, for example, chemical reaction data. Repeatability of same experiment versus real time update forecast is another difference.
- There is a wide range of variability in obtaining optimal parameters for the nonlinear problem. That is, the residual values may change between  $10^{-1}$  and  $10^{-14}$ .
- We have methods which are efficient for certain stiff and non-stiff applications (see Ascher and Petzold [3]). For an arbitrary day price prediction via the asset flow differential equations, we meet with both stiff problems (having widely varying time scales where the standard numerical methods may require extremely small step size  $h$ ) and non-stiff problems.

There are more factors affecting the success of the prediction besides the difficulties of the optimization process. Forecasting is often difficult in many disciplines. For example, weather forecasting (see Mak [24]) has been studied extensively for many decades with some success, and yet there are still many surprises. In the case of markets, forecasting is especially difficult since one is trying essentially to make a forecast that is better than the aggregate forecast of the market participants. The efficient market hypothesis asserts that this is not possible.

The studies using raw data (for example, the standard Box-Jenkins procedure) have shown that random walk is the best of the auto-regressive integrated moving average (ARIMA) models. Moreover, the ARIMA model has no realistic chance of showing a turning point. Furthermore, one of the recent studies (see Hutchinson [20], pp 106) could obtain success rates between 44.90% and 51.02% for performance of out of sample prediction by using financial time series analysis models such as random walk and auto-regressive/moving average (ARMA(0,1)).

To the best of our knowledge, this is the first study to find the next day price return direction for an arbitrary day with a significant right match probability greater than 52% by using the power of differential equations.

The remainder of the paper is organized as follows. In Section 2, we present our parameter optimization approach for the asset flow differential equations, the corresponding algorithm, and optimization results with three examples. In Section 3, our method of market price return prediction, two success tests and prediction results are discussed. The decision parameters for success performance are the direction of market and the absolute difference of predicted return and actual return. Section 4 concludes the paper. The Appendix includes the quasi-Newton method with our implementation choices.

## 2. Parameter optimization for differential equations

### 2.1. Optimization problem

We use a nonlinear computational optimization technique successively to evaluate the vector  $\bar{K}$  of four parameters  $(c_1, q_1, c_2, q_2)$  in the asset flow differential equations (AFDE) (see [11]). That is, the inverse problem of parameter identification is converted into an optimization problem to minimize a function in four variables by using nonlinear least-square curve fitting via the AFDE. We try to employ most of the data up to any given time in order to choose the parameters optimally. Then, we make a forecast for the next few days, and compare the forecasts with the actual values.

In practice, optimization problems may have several local solutions. However, optimization methods which seek global minima can confuse whether a point  $K^*$  that has been found is a local minimum or a global minimum. There is no strategy that will guarantee the number of necessary iterations to discover the neighborhood of the global optimum (see Bartholomew-Biggs [4], Chapter 23). Therefore, we use an initial parameter pool which has only fixed vectors chosen via a set of grid points in a hyper-box at the beginning of the optimization process for each fund. The second part of the pool is updated via previously found optimal parameters and specific to the fund's price behavior. Then, we pick the minimum of the resulting relative minimum function values and the corresponding optimal parameter to be used for the next day return prediction.

After presenting the proposed optimization algorithm in this section, we discuss the out-of-sample daily return prediction in Section 3.

### 2.2. The system of asset flow differential equations (AFDE)

#### 2.2.1. Notation

We define the variables in the system as follows

$P(t)$  : The market price (MP) of the single asset at time  $t$ .

$\frac{1}{P} \frac{dP}{dt}$  : The relative price change.

$P_a(t)$  : The fundamental value.

$V(t)$  : The net asset value (NAV) price at time  $t$ .

$M$  : All the cash in the system.

$N$  : The total number of shares.

$L := \frac{M}{N}$  : The liquidity value.  $L$  is a fundamental scale for price.

$B$  : The fraction of total funds in the asset.

$\zeta_1(t)$  : The trend-based component of the investor preference.

$\zeta_2(t)$  : The value-based component of the investor preference.

#### 2.2.2. Assumptions

- (A) We consider trading in a single stock and let  $B$  denote the fraction of total assets that are invested in the stock, so that  $1 - B$  is the fraction in cash. Let  $k \in (0, 1)$  denote the transition rate, or the probability that a unit of cash will be used to purchase stock (see Figure 1). Then the demand,  $D$ , is given by this transition rate times the fraction in cash, i.e.,  $D = k(1 - B)$ , while the supply,  $S$ , is  $S = (1 - k)B$  so that

$$\frac{D}{S} = \frac{k}{1 - k} \frac{1 - B}{B} \quad (1)$$

(B) The transition rate,  $k$ , is derived from the investor sentiment,

$$\zeta = \zeta_1 + \zeta_2, \quad (2)$$

by mapping the  $\zeta$  values in the range of  $(-\infty, \infty)$  into the interval  $(0, 1)$ . This is most easily done through a *tanh* function,

$$k(t) = \frac{1}{2} + \frac{1}{2} \tanh(\zeta), \quad (3)$$

although the linearization of  $\tanh(x) \cong x$  yields very similar results under the conditions that are relevant in practice. The sentiment,  $\zeta$ , is a sum of contributions due to motivations for buying and selling. We utilize two of these: momentum and valuation, denoted  $\zeta_1$  and  $\zeta_2$ , respectively. The momentum term consists of an integral of the derivative of price, with a weighting factor  $e^{-c_1 t}$  so that recent price changes have the largest weight.

$$\zeta_1 \equiv q_1 c_1 \int_{-\infty}^t e^{-c_1(t-\tau)} \frac{1}{P(\tau)} \frac{dP(\tau)}{d\tau} d\tau, \quad (4)$$

Similarly, the valuation is an integral over the relative deviation of price from the “fundamental value,” reflecting the fact that some time elapses for investors to act on undervaluation.

$$\zeta_2 \equiv q_2 c_2 \int_{-\infty}^t e^{-c_2(t-\tau)} \frac{P_a(\tau) - P(\tau)}{P_a(\tau)} d\tau \quad (5)$$

The constants  $1/c_1$  and  $1/c_2$  are the time scales, respectively, for the momentum and valuation strategies. The parameters  $q_1$  and  $q_2$  are the coefficients of the trend-based and value-based sentiment, respectively.

(C) The relative price change is proportional to the excess demand,  $(D - S)/S$ , and yields,

$$\frac{\tau_0}{P} \frac{dP}{dt} = \frac{D}{S} - 1. \quad (6)$$

### 2.2.3. The asset flow equations

The immediate implication of the assumptions above is a set of ordinary differential equations coupled with an algebraic equation.

1. Using the equations for demand and supply we write the price equation as

$$\frac{1}{P} \frac{dP}{dt} = f\left(\frac{D}{S}\right) = f\left(\frac{k}{1-k} \frac{1-B}{B}\right) \quad (7)$$

where  $f$  is an increasing function satisfying that  $f(1) = 0$  and taken as  $f(x) = \delta \log(x)$  for a constant amplitude  $\delta$ , for example 1, that scales time in equation (7).

2. The fraction of assets invested in the stock can change as assets flow between

## Transition Diagram

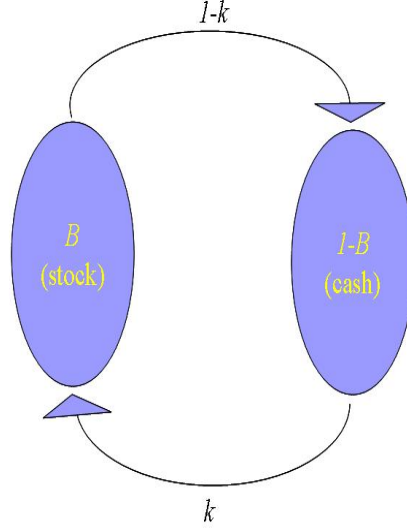


Figure 1. Transition.

cash and stock, and also through changes in the stock's price (See Figure 1):

$$\frac{dB}{dt} = k(1 - B) - (1 - k)B + B(1 - B)\frac{1}{P}\frac{dP}{dt} \quad (8)$$

3. The transition rate,  $k$ , satisfies the algebraic equation

$$k = \frac{1}{2} + \frac{1}{2} \tanh(\zeta_1 + \zeta_2). \quad (9)$$

4. Upon differentiating the integrals in the sentiment functions we obtain ordinary differential equations for these variables. The trend based component of the sentiment,  $\zeta_1$ , satisfies the differential equation

$$\frac{d\zeta_1}{dt} = c_1\left(\frac{q_1}{P}\frac{dP}{dt} - \zeta_1\right) \quad (10)$$

5. Similarly, the value based component satisfies

$$\frac{d\zeta_2}{dt} = c_2(q_2A(t) - \zeta_2). \quad (11)$$

The fundamental value, as perceived by the investors, needs to be specified. In principle one could choose this factor to be  $(V_t - P_t)/V_t$ . This would imply that investors are inclined to buy whenever the value of the closed-end fund is below the price, and likewise for selling. However, this is usually not the case, as many such funds trade at a chronic discount, and some at a premium. For example, the discount for some funds is often near 10%. The fact that the discount is 10% today does not mean people are eager to buy it due to undervaluation. However, if it goes to a 20% discount then some people look at that as a bargain. Similarly, there are funds that are usually at a 25% premium, so that a 10% premium is perceived to be an undervaluation. Hence, we assume that the investors will buy when the discount/premium is below a weighted average of the past ten days with

the most recent weighted highest. Using a discrete exponential weighting we let  $\sum_{k=1}^{10} e^{-0.25k} = 3.2318$  (see Caginalp and Ilieva [12]) related to the normalization and the relative valuation change

$$A_t = \frac{V_t - P_t}{V_t} - \left( \sum_{k=1}^{10} (3.2318)^{-1} \frac{V_{t-k} - P_{t-k}}{V_{t-k}} e^{-0.25k} \right) \quad (12)$$

for our discrete implementation of equation (11).  $q_2$  is multiplied by  $A(t)$  which is the difference between the discount at  $t$  and the exponentially weighted average value of a discount that persists.

The model used above (see Caginalp and Balenovich [11] and references therein) generalizes the classical price adjustment equations in two important aspects. First, there is the assumption of a finite asset base, rather than the classical assumption of infinite arbitrage of classical finance. Second, the motivation for buying or selling an asset can have various origins beyond valuation.

An implication of the finiteness of assets is the concept of “liquidity price” that was introduced in (Caginalp and Balenovich [11]) and defined as the total available cash divided by the number of shares in the system. A mathematical analysis of these equations leads to the conclusion [11] that in the absence of significant attention to value, the price tends to gravitate toward this liquidity price, a conclusion that was borne out in several experimental settings.

The system of equations (7-11) is a mathematically complete system that can be solved numerically with an ODE solver for suitable parameters satisfying that  $0 < k < 1$  and  $0 < B < 1$ . Previous studies have shown that oscillations tend to increase as the momentum coefficient,  $q_1$ , increases and the associated time scale  $1/c_1$  decreases. As  $q_2$  increases, prices tend to move closer to the fundamental value.

### 2.3. Non-linear least-square techniques with initial value problem (IVP) approach

Suppose we have a sequence of true daily market prices  $\bar{Z}(t_s)$  and net asset values  $\bar{V}(t_s)$ ,  $s = 1, \dots, m$  at times  $t_1, \dots, t_m$ . Using the differential equations, our goal is to choose the parameters  $\bar{K} = (c_1, q_1, c_2, q_2) \in \mathfrak{R}_+^4$  in an optimal manner.

$$\frac{dU}{dt} = \begin{bmatrix} \frac{dP}{dt} \\ \frac{dB}{dt} \\ \frac{d\zeta_1}{dt} \\ \frac{d\zeta_2}{dt} \end{bmatrix} = g(U, \bar{K}, t), \quad U(t_1) = \begin{bmatrix} \bar{Z}(t_1) \\ 0.5 \\ 0 \\ 0 \end{bmatrix} \quad (13)$$

where  $g$  is a function obtained from the AFDE (7-11). We solve the IVP (13) above for  $U$  by using Runge-Kutta (RK4) method and an assumed value  $\tilde{K}$  of the parameter  $\bar{K}$  from the fund’s dynamic initial parameter pool  $K$ . We define  $F[\tilde{K}]$  such that

$$F[\tilde{K}] := \sum_{s=i}^{i+n-1} W(s-i+1) \{ \bar{Z}(t_s) - P(\tilde{K}, t_s) \}^2 \quad (14)$$

where  $F[\tilde{K}]$  represents the sum of exponentially weighted squared differences between the actual market price values  $\bar{Z}(t_s)$  and the computed market price values  $P(\tilde{K}, t_s)$  obtained from the first row vector of the numerical solution  $U$  of IVP (13) by picking the values at time  $t_s$ s where  $t_s \in [i, i + n - 1]$  for  $i$ th parameter vector. We try to minimize  $F[\tilde{K}]$  over  $\mathfrak{R}^4$  by using line search algorithms. Here,  $W$  is a special positive weighting vector of normalized exponentially increasing entries that we obtain by

$$W(s) = \frac{e^{-0.25(n-s+1)}}{2.51208223355669}$$

for  $s = 1..n$  and

$$\sum_{s=1}^n W(s) = 1.$$

For example,

$$W = (0.114051, 0.146444, 0.188038, 0.241445, 0.310022)^T$$

for  $n = 5$ .

The dynamical system (7-11) has four first order ordinary differential equations and one algebraic equation. It is non-linear in terms of the dependent variables. Moreover, there are products of optimization parameters in the system (7-11) e.g.  $c_1q_1$  in equation (10) and  $c_2q_2$  in equation (11). The optimization problem is a non-linear least-squares problem since the subfunctions in the equation (14) are not linear in the components of  $\bar{K}$ . Furthermore, for a financially meaningful model we require  $\bar{K} > 0$ .

In the calculations we use the initial conditions for  $P$  (of  $\bar{Z}(t_1)$ ),  $B$  (of 0.5),  $\zeta_1$  (of 0) and  $\zeta_2$  (of 0) (see equation (13)). Subsequently, these evolve from the dynamical system. We are not using  $M$  directly.

#### 2.4. Main optimization algorithm

Given an  $n$ -day period of market prices (MP) and net asset values (NAV) from day  $i$  to day  $i + n - 1$  as  $i$ th event where  $n = \tau_1 + 1$  and  $i > \tau_2$ , we compute optimal parameter vector  $\bar{K}_i$  for the period  $i$ . Then, we obtain  $m - i + 1$  optimal parameters for the overlapping periods such as  $[i, i + n - 1]$ ,  $[i + 1, i + n]$ , ...,  $[m, m + n - 1]$  for the MP sequence of size  $m + n - 1$ .

There is a tradeoff for selection of  $n$ . We choose  $n$  sufficiently small in order to use the daily market price and net asset value. Moreover, local price patterns which are related to 3 to 15 trading days on average (see Duran and Caginalp [17] and Caginalp and Balenovich[10]) can be exploited by small values of  $n$  during optimization and prediction processes. On the other hand,  $n$  should be large enough so that the parameter optimization process can capture the price trend reasonably. For example, we tested for  $n = 5$  and  $n = 10$ .

We implement a line search algorithm to obtain optimal parameters during the optimization process (see Duran [16]). Our algorithm uses a quasi-Newton method with weak line search for minimizing the sum of squares defined in (14) by using the AFDE. It has a fast rate of convergence and it is efficient. We mainly focus on the quasi-Newton method due to its advantages within our study. The Newton method was not preferred because it requires computation of a second derivative. Although

it is possible to use finite difference expressions, the calculation of derivatives is one of the most time consuming parts (see [4] and [29]). Even if in  $\mathfrak{R}^4$ , we should do it so many times for a large data set in a real time update forecast. Moreover, the approximation can be inaccurate.

The Appendix includes definition of a line search method, quasi-Newton method and backtracking line search algorithms, our implementation choices, the central difference formula, the BFGS formula, and the cost and convergence of the algorithms.

We present the optimization process via Algorithm 1 and the flowchart in Figure 2 with definition of the constants, variables, and functions in the algorithm.

$c_{NLS}$  : The computed error  $F[\tilde{K}]$  defined in the equation (14). It is the objective function to be minimized.

$\bar{Z}$  : A sequence of closed-end fund market prices. For example, daily closing prices during 1998-2006.

$\bar{V}$  : A sequence of closed-end fund net asset value prices. For example, daily data during 1998-2006.

$\tau_1$  : Period of event minus 1 over which optimal parameter vector is found.

$\tau_2$  : Long period of most recent days before the beginning of an event day. It is used to estimate the relative valuation changes defined in the equation (12).

$z_{fixed}$  : Number of parameter vectors in the initial fixed parameter pool.

$z_{max}$  : Maximum parameter pool size.

$h_{RK}$  : RK4 step size.

$i$  : Day index from the price list of a fund. It corresponds to the beginning of the current event period.

$i_{first}$  :  $i$  of the first event.

$K_{fixed}$  : The pool of initial parameters and  $K_{fixed}(i, :) \in [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \times [a_4, b_4]$  for  $i = 1..z_{fixed}$ .

$K$  : The pool of running initial parameter vectors. While the initial  $K$  is a  $z_{fixed} \times 4$  matrix  $K_{fixed}$ , it can be at most a  $z_{max} \times 4$  matrix.

$K_q$  : A candidate optimal parameter vector which is obtained by quasi-Newton method given the  $j$ th initial parameter from  $K$  for the  $i$ th event.

$\hat{K}$  : A matrix whose row vectors consist of positive  $K_q$  vectors which provide small errors  $F[K_q] < \epsilon_2$  for the  $i$ th event.

$K_{iGLOpt}$  : A positive optimal vector in the row vectors of  $\hat{K}$  which provides the minimum error  $F[K_q]$  for the  $i$ th event.

$K_{CEF}$  : A sequence of global optimal parameters for a certain closed-end fund (CEF).

$\epsilon_1$  : Threshold for the gradient, for example  $10^{-6}$ .

$\epsilon_2$  : Threshold for the  $c_{NLS}$  defined in (14) according to the exponential weights. For example,  $\epsilon_2 = 0.16$  means that the average error allowed per day for fitting (during optimization phase) is  $\sqrt{0.16}$  corresponding to \$0.40.

QN: A function call to obtain candidate optimal parameter vectors by using quasi-Newton weak line search with BFGS formula and a dynamic initial parameter pool.

$Q_{check}$  :  $Q_{check}$  takes -1 which indicates a failure during quasi-Newton function call related to gradient approximation or singularity at  $k = 0$ ,  $B = 0$ ,  $k = 1$ , or  $B = 1$ . Otherwise, it is 1 referring a success.

NLS: A function call to obtain the  $c_{NLS}$  corresponding to a candidate optimal parameter vector.

**Algorithm 1: The optimization process in the asset flow optimization forecast algorithm**

Inputs:  $\bar{Z}$ ,  $\bar{V}$ ,  $K_{fixed}$ ,  $\tau_1$ ,  $\tau_2$ ,  $i_{first}$ ,  $h_{RK}$ ,  $z_{max}$ ,  $\epsilon_1$ , and  $\epsilon_2$

Output:  $K_{CEF}$

1. Set  $n = \tau_1 + 1$ ,  $B = 0.5$ ,  $\zeta_1 = 0$ , and  $\zeta_2 = 0$
2. Set  $i_{last} = \text{length}(\bar{Z}) - n$ ,  $K = K_{fixed}$  and  $K_{CEF} = []$
3.  $l = \text{length}(K)$
4.  $z_{fixed} = l$
5. **for**  $i = i_{first} : i_{last}$  (event loop)
  - $Z = \bar{Z}_{i-\tau_2:i+\tau_1}$
  - $V = \bar{V}_{i-\tau_2:i+\tau_1}$
  - $\hat{Z} = Z_{\tau_2+1:\tau_2+n}$
  - $\hat{A} = \text{zeros}(n, 1)$
  - **for**  $s = 1 : n$  (relative valuation change loop)
    - for**  $k = 1 : \tau_2$  (chronic discount loop)
      - $u = s + \tau_2 - k$
      - $\hat{A}_s = \hat{A}_s + \frac{V_u - Z_u}{V_u} e^{0.25k}$
  - $A_s = \frac{V_{s+\tau_2} - Z_{s+\tau_2}}{V_{s+\tau_2}} - \frac{\hat{A}_s}{3.23180584357794}$
- Reset local variables  $\hat{K} = []$  and  $c_{locOpt} = []$
- $j = 0$
- **while** ( $j < l$ ) (multi-start initial parameter loop)
  - $j = j + 1$
  - $K_0 = K_{j,:}^T$
  - $[K_q, Q_{check}] = QN(K_0, t_1, t_2, h_{RK}, \hat{Z}, A, B, \zeta_1, \zeta_2, \epsilon_1)$
  - if** ( $(\text{length}(K_q) \neq 0) \& (Q_{check} == 1)$ )
    - $c_{NLS} = NLS(K_q, t_1, t_2, h_{RK}, \hat{Z}, A, B, \zeta_1, \zeta_2)$
    - if** ( $(c_{NLS} < \epsilon_2) \& (K_q > 0)$ )
      - $\hat{K} = [\hat{K}; K_q^T]$
      - $c_{locOpt} = [c_{locOpt}; c_{NLS}]$
- $s_{locOpt} = \text{size}(c_{locOpt})$
- $l_{locOpt} = s_{locOpt}(1)$
- **if** ( $l_{locOpt} > 0$ )
  - $c_{iGLOpt} = \text{min}(c_{locOpt})$
  - $j_{iGLOpt} = \text{find}(c_{locOpt} == c_{iGLOpt})$
  - $K_{iGLOpt} = \hat{K}_{j_{iGLOpt},:}$
  - if** ( $l == z_{max}$ )
    - $K = [K_{1:z_{fixed},:}; K_{iGLOpt}; K_{z_{fixed}+1:z_{max}-1,:}]$
  - elseif** ( $l == z_{fixed}$ )
    - $K = [K_{1:z_{fixed},:}; K_{iGLOpt}]$
    - $l = l + 1$
  - else**
    - $K = [K_{1:z_{fixed},:}; K_{iGLOpt}; K_{z_{fixed}+1:end,:}]$
    - $l = l + 1$
- $K_{CEF} = [K_{CEF}; K_{iGLOpt}]$

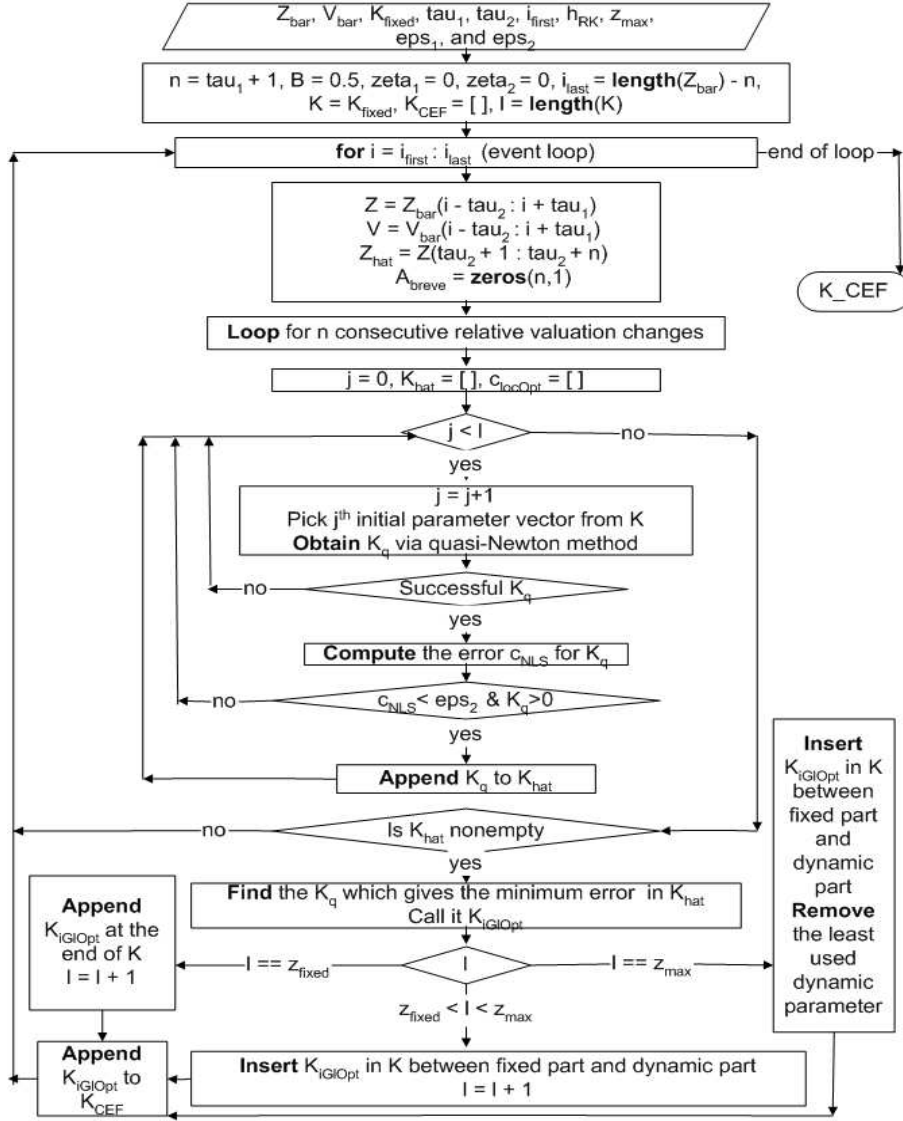


Figure 2. Flowchart for the evolution of optimal parameters during the optimization process in the asset flow optimization forecast algorithm.

## 2.5. Optimization results

We present our findings with the main optimization algorithm by the following three examples. The average maximum improvement factor (MIF) is used to measure the performance of the optimization process where

$$MIF = c_{\text{NLSmin}}/c_{\text{NLSinit}}. \quad (15)$$

Generally, the smaller MIF corresponds to a better performance which depends on the closeness of the initial parameter to the optimal one as well.

**Example 1.** Given the following actual market price, net asset value over the five trading days vector beginning on Friday for Alliance All-Market Advantage Fund (AMO) (a general equity fund (GEF)), and an initial pool having 56 parameter

Table 1. **The computational optimization by finding parameters in the asset flow differential equations for a small example.** Quasi-Newton method with weak line search is applied for the AMO fund data during 8.13.1999-8.24.1999.

Number of Events	8
Event Period	5-day
$h_{RK}$	0.05
$z_{fixed}$	56
$z_{max}$	80
$\epsilon_1$	$10^{-6}$
Prediction Attempt	100%
Average Number of QN Iteration	132
$\epsilon_2$	0.16
Average $c_{NLS}$	0.0572
Average MIF	57.26 %

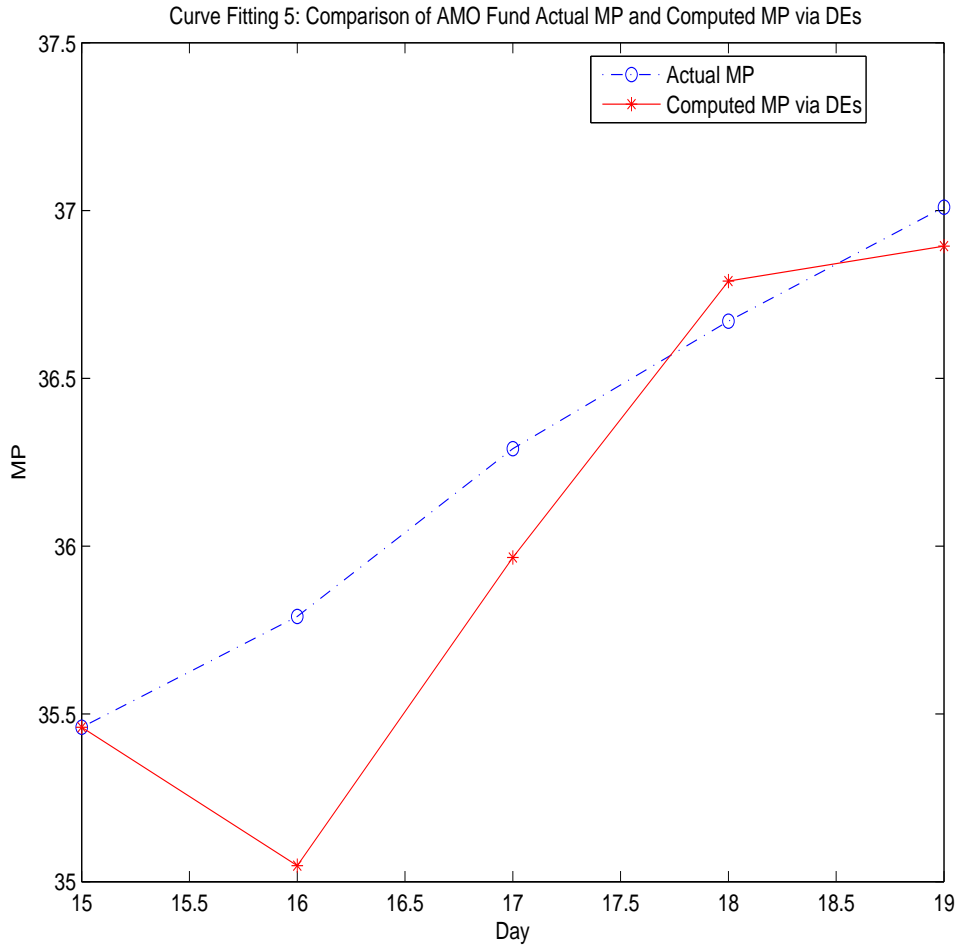


Figure 3. Curve fitting and getting optimal parameters for AMO MP's over the fifth 5-day period.

vectors, we find the first optimal parameter vector with event index 11 as follows.

The first 10-day MP and NAV are used to compute the relative valuation change. Here are the following 5-day MP and NAV prices.

$$\bar{Z}_{11:15} = (35.40, 35.62, 35.62, 35.68, 35.46),$$

$$\bar{V}_{11:15} = (43.95, 44.08, 44.75, 44.45, 43.93),$$

and

$$\text{Day} = (8.13.1999, 8.16.1999, 8.17.1999, 8.18.1999, 8.19.1999).$$

After applying the main optimization algorithm in subsection 2.4, we obtain 56 candidate optimal parameter vectors via  $QN$  function calls. We allow only the positive candidate vectors satisfying the threshold condition with  $\epsilon_2$ . Thus, we obtain a set of candidate vectors  $\hat{K}$  and the corresponding set of minimized functional values  $c_{locOpt}$ . Later, we find the minimum of  $c_{locOpt}$  and the related optimal parameter vector  $K_{iGLOpt}$ . After the curve fitting over the first 5-day period, the first optimal parameter vector will be used to predict next day (i.e. 8.20.1999) return. The optimal parameter vector is appended to the initial parameter pool so that the experience can be exploited for future optimizations.

Similarly, we obtain the second optimal parameter vector  $K_{iGLOpt}$  with event index 12 by using

$$\bar{Z}_{12:16} = (35.62, 35.62, 35.68, 35.46, 35.79)$$

and

$$\bar{V}_{12:16} = (44.08, 44.75, 44.45, 43.93, 44.34)$$

for AMO over

$$\text{Day} = (8.16.1999, 8.17.1999, 8.18.1999, 8.19.1999, 8.20.1999),$$

and the initial pool having 57 parameter vectors. The second optimal parameter vector can be used to predict next trading day (8.23.1999) return.

Table 2. **Initial parameters.**

Event #	Initial Parameters			
	$c_1$	$q_1$	$c_2$	$q_2$
11	0.501000	5.010000	0.005000	0.010000
12	0.501520	5.010123	0.033413	0.038930
13	0.001000	0.010000	1.005000	5.010000
14	0.001000	5.010000	2.000000	10.010000
15	0.501000	5.010000	1.005000	5.010000
16	0.001000	5.010000	1.005000	0.010000
17	0.001000	10.010000	1.005000	5.010000
18	0.001000	10.010000	0.505000	5.010000

Figure 3 shows one of the eight consecutive optimization processes in this small example. Table 1 summarizes the cost of the optimization process and MIF.

While Table 2 illustrates the initial parameter vectors which could lead to optimal parameters for the events from 11 to 18, Table 3 shows the resulting optimal parameter vectors for these events.

**Example 2.** We obtain the optimal parameters for six sample closed-end funds with event 5-day periods described by Table 4. If we cannot determine an optimal parameter satisfying the desired conditions, we skip the 5-day event and the next

Table 3. **Optimal parameters.**

Event #	Optimal Parameters			
	$c_1$	$q_1$	$c_2$	$q_2$
11	0.501520	5.010123	0.033413	0.038930
12	0.502606	5.010537	0.046891	0.055083
13	0.000248	0.009944	0.567693	4.935129
14	0.001555	379.573677	52.749315	8.049116
15	0.002802	419.708991	52.957131	8.322994
16	0.703218	5.117946	1.111443	1.458326
17	0.003087	133.925347	0.005573	649.593726
18	0.002073	565.436200	0.004883	320.677185

Table 4. **The computational optimization by finding parameters in the asset flow differential equations for a large sample data set.** Quasi-Newton method with weak line search is applied for a six sample closed-end funds data during 1998-2006.

Number of Events	8411
Event Period	5-day
$h_{RK}$	0.05
$z_{fixed}$	56
$z_{max}$	80
$\epsilon_1$	$10^{-4}$
Prediction Attempt	66.46 %
Average Number of QN Iteration	80
$\epsilon_2$	0.16
Average $c_{NLS}$	0.0124
Average MIF	32.16 %

day prediction. In this manner, we are able to render a prediction attempt 66.46% of the time.

Table 5. **The computational optimization by finding parameters in the AFDE for 10-day event period.** Quasi-Newton method with weak line search is applied for the APB data during the trading days 1.17.2002-6.20.2003.

Number of Events	339
Event Period	10-day
$h_{RK}$	0.05
$z_{fixed}$	56
$z_{max}$	80
$\epsilon_1$	$10^{-5}$
Prediction Attempt	100 %
Average Number of QN Iteration	38
$\epsilon_2$	1.00
Average $c_{NLS}$	0.0491
Average MIF	66.64 %

**Example 3.** We find the optimal parameters for Asia Pacific Fund (APB) (a world equity fund (WEF)) with event period of 10-day by following the instructions in Table 5. If we cannot obtain an optimal parameter satisfying the desired conditions, we use the most recent computed optimal parameter so that we have 100 % prediction attempt. Otherwise, the prediction attempt would be 73.75%.

### 3. Market price return prediction

During the past several decades, the dominant theory of finance has been the efficient market hypothesis (EMH). In its weak form the EMH asserts that any information relating to price cannot be used for excess profit since such information is readily available to anyone. In its stronger form EMH asserts similarly that all publicly available information cannot be used to increase profits beyond the risk premium inherent in that class of investments. Consequently, the best possible prediction that can be made for the price of a stock is given by

$$\frac{P_{t+1} - P_t}{P_t} = \beta r_M + \varepsilon_t. \quad (16)$$

In other words, the best predictor of tomorrow's price is today's price augmented by the tiny factor  $\beta r_M$  which represents the expected daily return for the overall market (i.e., a few percent divided by the 250 trading days per year) times the beta factor that scales the volatility of the stock relative to the overall market. The term  $\varepsilon_t$  is the excess return specific to the stock for day  $t$ . The mean of this term according to EMH must be zero for reasons stated above. Thus, we can state that neglecting a term of order  $(10\%)(1/250) = 1/2500$ , EMH asserts that the best forecast of tomorrow's price assuming knowledge of today's price is

$$P_{t+1} = P_t + \tilde{\varepsilon}_t, \quad (17)$$

i.e., random walk (plus a tiny upward drift term).

Practitioners in financial markets generally do not subscribe to EMH, and often believe that the price trend of the asset, for example, has an important bearing on future prices. However, these ideas are often difficult to test due to the presence of "noise" or fluctuations in the asset's value due to the randomness of world events. Consequently, academic studies tend to show either no measurable advantage to trading strategies or one that is smaller than trading costs, e.g. Poterba and Summers [30].

Caginalp and Laurent [13] performed the first scientific test providing strong evidence in favor of any trading rule or pattern on a large scale. They applied a non-parametric statistical test for the predictive capabilities of candlestick patterns using daily data for each stock in the S&P 500 during the time period 1992-1996. The out-of-sample tests indicate statistically significant profit of almost 1% during a two-day holding period. Moreover, Caginalp and Balenovich [10] develop a foundation for the technical analysis of securities by using a dynamical microeconomic model. They deal with a broad spectrum of patterns that are generated by the presence of two or more trader groups with asymmetric information, in addition to the patterns generated by the activities of a single group.

Rapach et al. [31] employ in-sample and out-of-sample procedures related to data mining for international stock return predictability with macro variables.

In this section, we study price forecast by solving the initial value problem (13) with the asset flow differential equations (AFDE) (7-11) for an arbitrary day independent from a pattern. In other words, we employ the dynamical microeconomic model (7-11) which provides valuable constraints analogous to conservation laws in physics, rather than the classical time series analysis with a single stage approach. Despite the difficulties, we provide out-of-sample predictions which are more successful than EMH.

### 3.1. Method description

The proposed out of sample prediction is performed in the following way. Given MPs and NAVs for an  $n$ -day period from day  $i$  to day  $i+n-1$  and the corresponding optimal parameter vector  $\bar{K}_i$  for the  $i$ 'th period computed via an optimization method in Section 2, we solve the initial value problem (13) with the asset flow differential equations (7 - 11) to predict MP value and return on day  $i+n$ .

### 3.2. Success tests

The out-of-sample predictions generated by this method can be tested against the actual returns in terms of both the absolute magnitude of the difference between the actual and predicted ones and in terms of the sign, i.e., the direction of the market. Knowing the direction of the market is often very useful in financial markets.

#### 3.2.1. Absolute difference of predicted return and actual return

Once we have a set of returns  $r^{PredDe}$  based on the optimized parameters for each day, the results can be compared with the actual daily returns  $r^{Actual}$  on the stock exchange. There are a number of methods for performing this comparison. These tests can be used to determine if  $|r^{PredDe} - r^{Actual}|$  has a median that is less than that of the default prediction, namely,  $|r^{PredRw} - r^{Actual}|$  where the first column  $|r^{PredDe} - r^{Actual}|$  consists of the absolute values of differences between the actual daily returns and the predicted daily returns via the differential equations and the second column  $|r^{PredRw} - r^{Actual}|$  has the absolute values of differences between the actual daily returns and the predicted returns via random walk. Non-parametric tests are preferable since they do not make any assumptions concerning the underlying probability distribution. In particular, we apply the Mann-Whitney U test (see Mendenhall et al. [26]) and the Wilcoxon rank sum test [26] to column 1 and column 2.

**Null Hypothesis,  $H_0$**  : The median absolute value of difference  $|r^{PredDe} - r^{Actual}|$  and the median absolute value of difference  $|r^{PredRw} - r^{Actual}|$  are equal.

**Alternative Hypothesis,  $H_1$**  : The median absolute value of difference  $|r^{PredDe} - r^{Actual}|$  is less than the median absolute value of difference  $|r^{PredRw} - r^{Actual}|$ .

#### 3.2.2. Prediction of relative price change direction

At this point, we obtain relative price changes for the actual MP and the predicted prices via the proposed method. Then, we count the number of matches, meaning that the direction of the prediction and actual MP agree. We obtain a new sequence such that the sequence element is 1 if there is a match. Otherwise, the sequence element is  $-1$ . We apply z-test to the sequence. According to EMH, the mean value of the sequence would be 0 as null hypothesis. The alternative hypothesis states that the mean value of the sequence is different from zero.

### 3.3. Prediction results

Here are three examples for our predictions as continuation of Example 1, Example 2, and Example 3 respectively:

**Example 4.** By using the 8 optimal parameters obtained in Example 1, we solve the initial value problem (13) with (7 - 11) to predict MP value and return for the next days from day 16 to day 23. In Figure 4, the following actual market price

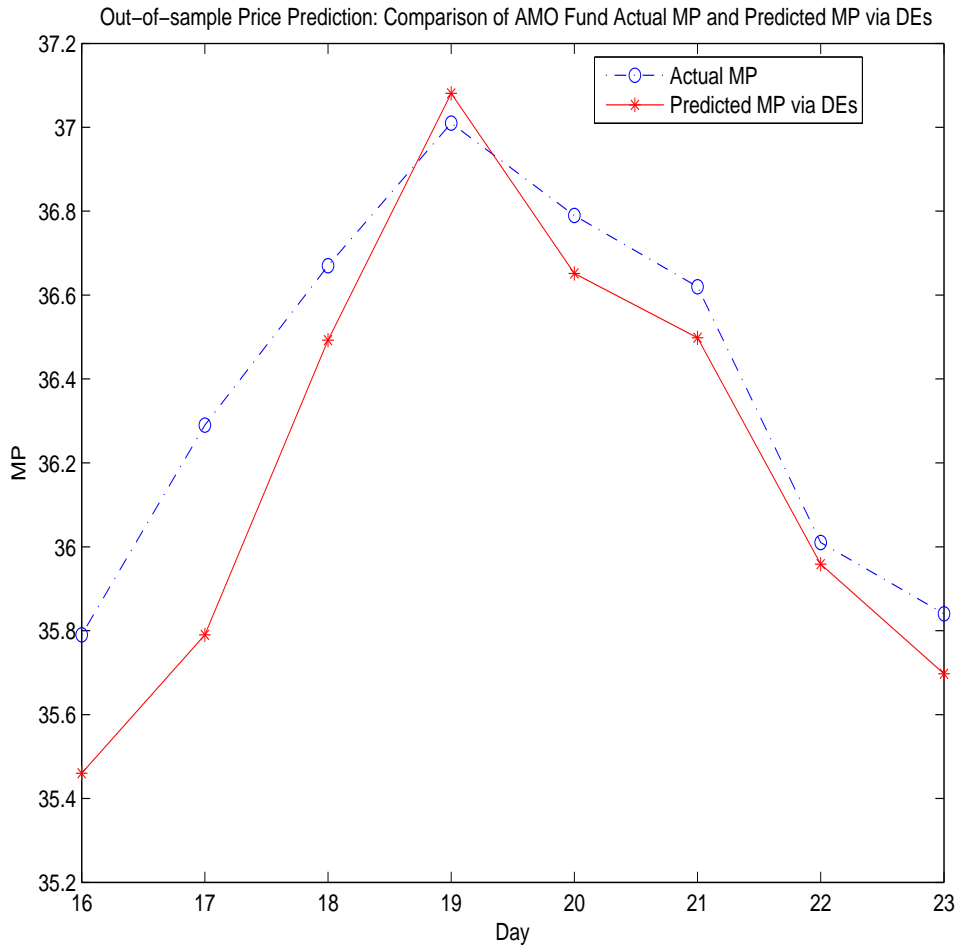


Figure 4. Prediction of AMO MPs over 8-day

and predicted MP via the asset flow differential equations are compared

$$\bar{Z}_{16:23} = (35.79, 36.29, 36.67, 37.01, 36.79, 36.62, 36.01, 35.84)$$

and

$$P_{16:23}^{PredDe} = (35.46, 35.79, 36.49, 37.08, 36.65, 36.50, 35.96, 35.70)$$

for the following trading days

Day =

(8.20.1999, 8.23.1999, 8.24.1999, 8.25.1999, 8.26.1999, 8.27.1999, 8.30.1999, 8.31.1999).

In Figure 5, for the same days as in Figure 4, the following actual return and predicted return via the asset flow differential equations are shown

$$r_{16:23}^{Actual} =$$

(0.009306, 0.013970, 0.010471, 0.009272, -0.005944, -0.004621, -0.016658, -0.004721)

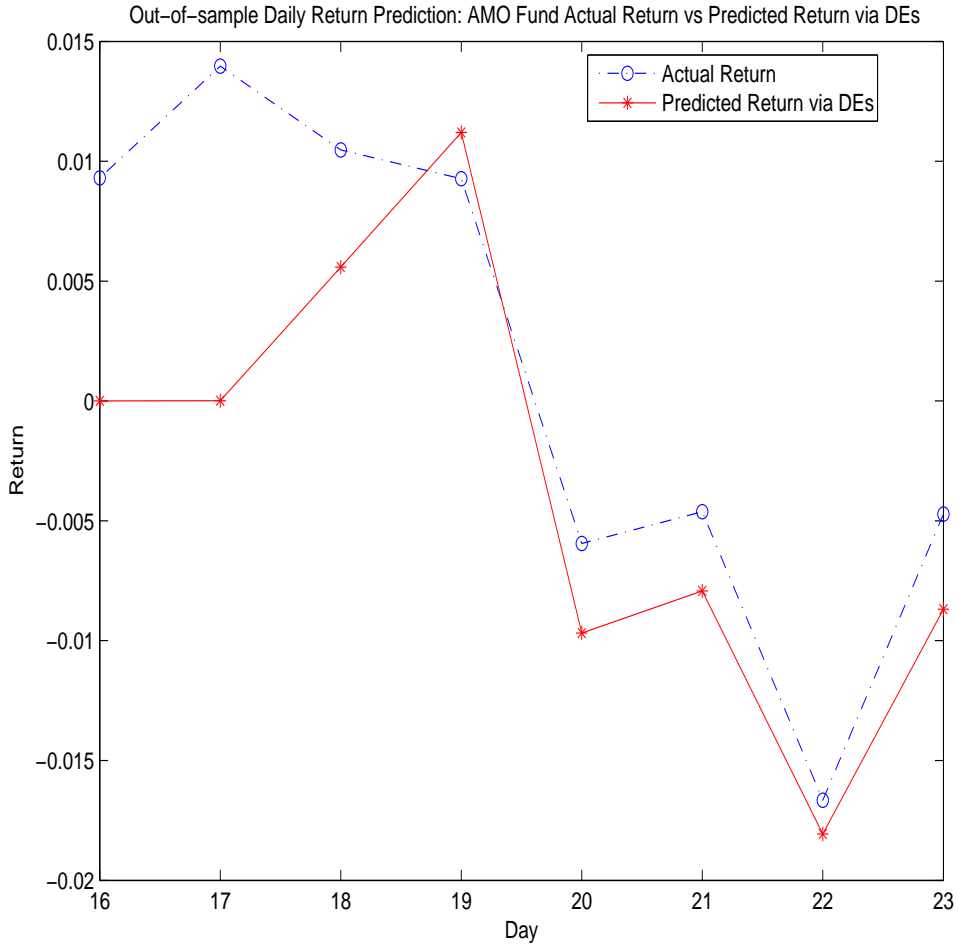


Figure 5. Prediction of AMO fund daily returns over 8-day

and

$$r_{16:23}^{PredDe} =$$

(0.000000, 0.000007, 0.005584, 0.011204, -0.009685, -0.007927, -0.018067, -0.008684).

In Figure 6, the absolute errors for the predicted returns via AFDE are

$$|r_{16:23}^{PredDe} - r_{16:23}^{Actual}| =$$

(0.009306, 0.013964, 0.004887, 0.001933, 0.003741, 0.003306, 0.001410, 0.003963)

and the absolute errors for the predicted returns via RW are

$$|r_{16:23}^{PredRw} - r_{16:23}^{Actual}| =$$

(0.0090682, 0.0137323, 0.0102331, 0.0090338, 0.0061824, 0.0048589, 0.0168957, 0.0049590).

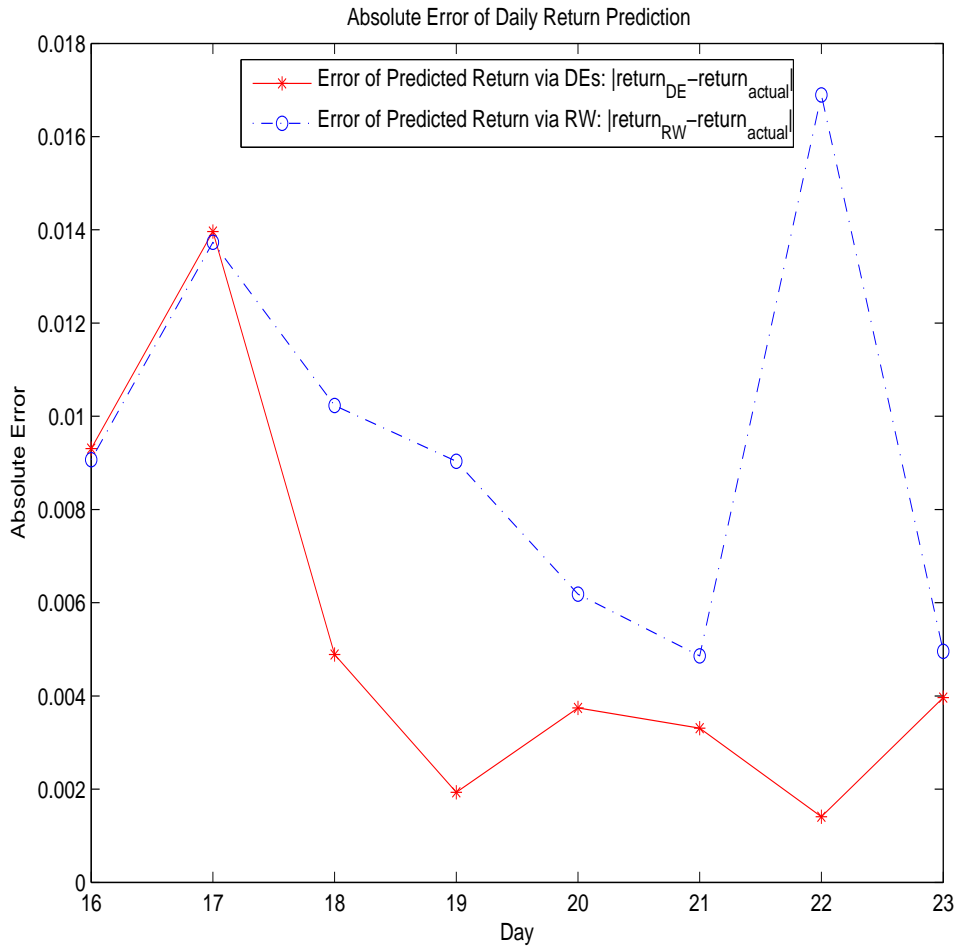


Figure 6. Absolute error of AMO fund predicted returns over 8-day.

After day 17, the absolute errors for the predicted returns via AFDE are less than that of RW.

For example, the MP and return on day 20 is predicted (see Figure 4 and Figure 5 respectively) by using initial conditions on day 19 and the computed optimal parameter vector

$$(0.002802, 419.708991, 52.957131, 8.322994)$$

for 5-day period from day 15 to day 19 as in Figure 3 and Table 3. It is remarkable to predict such a reversal in MP and sign of return on day 20 after a 3-day rise trend in MP. This successful prediction cannot be expected from a prediction via pure curve fitting.

By using Mann-Whitney U test for 8 events, we obtain

$$\text{median}(|r_{16:23}^{\text{PredDe}} - r_{16:23}^{\text{Actual}}|) = 0.00385$$

and

$$\text{median}(|r_{16:23}^{\text{PredRw}} - r_{16:23}^{\text{Actual}}|) = 0.00905.$$

That is, the error via the asset flow differential equations is less than half of the error via RW. Point estimate for  $ETA_1 - ETA_2$  is -0.00422. But, computed

$ETA_1 - ETA_2$  is -0.0052. 95.9 % CI for  $ETA_1 - ETA_2$  is  $(-0.00883, 0.00003)$ . The rank sum  $W = 49.0$ . Test of  $ETA_1 = ETA_2$  vs  $ETA_1 < ETA_2$  is significant at 0.0260. Since  $0.0260 < 0.05$ , we can reject  $H_0$  at the 0.05 level for this small example. Moreover, the prediction success of MP return direction by the asset flow differential equations is 100 %.

**Example 5.** We predict the next day MP return by using the optimal parameters obtained in Example 2. We apply Mann-Whitney U test and have

$$median(|r^{PredDe} - r^{Actual}|) = 0.00554$$

and

$$median(|r^{PredRw} - r^{Actual}|) = 0.00577$$

for the 5590 prediction attempts. Point estimate for  $ETA_1 - ETA_2$  is  $-0.00018$ . 95.0% CI for  $ETA_1 - ETA_2$  is  $(-0.00036, -0.00001)$ . The rank sum  $W = 30898021.0$ . Test of  $ETA_1 = ETA_2$  vs  $ETA_1 < ETA_2$  is significant at 0.0193. The test is significant at 0.0193 also when adjusted for ties. Therefore, we can reject the null hypothesis  $H_0$  at the 0.05 level for this sample portfolio.

When we apply Wilcoxon rank sum test, the p-value is 0.0386, the z-val is  $-2.0681$ , and the rank sum is 30898023.0. Thus, we can reject the null hypothesis  $H_0$  at the 0.05 level by using Wilcoxon rank sum test, as well.

The prediction success of relative price change direction by the AFDE is 63.33% (with 3540 direction matches out of 5590 prediction attempts). When we apply the z-test to the direction match sequence of -1 and 1, we obtain a mean value of 0.2666, p-value of 0, with a 95.0% CI of  $(0.2403, 0.2928)$ , and z-value of 19.9288. Therefore, we can reject the null hypothesis. Moreover, the success of prediction that the price will be non-increasing or non-decreasing is 69.84% with 3904 matches out of 5590 prediction attempts. According to the z-test, the mean value is 0.3968, the p-value is 0, 95.0% CI is  $(0.3706, 0.4230)$ , and z-val is 29.6658. Again, we can reject the null hypothesis.

While the success of this method is encouraging, more large scale studies are needed before concluding that this procedure in itself can be used profitably.

**Example 6.** We obtain MP and return prediction of APB for 10-day event period by using optimal parameters obtained in Example 3 and Table 5. We compare the predicted returns with the actual returns during 2.19.2002-6.23.2003. According to Mann-Whitney U test for 339 events,

$$median(|r^{PredDe} - r^{Actual}|) = 0.00846$$

while

$$median(|r^{PredRw} - r^{Actual}|) = 0.00871.$$

Point estimate for  $ETA_1 - ETA_2$  is 0.00020. 95.0% CI for  $ETA_1 - ETA_2$  is  $(-0.00089, 0.00129)$ . The rank sum  $W$  is 116154.0. For the test of  $ETA_1 = ETA_2$  vs  $ETA_1 < ETA_2$ , we cannot reject the null hypothesis for this example by using the method of full prediction attempt via 10-day event period since  $W$  is  $> 115090.5$ , although

$$median(|r^{PredDe} - r^{Actual}|) < median(|r^{PredRw} - r^{Actual}|).$$

The prediction success of relative price change direction by AFDE is 54% which

is smaller than that of Example 5 because of the rate of prediction attempt, larger event period and larger  $\epsilon_2$ . But, it is still greater than 50%.

#### 4. Conclusion

A nonlinear computational optimization algorithm combining quasi-Newton weak line search with the BFGS formula can be used successfully to determine the optimal four parameter set in the asset flow differential equations. Using these optimal parameters and the price history in the differential equations we can forecast the price for the next day. In this way we have a set of out-of-sample predictions that can be tested against the actual prices. For a sample set of 6 funds the predictions are compared with the default theory of random walk (EMH). The Mann-Whitney U test and Wilcoxon rank sum test show that the out-of-sample prediction outperforms EMH and we can reject the null hypothesis  $H_0$  at the 0.05 level for this sample portfolio. The forecasts are even better in terms of predicting the direction of prices (higher versus lower for the next day).

The threshold for the gradient should be sufficiently small. But, decreasing the threshold from  $10^{-4}$  to  $10^{-6}$  just increases the average number of quasi-Newton iterations from 89 to 156 without significant improvement in minimization for a large sequence of data. So, we believe that the threshold values between  $10^{-4}$  and  $10^{-5}$  are reasonable for gradient without perfect line search, in practice.

One of the novel and important components of the proposed algorithm is the dynamic initial parameter pool. The fixed part of the pool consists of the expected initial vectors. The dynamic part of the pool is updated via previously found optimal parameters and it is specific to the fund's price behavior. The overall pool provides a stable number of quasi-Newton iterations because experience is employed and the impact of most recent events are dominated.

By reactive evaluation of the financially meaningful optimal parameters employing most of the data up to any given time, we get a stable 32% average maximum improvement factor defined in equation (15) and a reasonable average daily deviation in market price return during the curve fitting for a sample large data set.

We need a reasonable minimization during the preceding period for a successful next day price return prediction. Sometimes it is possible to get a better curve fitting locally if one were to ignore the intrinsic constraints. However, it does not imply there would always be a better prediction. For example, some vectors with negative parameters may provide smaller sum of squares. But, the negative parameters are not meaningful financially in the model. Moreover, while minimizing the sum of squares, we place exponential weights on the most recent price changes which is important in terms of investor strategy. Furthermore, there is a trade off between trend curve fitting and de-trended curve fitting. As shown in (Caginalp and Balenovich [10]) and (Duran [16], Chapter 2), there are various price return patterns which are relevant for 3 to 10 trading days. They can be captured by de-trended curve fitting. On the other hand, trend curve fitting should not be neglected because the percentage of momentum traders is significant. There are other constraints such as finiteness of traders' assets [11] as well. Also, the time scalings to reflect the current reaction speed of momentum traders and value based traders should be handled automatically. Therefore, the dynamical microeconomic model (7-11) which combines several factors is more successful than pure curve fitting.

The procedures we use are quite general and can be expanded in several directions. As noted earlier, there is a growing body of research in behavioral finance that is uncovering motivations beyond valuation. Momentum, or the tendency to

buy when prices are rising, is already incorporated into the system of equations. Other motivations can easily be incorporated into the sentiment function (see equation (2)) in a similar way. The optimization problem then involves one additional parameter for each motivation.

Closed-end funds provide a useful data set to test these optimization methods. The extension to a broad set of assets such as ordinary stocks can be implemented by using standard valuation methods in place of the net asset value.

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## Appendix A. Quasi-Newton method for minimizing the sum of squares.

A line search method (see [29], Chapter 3) computes a search direction  $P_k$  and a step length  $s_k$  to move along that direction, at each iteration given by

$$\bar{K}_{k+1} = \bar{K}_k + s_k P_k. \quad (\text{A1})$$

The choices of  $P_k$  and  $s_k$  affect the success of the line search method. In particular,  $P_k$  needs to be a descent direction satisfying that  $P_k^T \nabla F_k < 0$ , so that it is guaranteed that  $F$  defined in (14) can be decreased along this direction. Also,  $P_k$  is of the form

$$P_k = -B_k^{-1} \nabla F_k, \quad (\text{A2})$$

where  $B_k$  is a symmetric and nonsingular matrix. In Newton's method  $B_k$  is the exact Hessian  $\nabla^2 F(\bar{K}_k)$ . A perfect line search terminates at a point when the direction of search is perpendicular to the gradient vector.

In a quasi-Newton method, the inverse of Hessian matrix  $\nabla^2 F(\bar{K}_k)^{-1}$  is approximated by using a positive definite matrix  $H_k$ , instead of computing exact second derivatives. The second derivative information is developed by updating the approximate matrix on each iteration.  $P_k$  is a descent direction, since  $H_k$  is positive definite and  $P_k^T \nabla F_k = -\nabla F_k^T H_k \nabla F_k < 0$  is obtained by using (A2) (see [29], Chapter 3).

### Algorithm 2: Quasi-Newton method

1. Choose an initial parameter vector  $\bar{K}_0$  as an estimate of  $\bar{K}$  that would minimize  $F(\bar{K})$ .
2. Choose initial symmetric positive definite matrix  $H_0$  (Identity matrix  $I$  can be taken as  $H_0$ ).
3. Set convergence tolerance  $\epsilon_1 = 10^{-4}$  or set a maximum number of iterations.
4. While  $\|\nabla F(\bar{K}_{k+1})\| > \epsilon_1$ 
  - Set  $g_k = \nabla F(\bar{K}_k)$
  - Compute the search direction  $P_k = -H_k g_k$
  - Find the candidate step length  $s_k$  by using backtracking line search algorithm where sufficient decrease condition is obtained for  $F(\bar{K}_k + s_k P_k)$ .
  - Set  $\bar{K}_{k+1} = \bar{K}_k + s_k P_k$ ,  $\beta_k = g_{k+1} - g_k$ ,  $\delta_k = \bar{K}_{k+1} - \bar{K}_k$
  - Get a new positive definite matrix  $H_{k+1}$ , by using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) formula (A6), such that

$$H_{k+1} \beta_k = \delta_k \quad (\text{A3})$$

5. End (while)

The gradient  $(\nabla F(x))$  is approximated by using the central difference formula (see [29], Chapter 7)

$$\frac{\partial F}{\partial x_i}(x) \approx \frac{F(x + \epsilon_3 e_i) - F(x - \epsilon_3 e_i)}{2\epsilon_3} \quad (\text{A4})$$

for the partial derivatives, where

$$\frac{\partial F}{\partial x_i}(x) = \frac{F(x + \epsilon_3 e_i) - F(x - \epsilon_3 e_i)}{2\epsilon_3} + O(\epsilon_3^2), \quad (\text{A5})$$

$\epsilon_3 = u^{1/3}$ , unit roundoff  $u$  is about  $1.110223e - 016$  for double-precision arithmetic, and  $e_i$  is the  $i$ th unit vector.

### Backtracking line search

The backtracking method provides either that the selected step length  $s$  is at least a fixed value ( $\bar{s} = 1$ ), or that it is sufficiently short to satisfy the sufficient decrease condition but not too short (see [29]).

*Algorithm 3:*

1. Set  $\bar{s} = 1$  and choose  $\sigma, \theta \in (0, 1)$
2. Set  $s = \bar{s}$
3. Repeat until  $F(\bar{K}_k + sP_k) \leq F(\bar{K}_k) + \theta s(\nabla F(\bar{K}_k))^T P_k$ 
  - Set  $s = \sigma s$
4. End (repeat)
5. Return with  $s_k = s$ .

**The BFGS formula** (see Broyden [8] and [9])

$$H_{k+1} = H_k - \frac{H_k \beta_k \delta_k^T + \delta_k \beta_k^T H_k}{\delta_k^T \beta_k} + \left(1 + \frac{\beta_k^T H_k \beta_k}{\delta_k^T \beta_k}\right) \frac{\delta_k \delta_k^T}{\delta_k^T \beta_k} \quad (\text{A6})$$

By using the formula (A6), positive definite matrix  $H_{k+1}$  is obtained when the curvature condition  $\delta_k^T \beta_k > 0$  is satisfied (see [4]). However, sometimes the curvature condition which rules out unacceptably short steps may not hold, even for the iterates close to the solution. In practice, to deal with the special cases where  $\delta_k^T \beta_k$  is negative or too close to zero, the BFGS update (A6) is skipped by setting  $H_{k+1} = H_k$ . However, it should not be done often (see [29]). Within our study we check the  $\delta_k^T \beta_k$  and update  $H_{k+1}$  by identity matrix or set  $H(i, i) = i/2$  to handle the special cases described above. We allow such cases limited times (at most five times) and try another initial parameter vector.

Each iteration of the quasi-Newton method can be done at a cost of  $O(n^2)$  arithmetic operations in addition to the function and gradient evaluations (see [29]) where  $n$  is the number of parameters namely 4 in (14). The algorithm has a super-linear rate of convergence (see [29] for the related convergence theorems and proofs under ideal mathematical assumptions). Although Newton's method converges quadratically, it is more costly per iteration. Moreover, rounding errors sometimes may mean that such theoretical convergence rates are not realized in practice (see [4] and [19]). Although the errors in computed values of  $F$ , and the entries of  $\nabla F$  and  $\nabla^2 F$  in double precision arithmetic are usually negligibly small, they can be significant when  $\nabla F$  is near zero (see [4]).