Asset price dynamics with heterogeneous groups

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Abstract

A system of ordinary differential equations is used to study the price dynamics of an asset under various conditions. One of these involves the introduction of new information that is interpreted differently by two groups. Another studies the price change due to a change in the number of shares. The steady state is examined under these conditions to determine the changes in the price due to these phenomena. Numerical studies are also performed to understand the transition between the regimes. The differential equations naturally incorporate the effects due to the finiteness of assets (rather than assuming unbounded arbitrage) in addition to investment strategies that are based on either price momentum (trend) or valuation considerations. The numerical studies are compared with closed-end funds that issue additional shares, and offers insight into the strategies of investors.

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1. Introduction

Classical economics stipulates that an equilibrium price is attained when supply and demand are equal. The neoclassical approach to the dynamics of asset prices assumes that prices return to equilibrium at a rate that is proportional to the extent to which they are out of equilibrium. There are three tacit assumptions in the classical and neoclassical descriptions of asset prices: (i) the supply and demand are based upon the value of the asset; (ii) there is a general agreement on the valuation of an asset among market participants since the information is public; and (iii) there is a huge amount of capital that would quickly take advantage of any discrepancies from a realistic value of the asset. While these classical academic theories are sometimes a useful idealization of asset market dynamics, they neglect several key aspects that are routinely examined by practitioners.

First, it has been known for some time that diverse motivations underlie the buying and selling of shares. An important motivation is often due to the price trend that is evident to traders. Sometimes called the momentum effect, this has complex origins in terms of trader psychology and strategy. For example, it may be purely rational self-interest in the case of a money manager who had previously minimized his portfolio in a sector where prices are rising rapidly. As his portfolio lags the averages, he would find it difficult to maintain his investors, or even his job, by ignoring the trend. The motivation could be regret avoidance on the part of an individual investor who does not want to sell a rising stock in anticipation of the regret he would feel upon observing his stock at a higher price after he has sold it. As a stock is declining in price, the motivation to sell could arise in a purely rational way if the investor receives a margin call, i.e., the broker informs him that he must send more funds to cover borrowed money to avoid liquidation of the position, otherwise the position will be liquidated. Alternatively, the selling could arise from fear of further declines in price, or the attempt to avoid the regret of not having sold sooner.

Secondly, the concept of a unique price determined by the set of all available information is an interesting idealization. It
is based on the idea of each trader realizing that other traders know what he knows, etc. This game theoretic analysis is assumed to be iterated indefinitely and to lead to a unique fixed point. Of course this is an oversimplification both theoretically and practically. At the theoretical level, there are many simple games for which there are many equilibrium fixed points. Furthermore, the analysis is based not only upon the self-optimization of participants, but on the premise that they will rely on the self-optimization of others. When tested experimentally, this premise has failed to hold. On the practical level, we see each day that differing views are held by different market participants. Even for currencies, large and dominant players such as the major money center banks often maintain substantially divergent views on the levels they expect to prevail. Of course, they are aware of one another’s views and resources, and place their trades accordingly. Nevertheless, the trading price is more aptly described as a tenuous steady state due to a temporary balancing of forces between two parties, than a stable equilibrium characterized by a return to a particular level upon perturbation. The difference between a “comfortable”, stable equilibrium and an uneasy steady state is more than an academic issue. In the former, one expects that a brief imbalance of the asset will quickly return to equilibrium. In the latter, such a disturbance could lead to participants changing positions as they react to new and perhaps ambiguous information. The change in positions (e.g., selling the asset) leads to an acceleration in the same direction, which in turn is interpreted as new information, etc. Thus an important aspect of price dynamics involves the interaction between two or more groups with differing valuations of the asset, and different strategies or motivations, as well as their resources.

The third assumption in classical efficient market theory is that there is – for all practical purposes – an infinite amount of “arbitrage” capital that is ready to exploit any deviations from the realistic price. On a practical level there is ample evidence that practitioners have no faith in this assumption. For example, in the initial or secondary public offering market there is always analysis and speculation as to whether there is enough demand for the shares. In particular, closed-end funds (see Section 5) that specialize in a single country or region, for example, typically would be eager to issue more shares without bound, since the fees of the parent company are based upon the total assets under management. If they believed that arbitrage capital were adequate to maintain a trading price that is not too far below the net asset value, they would not hesitate to issue shares relentlessly. We will return to this important example after presenting our model, and derive a key quantitative relationship.

The high-tech bubble of 1998–2000 is a good illustration of the issues discussed above. A wide gulf erupted between practitioners who measured valuation carefully and others who claimed that these “old fashioned metrics” were outdated in this new era. The two groups often differed by a factor of 100 in terms of the price per share that they were willing to pay for the stocks. The assets of the latter group far exceeded the few lonely voices for value-based investments, and prices soared. In reality the capital available for selling short the shares was very limited compared to the huge supply of cash that was generated by a public eager to participate. Eventually, the supply of cash of this group of investors – while not exhausted – diminished to the point at which the selling (due to a number of factors, including insiders who sought to limit their holdings) eclipsed the buying. Once prices started to decline, they accelerated due to two reasons. (i) The downward trend led to selling by those focusing on the trend — for essentially the same reason that they bought when prices were rising. (ii) Under most circumstances, as prices fall, there are different groups who are buying at the lower prices. However, the huge divergence of opinion on the internet/high-tech stocks meant that value-based buyers would not consider these stocks a bargain at even half price. Hence, prices eventually plummeted to a tiny fraction of their peak prices. This great bubble remains as a serious challenge to the efficient market theorists, particularly since it occurred at a time and place where so much information was readily available — unlike other great bubbles such as the Dutch tulip bulb craze and the English South Sea Bubble of the 16th and 17th centuries.

The model we present is based on basic microeconomic principles of adjustment to supply and demand. Unlike the neoclassical models, however, the supply and demand may depend upon the price trend in addition to the valuation. Also, there is the assumption of a finite supply of cash so unlimited arbitrage is not possible. This model generalizes the equations presented in [4] (which has its roots in [3]) by assuming the existence of two or more groups with disparate motivations and assessments of fundamental value. After deriving this generalization, we study the price dynamics analytically and numerically in order to understand the price dynamics when two groups assess the value of a stock differently upon receiving new information.

2. The mathematical model for conserved systems

We consider a system in which a single asset is traded by any number of groups of investors. For simplicity, we discuss two groups, and the generalization to more groups is nearly identical. Each group will be characterized by its own parameters and assessments of valuation. We let \( N_1(t) \) and \( N_2(t) \) be the number of shares owned at time \( t \) by groups 1 and 2, respectively. Similarly, \( M_1(t) \) and \( M_2(t) \) denote the cash positions. In the simplest case of conserved cash and shares in the system one would have the identities

\[
M_1(t) + M_2(t) = M_0 \tag{2.1}
\]
\[
N_1(t) + N_2(t) = N_0 \tag{2.2}
\]

where \( M_0 \) and \( N_0 \) are constants. In other situations one can consider the issuing of new shares or influx of new cash so that there is a net change in the total shares or cash in the system. This “conserved” approach differs from the original equations in which the shares and cash are not conserved. The difference between the two approaches can be illustrated by an analogy with the thermodynamics of fluid in a bottle. If the bottle is an ideal thermos, so no heat is lost or gained, then there is conservation of energy within the bottle. This corresponds to our current approach. On the other hand, one can have, for example, a warm closed bottle that is placed in a large pool.
of water. Then energy is no longer conserved within the bottle (although it is conserved for the larger system including the pool), but the temperature on the surface of the bottle equals that of the fluids inside and outside the bottle. This is analogous to the asset price model presented in [5].

Using the same formalism as in [4] we assume that relative price change is proportional to the excess demand, or an increasing function of excess demand:

$$\tau_0 \frac{1}{P} \frac{dP}{dt} = \frac{D}{S} - 1, \quad \text{or} \quad \tau_0 \frac{1}{P} \frac{dP}{dt} = f(D/S),$$

(2.3)

where $D$ is demand and $S$ is supply of the asset, and $f$ is an increasing function such that $f(1) = 0$ and $f'(1) = 1$; e.g., $f(x) = \log x$. In the simplest case where no additional cash or shares are introduced into the system (i.e., when (2.1) and (2.2) apply), one has

$$D = k_1 M_1 + k_2 M_2,$$

(2.4)

where $k_1$ and $k_2$ are transition rates or probabilities for Groups 1 and 2, respectively. In other words, $k_1$ represents the probability that a dollar of cash belonging to Group 1 is submitted to the market for purchase of the asset. Similarly, the supply of asset offered for sale, $S$, satisfies

$$S = (1 - k_1) N_1 P + (1 - k_2) N_2 P.$$

(2.5)

In other words the probability of a dollar of asset being sold by Group 1, for example, is $(1 - k_1)$, the complement of the probability that a dollar of cash is submitted for purchase, namely, $k_1$. Using (2.1), (2.2), (2.4) and (2.5) in (2.3) we obtain

$$\tau_0 \frac{1}{P} \frac{dP}{dt} = \frac{k_1 M_1 + k_2 (M_0 - M_1)}{(1 - k_1) N_1 P + (1 - k_2) (N_2 - N_1) P} - 1.$$

(2.6)

In the absence of shares added or subtracted from the system, a change in their number can occur only through buying or selling, which occur at the rates $k_i$ and $(1 - k_i)$ respectively for each group, yielding,

$$\frac{dN_i}{dt} = k_i M_i - (1 - k_i) N_i P$$

(2.7)

$$\frac{dM_i}{dt} = -k_i M_i + (1 - k_i) N_i P.$$

(2.8)

The conditions (2.1) and (2.2) when differentiated yield $\frac{d}{dt}(M_1 + M_2) = 0$ and $\frac{d}{dt}(N_1 + N_2) = 0$, so that summing either of the two Eqs. (2.7) or (2.8) over $i = 1, 2$ implies the identity

$$P = \frac{k_1 M_1 + k_2 M_2}{(1 - k_1) N_1 + (1 - k_2) N_2}.$$

(2.9)

While this equation is valid as $P(t)$, $M_i(t)$ and $N_i(t)$ vary in time, it also yields a steady state price that satisfies (2.6) and establishes this price level in terms of the assets of each group and the basic transition probabilities or flow rates, $k_i$. Note that (2.6) is just the assertion that the rate of change of price is proportional to the magnitude of the deviation from this identity, and that the direction of the change is toward restoring (2.6).

Continuing to use the assumption that there is no influx of shares or cash, we see that the right hand sides of Eqs. (2.7) and (2.8) are identical except for the sign, and we can write

$$\frac{dM_i(t)}{dt} = -P(t) \frac{dN_i}{dt}.$$

(2.10)

This means that $N_1$ and $M_1$ can only change at one another’s expense (and similarly for $N_2$ and $M_2$). In other words, the change in the amount of cash for each group is due solely to purchasing shares. Note that this equation, like (2.7) and (2.8), does not involve the time scale $\tau_0$. Eq. (2.10) also follows directly from the basic assumption of the model that each trader can only exchange cash for asset at the current price.

The motivations and strategies of the traders are represented throughout the transition rates, $k_i$, which need to be specified. In principle these quantities can describe a broad range of investor strategies and motivations. For example, the $k_i$ could depend upon (a) the deviation of the price from a calculated fundamental value; (b) the price history, particularly the overall price trend in accordance with the investor group’s time scale; (c) the price and trading history, e.g., the price at which the asset was purchased, or the high or low price within a time frame such as one year; (d) inherent behavioral biases, such as the overweighting of small probabilities of negative outcomes (fear), or positive outcomes (hope); (e) a utility function that is either risk neutral, risk averse or risk seeking, so that the function is linear, concave or convex.

Of these motivations, (a) is standard in economics and conventional finance, while traders have long acknowledged the importance of (b), the price trend. In particular, even if one is entirely “fundamentalist” so that only value of the asset motivates decisions, there are practical constraints that would force one to close a position due to the price trend. If one has bought stocks on margin (i.e., partly on borrowed money) and the stock price drops below a certain level, the position must be liquidated (by Federal regulations in the US) unless the investor provides additional cash. Similarly, if an investor has a short position then the position would be similarly closed if the price rises above a particular mandated level. In addition to these practical considerations, there are also motivations for both individuals and professionals. As prices rise, for example the investor anticipates further increases in wealth and wishes to avoid the regret of selling too early. In this paper we will model the motivations (a) and (b). However, the model is sufficiently general to incorporate other motivations and behavioral effects such as (c), (d) and (e). The motivation associated with (c) will be considered in a future work and is also closely related to regret avoidance. The motivations related to (d) have been discussed in behavioral finance by Lopes [8] and others. The utility function and various nonlinearities have been suggested by Kahneman and Tversky [7] and others (see Shefrin [9]). An early mathematical model [5] discussed incorporating such effects into a system of differential equations.

We define the functions $k_i$ as in earlier papers with each group now having its own rate. The $k_i$ is defined in terms of the sentiment functions, $\xi_1^{(i)}$ and $\xi_2^{(i)}$, that quantify the $i$th investor group’s sentiment toward the price trend and deviation from
fundamental value, respectively:

\[ \xi_1^{(i)}(t) := q_1^{(i)}c_1^{(i)} \int_{-\infty}^{t} \frac{1}{P(\tau)} \frac{dP(\tau)}{d\tau} e^{-c_1^{(i)}(t-\tau)} d\tau \]  
(2.11)

\[ \xi_2^{(i)}(t) := q_2^{(i)}c_2^{(i)} \int_{-\infty}^{t} \frac{P_a^{(i)}(\tau) - P(\tau)}{P_a^{(i)}(\tau)} e^{-c_2^{(i)}(t-\tau)} d\tau. \]  
(2.12)

Here the parameters \( c_1^{(i)} \) and \( c_2^{(i)} \) are the inverses of the time scales for the two motivations, while the \( q_1^{(i)} \) and \( q_2^{(i)} \) scale the magnitudes. The functions \( P_a^{(i)} \) represent the fundamental value that Group \( i \) assigns to the asset. In other words, the different groups may view the valuation differently. Differentiating these expressions, one can write the differential equations

\[ \frac{d\xi_1^{(i)}}{dt} = c_1^{(i)} \left\{ q_1^{(i)} \frac{dP}{d\tau} - \xi_1^{(i)} \right\} \]  
(2.13)

\[ \frac{d\xi_2^{(i)}}{dt} = c_2^{(i)} \left\{ q_2^{(i)} \frac{P_a^{(i)}(\tau) - P(\tau)}{P_a^{(i)}(\tau)} - \xi_2^{(i)} \right\}. \]  
(2.14)

The transition rate functions \( k_i \) are specified by mapping the sums \( \xi^{(i)}(t) := \xi_1^{(i)}(t) + \xi_2^{(i)}(t) \) which take on values on \((-\infty, \infty) \) to \((-1, 1) \) through the function

\[ k_i := \frac{1}{2} \left\{ 1 + \tanh(\xi_1^{(i)} + \xi_2^{(i)}) \right\} \]  
(2.15)

which provides the simplest vehicle for accomplishing this.

As discussed above, other motivations can be incorporated into (2.15). As the paradigm of behavioral finance advances, it is likely that decision theory research will find evidence for additional terms \( \xi_j^{(i)} \), \( j = 3, 4, \ldots \), in the general sentiment function,

\[ \xi^{(i)}(t) = \sum_{j=1}^{m} \xi_j^{(i)}(t). \]  
(2.16)

Furthermore, the structure of the sentiment function could be nonlinear rather than simply linear additive. Also, while (2.11) and (2.12) are symmetric (in the interest of simplicity) with respect to gains and losses, or overvaluation and undervaluation, the functions can readily be modified to account for asymmetry.

Within this general structure, we can write down a system of differential and algebraic equations that can be solved numerically. Since (2.1) and (2.2) express \( M_2 \) and \( N_2 \) in terms of \( M_1 \) and \( N_1 \) we need only solve the differential equations for Group 1. One can consider the system consisting of (2.1) and (2.2) plus (2.6), (2.7), (2.10) and (2.13)–(2.15) for Group 1. A more general price equation based upon (2.3) can be used in place of the simpler (2.6).

3. Systems with influx of cash or shares

We now generalize this system of equations to incorporate exogenous cash or share influx or outflow. As in the equations of Section 2, the trading is assumed to conserve cash and shares. However, there may be cash gained or lost by a particular group of investors outside of the trading mechanism due to exogenous reasons. Similarly we consider changes in the number of shares traded, a situation that occurs when a corporation decides to buy back some of its own shares, or issues more shares in order to finance additional investment. The change in the number of shares has an immediate impact on the fundamentals, for example, a stock buyback by an industrial company raises the earnings per share, so that the fundamental valuation parameters such as the \( P/E \) ratio (price per share/earnings per share) improves immediately. On the other hand, there is sometimes the criticism – reflected in the analysts’ estimates of long term growth – that the company is unable to use its extra cash toward improving the business and its gross earnings.

In order to incorporate the exogenous changes to the cash and share holdings of each group in our system of equations we write

\[ M_0(t) = M_0^{\text{orig}} + M_i^{\text{add}}(t) + M_2^{\text{add}}(t) \]  
(3.1)

\[ N_0(t) = N_0^{\text{orig}} + N_1^{\text{add}}(t) + N_2^{\text{add}}(t) \]  
(3.2)

so (2.1) and (2.2) remain valid, but with \( M_0(t) \) and \( N_0(t) \) both depending on time. Here we assume that \( M_0^{\text{orig}} \) and \( N_0^{\text{orig}} \) are the constant values of total cash and shares that are available at the initial time. The remaining terms such as \( M_i^{\text{add}} \) are functions of \( t \) that define the exogenous changes to the cash supply of group \( i \), and analogously the additional shares for \( N_i^{\text{add}} \). For example, if this group receives an additional amount of cash (excluding the cash proceeds of the sale of stock) at time \( t_1 \) then we define

\[ M_i^{\text{add}}(t) := \begin{cases} 0 & \text{if } t < t_1 \\ M_i^{\text{add}}(t_1) & \text{if } t \geq t_1. \end{cases} \]

Accordingly, we need to modify (2.7) and (2.8) as

\[ \frac{dM_i}{dt} = -k_iM_i + (1 - k_i)N_iP + \frac{dM_i^{\text{add}}}{dt} \]  
(3.3)

\[ P \frac{dN_i}{dt} = k_iM_i - (1 - k_i)N_iP + P \frac{dN_i^{\text{add}}}{dt} \]  
(3.4)

in order to account for the changes in cash and shares that are due to the exogenous changes.

Note that \( M_0(t) \), \( N_0(t) \), \( M_i^{\text{add}}(t) \) and \( N_i^{\text{add}}(t) \) can be regarded as known functions of \( t \). Hence, even though \( M_0 \) and \( N_0 \) are no longer constants, we can still use (2.1) and (2.2) to determine \( M_2(t) \) from \( M_1(t) \) and \( N_2(t) \) from \( N_1(t) \). Also, by summing (3.3) and (3.4) we see that

\[ \frac{dM_i}{dt} + P \frac{dN_i}{dt} = \frac{dM_i^{\text{add}}}{dt} + P \frac{dN_i^{\text{add}}}{dt} \]  
(3.5)

so that we can eliminate computation of either \( M_i \) or \( N_i \) for each \( i \). The set of equations (2.6), (3.1)–(3.3), (3.5) and (2.13)–(2.15) then provide a complete system that can be studied subject to initial conditions with \( M_0(t) \) and \( N_0(t) \) depending on time.

Example 1. We consider an asset that is traded by two groups, both of which initially assess the value of the asset at \( P_0(t) = P_0 = \text{Constant} \). At time \( t_1 \) there is an announcement by the
company that is interpreted differently by the two groups. One believes there is no change in the value, yielding

\[ P_a^{(1)}(t) := P_0 \]

for the first group. The other group believes that it lowers \( P_a(t) \) perhaps by a factor \( b \) such as two-thirds, so the function representing their valuation of the asset is given by

\[ P_a^{(2)}(t) := \begin{cases} P_0 & \text{if } t < t_1 \\ bP_0 & \text{if } t \geq t_1. \end{cases} \]

This poses two questions. First, suppose that we know the resources of both groups (i.e., cash and share position) as well as their parameters \( c_j^{(i)} \) and \( q_j^{(i)} \). What is the equilibrium (more properly the steady state) price that evolves from the differential equations? Note that there is no exogenous change in the shares or cash, so that the equations of Section 2 apply.

A second question is an inverse problem. Suppose that we observe the steady state price, \( P_0 \), before the announcement, and the steady state price, \( P_1 \), afterwards. What can we deduce about the resources of the two groups under the assumption that the groups are similar in characteristics such as trend and fundamentals orientation (i.e., their parameters \( c_j^{(i)} \) and \( q_j^{(i)} \) are similar)?

Example 2. Suppose there are two groups of investors during a secondary issue that adds a significant number of shares. Group 1 consists of current shareholders who already have, for example, 80% of the available resources in the asset. They have an assessment of value that is unchanged by the additional shares. Group 2 consists of potential investors who have their available resources in cash, but also have an assessment of value that is based on the liquidity price. Thus with the addition of shares, they are not willing to buy until the price is close to theirs. If we know the relative resources of the two groups, can we determine the new steady state price? How quickly does it attain that new price? These are the questions addressed in this section.

Example 2 continues to maintain its valuation while Group 2 changes its valuation to \( P_a^{(2)} \). After a transitional time, the price will approach a new equilibrium price, \( P_1 \). We study this issue first analytically in this section to determine \( P_1 \), and numerically in the next section to calculate the transition in price between \( P_0 \) and \( P_1 \) under a variety of parameter choices.

We now determine the equation satisfied by \( P_1 \) and the parameters in the system. (The steady state price, \( P_0 \), that is attained prior to the announcement will also follow as a consequence.) Since the price is not changing in time, one has \( \zeta_1^{(i)}(t) = 0 \) by (2.11). Similarly, (2.12) implies

\[ \zeta_2^{(i)} = q_2^{(i)} \left(1 - \frac{P}{P_a^{(i)}}\right) \]

so that the two inverse time scales, \( \zeta_1^{(i)} \), along with the parameters \( c_1^{(i)} \) and \( q_1^{(i)} \) are irrelevant for this calculation. Using (4.1) in (2.15) together with the approximation \( \tanh x \approx x \) for small \( x \), we can write,

\[ k_i \approx \frac{1}{2} + \frac{q_2^{(i)}}{2} \left(1 - \frac{P_1}{P_a^{(i)}}\right). \]

We use the notation \( L := M_0/N_0 \), \( M_0 = M_1(t) + M_2(t) \), \( N_0 = N_1(t) + N_2(t) \), and

\[ A_i := q_2^{(i)} \left(1 - \frac{P_1}{P_a^{(i)}}\right), \quad x_i := \frac{M_i}{M_1 + M_2}, \]

\[ y_i := \frac{N_i}{N_1 + N_2}. \]

One has then \( x_1 + x_2 = 1 \) and \( y_1 + y_2 = 1 \). Using (2.9) we can express \( P_1 \) as

\[ \frac{P_1}{L} = \frac{(1 + A_1)x_1 + (1 + A_2)x_2}{(1 - A_1)y_1 + (1 - A_2)y_2} = \frac{1 + A_1x_1 + A_2x_2}{1 - A_1y_1 - A_2y_2} \]

yielding a quadratic that one can solve for \( P_1 \). However, if \( A_i \ll 1 \) (e.g. the equilibrium price is not far from the fundamental value) we can write the first terms in the Taylor series and obtain a simpler expression,

\[ \frac{P_1}{L} \approx 1 + (A_1z_1 + A_2z_2) \]

where \( z_i := x_i + y_i \) (so that \( z_1 + z_2 = 2 \)). Note that in the event that the second group has no assets (i.e., does not exist), then
(4.5) can be solved for $P_1$ to yield the equation for the single group given by (2.15) of [4], namely,

$$P_0 = \frac{1 + 2q}{1 + 2qL/P_a}$$  \hspace{1cm} (4.6)$$

where we use $P_0$ for the steady state price for a single group, analogous to $P_1$ for the two group case.

Rewriting (4.5) by expanding $A_i$ we have

$$P_1 = \frac{1 + \sum q_2 (^{(i)} z_i)}{1 + \sum q_2 (^{(i)} z_i) L/P_a (^{(i)})}.$$  \hspace{1cm} (4.7)$$

If the two groups differ in their assessments of valuation by a factor $b$, i.e., $P_a (^{(1)}) = P_a$ and $P_a (^{(2)}) = bP_a$, but both groups have the same coefficient $q_2 (^{(i)}) = q$, so that they are similar in terms of their emphasis on value, then (4.7) reduces to

$$P_1 = \frac{1 + 2q}{1 + q \frac{2L}{P_a} (1 + \frac{1}{2}(b - 1)z_2)}.$$  \hspace{1cm} (4.8)$$

so that the term $\frac{1}{2}(b - 1)z_2$ expresses the deviation from the single group model.

The quantity $L := M_0/N_0$ which was introduced in [4] is a significant quantity in terms of asset price dynamics. Along with the trading price and the fundamental value of a share, $L$ also has units of dollars per share. It has been argued that whenever there is little focus on value, prices tend naturally toward $L$. In the case of a single group, formula (4.6) establishes a steady state price $P_0$ as an interpolation between the fundamental value, $P_a$, and the liquidity price, $L$, with $q$ (which is an abbreviation for the value coefficient $q_2$) as an interpolation parameter. When $q$ approaches zero (i.e., no attention to value), the steady state price, $P_0$, approaches the liquidity price, $L$. In the limit $q \rightarrow \infty$, it approaches the fundamental value, $P_a$. For the (two or more) groups situation we have the analogous result expressed in (4.7).

We consider now a situation that is similar to the one above in that the two groups have a different assessment of value after a particular announcement. Once again, both groups initially estimate the same valuation $P_a$. After the announcement one still has $P_a (^{(1)}) = P_a$ for the first group. However, in a situation involving a secondary issue (i.e., an increase in the number of shares) or stock buyback, there may be another group that feels the liquidity (which we are defining as $L$) is the dominant force in the market. Hence we write $P_a (^{(2)}) = L$ and assume that both groups pay the same level of attention to their respective valuation, so $q_2 (^{(i)}) = q$ again. Then we have, from the relations above, the identity

$$P_1 = \frac{1 + 2q}{1 + q \frac{2L}{P_a} + qz_2}.$$  \hspace{1cm} (4.9)$$

Note that as $z_2$ approaches zero, $z_1$ approaches 2, leading to the single group price once again (with valuation at $P_a$). Similarly, as $z_1$ approaches zero, the second group is the entire investor body, and $P_a = L$, leading to $P_1 = L$. Hence the relative size of this second group interpolates between $P_a$ and $L$ through the scaling function given by (4.9).

Prior to the announcement at time $t_1$, the two groups both valued the asset at $P_a (^{(i)}) = P_a$. Assuming the same coefficient $q_2 (^{(i)}) = q$ for both groups, we have the steady state price, $P_0$, given by

$$P_0 = \frac{1 + 2q}{1 + 2qL/P_a}.$$  \hspace{1cm} (4.10)$$

Hence the ratio of the steady state price attained prior to the announcement to the price later is given by the ratio of the right hand sides of (4.9) to (4.10):

$$\frac{P_1}{P_0} = \frac{1 + 2qL/P_a}{1 + qz_1L/P_a + qz_2} = \frac{1 + 2qL/P_a}{1 + 2qL/P_a + qz_2(1 - L/P_a)}.$$  \hspace{1cm} (4.11)$$

An evaluation of these parameters in a practical situation can be implemented as follows. Using optimization techniques, one can determine the $q_1$ (which we do not need to determine the values $P_1$ and $P_0$, but only for the transition between them) and $q_2$ from the data using the system of differential equations. Note that the function $P_a (t)$ is assumed to be exogenous (as discussed in the closed-end fund example in Section 5 below). Using $q_2 := q$ we see that (4.10) allows us to evaluate $L$ as a function of $q$ provided $P_a (t) = P_a$ and $P (t) = P_0$ have reached a steady state, yielding

$$L = P_0(1 + 2q(1 - P_0/P_a))^{-1}.$$  \hspace{1cm} (4.12)$$

Some time after the announcement at time $t_1$ we can evaluate $z_2$ once the price has settled into a steady state. Alternatively, we can estimate $P_1$ with an exogenous estimate of $z_2$.

An immediate application of these methods is to closed-end funds. Unlike open-end funds, whose prices are equal to the net asset value (and redeemable as such), closed-end funds trade on the exchanges like ordinary shares of individual companies. Hence the fund could trade at a discount or premium relative to the net asset value. These funds report weekly their net asset value per share, which may be regarded as $P_a (t)$, although the situation may be complicated by other factors such as tax liabilities, for example. Meanwhile the trading price constitutes the $P (t)$ in our models. In recent years, a plethora of “exchange traded funds” that are similarly traded has also been introduced. Thus these funds provide an important testing ground for theories and ideas on price movements.

For open-end funds, the purchase or redemption of funds leads to the increasing or decreasing in the number of shares. Closed-end funds begin with a particular number of shares sold to a group of investors. In some cases the fund management chooses to increase the number of shares. If the market were completely efficient (thereby assuming an infinite amount of capital for arbitrage) there would be no change in the discount from net asset value. In practice the discount typically widens. The management’s fees are usually proportional to the total net asset value of all shares. Hence the shareholders lose some market value, while the management enjoys an increase in
Fig. 1. $P_t/P_0$ as a function of the relative assets $z_2$ (left), and as a function of $q$ (right).

Using (4.11) we address the first of these questions. Suppose that in the rights offerings the management increased the number of shares by 1/2. Assuming that the liquidity value was near $P_a$ prior to the new shares (i.e., the original $L$ and $P_a$ were comparable before the new shares) one has $L/P_a = 2/3$. If we use, for example, $q = 1/3$ along with $z_1 = z_2 = 1$, then $P_1/P_0 = 0.93$. For $q = 1$ it is $P_1/P_0 = 0.87$. If one has an idea of the ranges of $q$ and the relative assets of the group that believes that the fundamental value has shifted to the liquidity value then we obtain a range of values for the emerging steady state price $P_t$. For $L/P_a = 2/3$ and $q = 1$, one has the graph shown in Fig. 1 (left) for $P_1/P_0$ as a function of $z_2$.

For $L/P_a = 2/3$ and $z_1 = z_2 = 1$, one has the graph in Fig. 1 (right) for $P_1/P_0$ as a function of $q$. Note that in the limit $q \to \infty$ this approaches 4/5.

Substituting the value $L/P_a = 2/3$ we can plot (4.11), i.e.,

$$
\frac{P_1}{P_0} = \frac{1 + 4q/3}{1 + 4q/3 + qz_2/3}
$$

as a function of $q$ and $z_2$ as shown in Fig. 2.

Typically, the announcement is made several weeks prior to the actual issuing of shares. We examine below the effect of this event which occurs at time $t_2$.

Before performing an analysis of this second event, we compare the results above with data from the specific case of
the closed-end fund Central Europe and Russia (CEE) in 2004. In early January 2004, the net asset value and trading price were approximately equal. In keeping with our theory we assume that the liquidity price, $L$, was also similar. On January 9, 2004 an announcement was made that a new rights offering would add 1/3 more shares. The trading price (and the discount) began to decline immediately. Although the actual shares would not be issued until March 29, 2004, the “rights” were distributed to the shareholders on February 19, 2004. For every three shares an investor held, he was given one transferable right. On February 19 these rights began trading, so that for all practical purposes the new shares had entered the market on this date. Fig. 3 shows the percentage discount, \( (P(t) - P_a(t))/P(t) \), during this period. In this example, the discount moved from 0% to about 10% almost immediately after the announcement. This seems to indicate that a significant percentage of investors had shifted their valuation to the liquidity price. When the additional shares began trading on February 19, 2004, it was widely anticipated, and any efficient market analysis would lead to the conclusion that it should not affect the trading price. Yet the data shows a sharp drop soon after this date. Eleven days after this date, the discount exceeded 20%, despite the fact that all of the information had been previously disseminated. This second drop is a demonstration of the importance of the “liquidity” price of an asset, a concept that is central to our differential equations approach. On March 29, 2004 it appears that the discount narrows from about 20% to 10%. However, this is largely due to the fact that the cost and dilution effect of the new shares – which are known in advance – have been taken from the shareholders’ assets on this day. So the narrowing of the discount at this stage offers no benefit to the shareholders. A number of closed-end funds experienced similar events.

5. Change in number of shares

Using the example above as motivation (the actual situation has several additional features), we consider the influx of shares (in exchange for cash) at a particular time \( t_2 \) which is not accompanied by any announcement. We assume two groups initially, with Group 1 valuing the asset at \( P_a \) while Group 2 values the asset at \( bP_a \) (with \( b < 1 \)). The total number of shares and cash, respectively, is \( N_0 \) and \( M_0 \), so \( L_0 := M_0/N_0 \). We assume that the price, \( P(t) = P_0 \) is at a steady state prior to \( t_1 \) and that \( q_2(i) = q \) for both groups. Using (4.6) one can calculate \( L_0 \) if \( q \) has already been calibrated from the prior data. We assume for simplicity that \( L_0 = P_a = P_0 \).

Next, we suppose that at time \( t_2 \), a total of \( N_0/3 \) shares have been added to the system. We assume that Group 1 rebalances its portfolio, but there are nevertheless \( N_0/3 \) more shares in the system, without the addition of any new money. In other words, if US investors had, say, $200 million for investment toward this purpose, they do not have any more after the announcement or the issuing of new shares. Then we can write

\[
\begin{align*}
x_1 &= \frac{M_1}{M_0}, & y_1 &= \frac{N_1}{4N_0/3}, \\
x_2 &= \frac{M_2}{M_0}, & y_2 &= \frac{N_2}{4N_0/3}
\end{align*}
\]

(5.1)

and let \( L_{\text{new}} \) be the new liquidity price given by the new cash/(number of shares):

\[
L_{\text{new}} = \frac{M_0}{4N_0/3} = \frac{3}{4} L_0.
\]

(5.2)

Hence, the new liquidity value is only three-quarters of the previous, due to the influx of new shares. We can now use (4.8) with \( L_{\text{new}} \) in place of \( L \) to obtain the new equilibrium price, \( P_1 \):

\[
\frac{P_1}{0.75L_0} = \frac{1 + 2q}{1 + 2(0.75)q(1 + \frac{1}{2}(1 - 1)z_2)}.
\]

(5.3)

Note that for \( q = 1 \) and \( z_2 = 0 \) we have from (5.3) that \( P_1 = 0.9L_0 \). For \( q = 1, z_2 = 1 \) and \( b = 0.8 \) (i.e., the assets of the two groups are comparable and the second group has a valuation of 80% of the first group) we have \( P_1 = 0.837L_0 \).
We calculate the time evolution of the price until the announcement time and define the ratio $P_1 / P_{0L}$ as a function of the relative assets $z_2$ for the values of $q = 1$ (left), and $q = 1/3$ (right).

From Fig. 4, the graph on the left in Fig. 5 shows the price $P(t)$ as a function of time $t$ for times $t < t_2$ with $q_2 = 0.3$.

These calculations suggest that the influx of additional shares in significant quantities is very likely to suppress the price by a large amount, even if there is no change in the valuation or the perception of value. In other words, the closed-end fund loses up to 25% of its trading price due to the additional influx of shares, even though both groups perceive the valuation to be much higher. This result highlights the key advantage of this methodology. Methods of classical finance are not useful in this situation as they predict that a discount of zero percent would continue regardless of the number of shares issued by the management. This is due to the concept of infinite arbitrage that indicates any discount would be immediately eliminated as investors quickly buy up shares that were at a discount from net asset value. Our methodology, however, allows one to determine the discount as a function of a limited number of parameters that can be calibrated through optimization from previous data, and from other related situations (e.g., similar closed-end funds).

6. Numerical computations

In this section we perform numerical computations on the problems discussed in Sections 4 and 5. Also, using the numerical methodology we can calculate the price dynamics that include both of the effects: the announcement of additional shares and, later, the influx of the actual shares, and the transition between these steady states. Hence we assume that an announcement is made at time $t_1$ that there will be 1/3 more shares that are to be distributed at time $t_2$. Prior to $t_1$ we assume (as above) that $L_0 = P_0 = P_a^{(i)} =: P_a$. For $t_1 < t < t_2$ Group 1 continues to value the asset at $P_a$ while Group 2 values it at the anticipated new liquidity price, $L_{new} = 3L_0/4$ (see (5.2) above). At time $t_2$ the additional shares are distributed into the system so that the new total number of shares increases to $4/3N_0$.

In terms of initial conditions we assume that $z_1 = 0$ since the price is assumed to be constant prior to the announcement, and $z_2 = 0$ since the price is assumed to be at the fundamental value. Both groups are assumed to have the same $q_1$ and $q_2$, which vary in the range (0,1). Also, we vary the assets of each group initially, ranging from Group 2 owning all of the assets to just 1/3 of the assets.

In each of the computer runs below, Eqs. (2.1), (2.2), (2.6), (2.7), (2.10) and (2.13)–(2.15) were used, and we define $t_1 := 9$ as the announcement time and $t_2 := 18$ as the time at which the new shares enter the market. We used the Matlab ODE package for these computations. The ODE package can solve problems $M(t, y) \cdot y' = F(t, y)$ with mass matrix $M$ that is nonsingular, or singular. More details can be obtained from the help menu in Matlab.

In Fig. 5, we calculate the time evolution of the price until the new shares are distributed into the system, i.e., $P(t)$ is calculated as a function of $t$ for times $t < t_2$. Recall that both groups value the asset at $P_a := P_a^{(i)} = L_0$ prior to the announcement made at $t = t_1$. Once the announcement is made, the first group continues the same valuation while the second
group changes its assessment to the anticipated new liquidity price, \( L_{\text{new}} = 3L_0/4 \). In this computer run, the parameters are set as \( c_1^{(i)} = c_2^{(i)} = 1 \), the value coefficient is \( q_2^{(i)} = 0.5 \) for both groups, and the trend coefficient \( q_1^{(i)} \) varies among 0.1, 0.3 and 0.5. For the initial assets position, we assumed that the first group owns 80% of the shares (i.e., \( N_1(0) = 80\% \) of \( N_0 \)) and 20% of cash supply (i.e., \( M_1(0) = 20\% \) of \( M_0 \)), i.e., the first group has 50% of the total assets in the system initially. The rest belongs to the second group. The numeric calculation shows that the equilibrium price is \( P_1 = 4.7091 \) (see the graph for times around 15 to 17, or so). Using (4.7) one can calculate the asymptotic value of the equilibrium price as \( P_1 = 4.7087 \) under the conditions above. Note that the tiny difference between the analytical and computational values is a consequence of the shifting assets in time. To obtain the exact value analytically, one would need the relative assets of the two groups at times beyond \( t_2 = 18 \) rather than initial time.

Fig. 6 displays the price dynamics including the addition of new shares into the system, i.e., \( P(t) \) is calculated as a function of time that includes both of the effects: the announcement of more shares at time \( t_1 \) and the addition of new shares that are 1/3 of the total shares distributed into the system at time \( t_2 \), and the transition between these steady states. The conditions on these numerical calculations are as follows. Initial asset position for each group is the same as in Fig. 5. We use again the parameters \( t_1 := 9, \ t_2 := 18, \ c_1^{(i)} = c_2^{(i)} := 1 \) for both groups, the trend coefficient \( q_1^{(i)} \) varies among 0.1, 0.3 and 0.5 while the value coefficients are \( q_2^{(i)} = 0.3 \) (for the figure on the left) and \( q_2^{(i)} = 0.5 \) (for the figure on the right). From the figures above, one can extract that the equilibrium prices are \( P_1 = 4.7091 \) and \( P_1 = 4.7087 \) for \( q_2^{(i)} = 0.3 \) and \( P_1 = 4.6231 \) and \( P_2 = 4.0137 \) for \( q_2^{(i)} = 0.5 \). Now using (4.7) (the asymptotic) steady state price can be calculated as \( P_1 = 4.7087 \) and \( P_2 = 3.9452 \) for \( q_2^{(i)} = 0.3 \) and \( P_1 = 4.6231 \) and \( P_2 = 4.0137 \) for \( q_2^{(i)} = 0.5 \). Under the conditions above. Once again, the numerical and analytical results are as close as one can expect. Figs. 5 and 6 also illustrate the effects of the trend and value coefficients on the price evolution of a single asset. It can be seen from each figure that the steady state price is independent of the trend coefficient, \( q_1^{(i)} \), as the theory suggests (see the graphs for \( 15 < t < 17 \) and \( 28 < t < 30 \).

Fig. 7 shows how the variation in the \( c^{(i)} \)'s affects the price dynamics, i.e., we compare the effects of more and less delay on the price evolution. In these calculations, we again assume that the first group has 50% of the total assets in the system.
initially as in the calculations above. In each figure, the $c_j^{(i)}$'s vary among 0.1 and 1, the trend coefficient is $q_1^{(i)} = 0.5$ for both groups while the value coefficient $q_2^{(i)}$ is set as 0.1 for the figure on the left and 0.5 for the figure on the right. When the shareholders do not respond quickly (that corresponds to $c_i = 0.1$ in the graph), the price stabilizes late.

Fig. 8 discusses the effect of variation in the initial assets position of each group on the price dynamics. In this calculation, we assume that the second group initially owns 1/3, 1/2 and all of the assets in the system, respectively. The parameters are set as $c_1^{(i)} = c_2^{(i)} = 1$ and $q_1^{(i)} = q_2^{(i)} = 0.1$ for both groups. For example, when the second group has 1/3 of the initial assets, the price stabilizes at the values $P_1 = 4.9101$ before the influx of the new shares, and, later, $P_2 = 3.8613$. Again using (4.7) the analytical calculation yields the steady state prices as $P_1 = 4.9100$ and $P_2 = 3.8650$. Once again, the numerics are consistent with the analytical results.

To summarize, these examples illustrate two important aspects of our equations. First we consider the announcement of the issuing of new shares. We assume that some of the investors react immediately and lower their estimate of the valuation based on the anticipated lowering of the liquidity price (i.e., due to the addition of new shares). Other investors maintain their original valuation. We calculate the resulting steady state price. Then we have an influx of shares without any further changes in investor’s appraisal of the stock and without any changes in the basic value of the stock. We then observe (after time $t_2$) the direct effect of the additional shares on the price. The analytical relations of Sections 4 and 5 (that are confirmed by the numerical results) allow one to relate the drop in price to the parameters of the system. The numerical results also show the transitions between the time periods (a) before the announcement, (b) after the announcement and before the issuing of additional shares, and (c) after the issuing of shares.

7. Conclusion

We have extended the derivation of [4] to multiple groups with differing assessments of valuation of a particular asset, as well as differing coefficients of motivation and strategy. These concepts are intrinsic to markets, although classical finance is largely based on the assumption that all publicly available information leads to a unique valuation. Another limitation of classical finance methodology is the assumption of an infinite amount of arbitrage capital that would take advantage of any deviations from this unique true value. Our differential equations naturally incorporate the finiteness of assets. In fact, an important price – in addition to the trading price and the fundamental price – is the liquidity price, that is obtained by dividing the available cash by the number of shares issued. This concept was the key to the prediction that an increase in cash would lead to a larger price bubble in experimental asset markets. In many financial market situations such as initial and secondary public offerings of stock, the first question is whether there is enough of a demand for the issue. Another example that we have considered above is the issuing of additional shares by a closed-end fund for which the fundamental value is known unambiguously. For this particular example we have considered some typical parameters, such as the fraction of new shares issued, etc., in order to obtain the new equilibrium price, that would be impossible in the context of classical finance.

The results indicate that predictions of price movements can be made based upon parameters such as the coefficient, $q_2$, for the value based motivation, and the fraction of assets owned by groups that assign a particular value to the asset. Using a range of values for these parameters, we can obtain changes in price that are in qualitative agreement with observations. The next step toward quantitative and accurate forecasts would be to use the data up to the current time to optimize the parameters using a least squares procedure. In this way the parameter $q_2$, for example, is evaluated in an optimal manner. Using the data shortly after the announcement, one can similarly optimize on the fraction of assets owned by the group that continues to value the asset as before. The numerical solution of the system of ordinary differential equations can then be computed using a set of parameters that have been obtained from the data up to the particular point in time. Out-of-sample forecasts can then be made on the eventual price (steady state) after the announcement. After a few weeks, the actual steady state price is known, and can be used to determine the parameters (e.g., the fraction of assets belonging to each group) more accurately. When the additional shares enter the market there are no additional parameters to be determined, since the exact number of the shares is a known quantity. Hence a forecast can be made well before these shares enter the market.

This example illustrates the possibilities for this approach. Secondary offering of shares and differences of opinion are commonplace in the markets. Traditional finance offers no methodology to make any nontrivial forecasts in these situations. Similarly, for initial public offerings there are several time periods that are important. During the first thirty days, typically the underwriter effectively controls the price. There is also a “lock-up period” during which the insiders cannot sell their shares. When the lock-up period ends, a large number of shares have the potential to enter the market, creating an effect similar to the one in our example.
Even in the absence of additional shares, there is often the issue of different groups having very different views on the value of an asset. In some cases the differences in valuation are extreme. For example in the internet/high-tech bubble, there were shares trading for $200 that fundamental investors valued at $5. Since investors buying at $200 were often momentum players, falling prices usually induced selling rather than buying. But with the value investors not interested until the price fell to a few percent of the former levels, the decline became a free fall. Thus, the approach we develop has the potential to address quantitatively the issues that are trivialized in classical economics and finance.

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