

MULTI-GROUP ASSET FLOW EQUATIONS AND STABILITY

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ABSTRACT. We consider a two-group asset flow model of a financial instrument with one group focused on price trend, the other on value. We prove the existence of both stable and unstable regions for the system of differential equations and show that a strong motivation based on (particularly recent) price trend is associated with instability. Numerical computations using a set of typical parameters describe precise regions of stability and instability. A precise limiting connection between the discrete and differential equations is also established.

1. Introduction. Recent events in the world's financial markets have demonstrated the need for developing mathematical models that are capable of addressing issues of market dynamics and stability. Standard theories of asset management and options assume near equilibrium conditions. In particular, asset and option valuation theories are based on the equation

$$\frac{dP}{P} = \sigma dX + \mu dt \quad (1.1)$$

where P is the asset price at time t , while dX is a normal random variable (mean 0 and variance dt), σ^2 is the variance, and μ is the drift, so that μdt is the expected return on the investment in time dt .

This formalism is based upon the idea that a sufficiently large proportion of investors are informed of the realistic value of the asset and act solely on that basis. It assumes that there is an infinite amount of arbitrage capital that is ever-present to quickly exploit any deviation from fundamental value. However, it has been noted (see [26]) that there exist practical limitations to arbitrage in real markets. Thus, while an arbitrage opportunity might exist, arbitrageurs might not be able to act on it due to risk of losses and the need to liquidate the portfolio due to pressure from investors. Furthermore, on a larger time scale the assumption of rapid assimilation of information leads to an inverse relationship between volatility, defined as σ and mean expected return, μ . A standard optimization result ([2]) is that all investors should invest in a mixture of a risk-free asset (e.g., Treasury bills) and a single portfolio of risky assets consisting of stocks and bonds. The fraction of wealth invested in risky assets will differ among investors who differ in their utility (or preference) functions, and consequently, risk tolerances.

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The classical theories are idealizations that are limiting cases. In particular, the capital owned by knowledgeable investors focused on value is assumed to be infinite. Starting from a theory in which a crucial quantity is infinite, it is often difficult to design a unique generalization or modification in which it is finite. The expectation, or hope, in the research community is that more refined theories with smaller effects will augment an existing, established theory. Thus it is difficult, if not impossible, to build on a theory that does not recognize a key finite quantity. One must model such effects from first principles. Yet utilizing a perspective of basic modeling that does not emanate from a familiar theory encounters difficulties as a result of the sociology of scientific research.

Two perspectives in addressing this issue are noted below. (i) One can consider first the modeling of an experimental asset market, pioneered by Vernon Smith (2002 Economics Nobel Laureate) and collaborators, where one or two dozen participants trade an asset defined by the experimenter with the objective of earning real money based on their trades. Within this setting one knows precisely the quantity of cash, asset, payouts etc. Thus, one can develop and test mathematical models of these experiments. (ii) Upon generalizing such a model to several groups with varying motivations and assuming some randomness one can try to recover the classical models as a limit of the value oriented group having a preponderance of the assets. This is an issue that we consider in Appendix A of this paper.

We summarize an approach that has been developed by Caginalp and collaborators since 1990 (see www.ssrn.com). This model is flexible enough to account for a variety of factors that may affect investor sentiment, allow for multiple investor groups, and incorporate concepts such as the gradual diffusion of information across the investor population. If we focus on a simple experimental asset market setting, there are a set number of traders who trade a single asset whose worth is defined by its payouts or dividends. The experimenter has the option of allowing these dividends to be traded during the experiment, or deferred to the end of the experiment. Thus we may define M as the constant representing the total cash in the system (assuming for the time being that dividends are deferred), and N as the (constant) number of shares of the asset. We let

$$B := \frac{NP}{NP + M} \quad (1.2)$$

be the fraction of total wealth invested in the stock in terms of the trading price, $P(t)$, at time t . Then one also has

$$1 - B = \frac{M}{NP + M} \quad \text{and} \quad \frac{B}{1 - B} = \frac{NP}{M} =: \frac{P}{L}. \quad (1.3)$$

The liquidity, $L := M/N$, is a key variable introduced in [4] that has units of dollars per share. Thus, the quantities L , P , and the fundamental value of the asset, $P_a(t)$, determined by the definition of the asset, all have these units, and one can measure price in natural units of liquidity. The modeling of asset price dynamics starts by using a simple equation that stipulates that price moves in proportion to the imbalance between demand, D , and supply, S :

$$\tau P^{-1} \frac{dP}{dt} = \frac{D - S}{S}, \quad (1.4)$$

where τ is a time scale ([29]). Within a closed system one can express the demand for the asset as $D = k(1 - B)$ and the supply as $S = (1 - k)B$, where k is a

transition rate. One can interpret k as the probability of a unit of cash being submitted as a buy order per unit time. Classical game theory ([22], [14]) would suggest that k should depend solely on the valuation, since each trader is aware of the same information. However, the dependence of k on other quantities such as the trend in price (price trend or trend) can be determined experimentally ([8]) and empirically (in the case of world markets) ([5], [23]). Given a particular motivation, one can use the discrete version of these differential equations in conjunction with statistical methods to determine whether there is statistical support for it ([6]). With these definitions, one can rewrite 1.4 as

$$\tau P^{-1} \frac{dP}{dt} = \frac{k(1-B)}{(1-k)B} - 1. \quad (1.5)$$

The key issue is the dependence of k on investor strategy and behavior. The investor sentiment can be described by a function $\zeta(t)$ with range in \mathbb{R} , and can be written in terms of a finite set of components: $\zeta(t) = \zeta_1(t) + \zeta_2(t) + \dots + \zeta_n$. Since k must have values in $[0, 1]$ a simple way to map values of ζ into k is through a function such as

$$k := \{1 + \tanh(\zeta_1 + \zeta_2 + \dots + \zeta_n)\} / 2. \quad (1.6)$$

Considering valuation, $P_a(t)$, and just one other motivation, namely trend, we have

$$\begin{aligned} \zeta_1(t) &= q_1 c_1 \int_{-\infty}^t e^{-c_1(t-\tau)} \frac{1}{P(\tau)} \frac{dP(\tau)}{d\tau} d\tau; \\ \zeta_2(t) &= q_2 c_2 \int_{-\infty}^t e^{-c_2(t-\tau)} \frac{P_a(\tau) - P(\tau)}{P_a(\tau)} d\tau \end{aligned} \quad (1.7)$$

where q_i and c_i characterize the magnitudes and time scales of the two motivations for investing [3]. The first of these indicates that investors are influenced by the change in price but are more strongly influenced by recent changes. A small value for the time scale $1/c_1$ indicates that investors are focused on short term changes in the trend. The $\zeta_2(t)$ parameter represents investors' focus on the deviation between the asset price and its fundamental value. Again, more recent changes have a greater effect on investor decisions. A large value for $1/c_2$ indicates that investors take action slowly when there is an over- or undervaluation. An alternative explanation for the $1/c_i$ parameters is that they regulate the rate at which information diffuses to the investor population. The magnitudes of q_1 and q_2 represent the significance of these motivations in trading decisions for the aggregate (homogeneous) investor population. Note that other motivations, $\zeta_3(t)$, $\zeta_4(t)$, etc. are possible. For example, statistical studies ([5]) have shown that the M2 Money Supply and asset price volatility also impact investor decisions to buy and sell.

One can also show (see equations 2.6 and 3.5 in [4]) that the change in the fraction of wealth invested in the asset is given by

$$\frac{dB}{dt} = k(1-B) + (k-1)B + B(1-B) \frac{1}{P} \frac{dP}{dt}. \quad (1.8)$$

The system of equations 1.5 - 1.8 is a complete set of ODEs that have been studied in a number of papers. We will study generalizations of these equations to two or more groups. One can simplify these equations by taking the limit of short time

scale and using the approximation $\tanh(x) \simeq x$, leading to the approximation

$$\frac{k}{1-k} = 1 + 2\frac{q_1\tau}{P} \frac{dP}{dt} + 2q_2 \left(1 - \frac{P}{P_a}\right). \quad (1.9)$$

One of the concepts that has emerged from this analysis is that the equilibrium price depends not only on P_a (which would be the case in classical economics) but also on the liquidity, L . The equilibrium price was found to be

$$\frac{P_{eq}}{L} = \frac{1 + 2q_2}{1 + 2q_2 \left(\frac{L}{P_a}\right)}. \quad (1.10)$$

When q_2 is large (i.e. there is a great deal of importance attached to valuation), the equilibrium price, P_{eq} , is close to P_a . However, if q_2 is small, the equilibrium price is close to L , as the valuation is marginalized.

This approach to modeling incorporates the issues of trend, overreaction, underreaction, disparate information and motivation of distinct groups as well as the effect of finiteness of assets within a single mathematical model. Thus it is evident that these effects arise from the dynamics of trading when the assumptions are more realistic than the classical idealizations. In the years since 1990 several works have implemented various aspects of this approach. In particular, [17] models overreaction, underreaction, and momentum via the interaction between two heterogeneous investor groups: news watchers and momentum traders. In other words, members of the fundamental group are called “news watchers” since they update the fundamental value via news reports. Similarly, an asset price’s tendency to follow a recent trend in price has been studied ([18], [19]) along with the competing concepts of overreaction and underreaction.

In addition several models including [9] attempt to explain momentum profits by investors’ inherent biases in interpreting information. They consider overconfident investors who overreact to private information. Within our models, this corresponds to one group setting a higher price on the fundamental value. The Barberis, Shleifer, and Vishny [1] model incorporates the psychological concepts of representativeness ([28]) and conservatism ([13]). Grinblatt and Han [15] explained momentum via the concept of market disposition, i.e. investors are more likely to sell to realize a gain than a loss ([25], [24]).

Note that trend effects need not be exclusively due to overreaction or underreaction. For example, due to the finiteness of assets, a large buyer attempting to build a position without drastically pushing up prices could cause an upward trend in price. Another possible explanation could be the observation that others are profiting, which causes even the value-based investor to act ([6]).

Duran and Caginalp [12] utilized optimization methods to estimate the parameters of the asset flow differential equations using market data. These equations were applied to typical secondary stock offerings by Caginalp and Merdan [7], who obtained quantitative results relating an influx of shares with lower prices within this model. A sensitivity analysis of parameters was performed in [11]. Recently, liquidity issues were further explored in [20] and [21].

In this paper we present a detailed derivation of the discrete and continuum equations for asset dynamics with disparate groups with differing assessments of valuation and motivational characteristics. We determine the region of equilibrium points given the basic parameters governing the investor group motivations and assets (see Theorem 4.1 in Section 4). Theorem 4.2 (see Section 4) establishes

regions of stability and instability. In Section 5 numerical computations for typical parameters display precise regions of stability and instability.

2. The discrete equations. We consider two groups of investors (that can easily be generalized to an arbitrary number of groups) so that group i ($i = 1, 2$) is endowed at time T_j with $N^{(i)}(T_j)$ shares and $M^{(i)}(T_j)$ in cash. The probability that a unit of cash is submitted by investor group i for purchase during the time interval (T_j, T_{j+1}) of the single equity in the system is given by $k_*^{(i)}$ and may depend on many factors including the price history, as discussed below. Similarly, $\tilde{k}_*^{(i)}$ is the probability that a unit of stock will be submitted for sale. The demand (in terms of dollars) at time interval $\delta T := T_{j+1} - T_j$ (assume a uniform spacing so that δT is independent of j) is given by $D(\delta T) = k_*^{(1)}(\delta T)M^{(1)} + k_*^{(2)}(\delta T)M^{(2)}$ while the supply (in terms of dollars) is $S := \tilde{S}P$ with $\tilde{S}(\delta T) := \tilde{k}_*^{(1)}(\delta T)N^{(1)} + \tilde{k}_*^{(2)}(\delta T)N^{(2)}$. An equilibrium trading price is one for which the supply, $\tilde{S}(T_j)$, and demand, $D(T_j)$, balance, i.e.

$$P(T_j) = F\left(T_j; k_*^{(i)}(P(T_0), \dots, P(T_j)), M^{(i)}(T_j), N^{(i)}(T_j)\right) \quad (2.1a)$$

where

$$\begin{aligned} & F\left(T_j; k_*^{(i)}(P(T_0), \dots, P(T_j)), M^{(i)}(T_j), N^{(i)}(T_j)\right) \\ & := \frac{D(T_j; \dots)}{\tilde{S}(T_j; \dots)} \\ & = \frac{k_*^{(1)}M^{(1)}(T_j) + k_*^{(2)}M^{(2)}(T_j)}{\tilde{k}_*^{(1)}N^{(1)}(T_j) + \tilde{k}_*^{(2)}N^{(2)}(T_j)} \end{aligned} \quad (2.1b)$$

and, for brevity, we have replaced $k_*^{(i)}(P(T_0), \dots, P(T_j))$ by $k_*^{(i)}$ on the right hand side. We define τ as a relaxation time that determines the rate at which a non-equilibrium situation returns to equilibrium. For simplicity we can assume $\tilde{k}_*^{(i)} := 1 - k_*^{(i)}$ though the modeling is similar in the more general case. Next, we describe the dynamics of price. Similar to [16] we postulate that the change in price in a unit time is proportional to the extent to which it is away from equilibrium:

$$\frac{1}{P(T_j)} \frac{P(T_{j+1}) - P(T_j)}{T_{j+1} - T_j} = \frac{1}{\tau} \left(\frac{F(T_j)}{P(T_j)} - 1 \right), \quad (2.2)$$

where we have suppressed the dependence of F on the other variables. As in earlier papers (see [7] and references therein) we consider a limiting form of this microeconomic equation, i.e. 3.2 below. In particular, we derive this limiting form precisely in Section 3. Note that by using the definition of F , we see that the right hand side is the excess demand, $(D - S)/S$. Hence, this equation is compatible with basic price theory ([29]) as well as observations in experimental asset markets ([27]).

The time scale. Since δT is simply the time interval on which we choose to perform the computations in the discrete case, a natural choice for δT is given by $\delta T := \tau$ since the imbalance is restored on this time scale. Then 2.2 is simply

$$P(T_{j+1}) = F(T_j). \quad (2.3)$$

This indicates that if the supply and demand do not balance, then a new price, defined as $P(T_{j+1})$, is discovered. This price restores the balance between supply and demand, i.e.

$$\frac{k_*^{(1)} M^{(1)} + k_*^{(2)} M^{(2)}}{\left(\tilde{k}_*^{(1)} N^{(1)} + \tilde{k}_*^{(2)} N^{(2)}\right) P(T_{j+1})} = 1, \quad (2.4)$$

within a time frame that is the natural relaxation time for the problem.

Equation 2.2 states that if the price is not at an equilibrium value then it moves toward this price on a time scale given by τ . We can regard $\delta T := T_{j+1} - T_j$ as the time interval between trades in a discrete system, e.g. a market in which bids and asks are submitted at noon each day or an experimental setup with discrete periods.

The choice $\delta T = \tau$ is perhaps the most natural and closest to an efficient market since it suggests that the basic time scale for readjustment to equilibrium is the one period time scale defined by the market institution.

Prior to focusing on $\delta T = \tau$ we consider the implications of choosing $\delta T := a\tau$ for $a \neq 1$. In this case 2.2 can be written as

$$\frac{P(T_{j+1})}{P(T_j)} - 1 = a \left(\frac{F(T_j)}{P(T_j)} - 1 \right) \quad (2.5)$$

or

$$P(T_{j+1}) = F(T_j) + \delta \{F(T_j) - P(T_j)\} \quad (2.6)$$

where $\delta := a - 1$.

For $-1 < \delta < 0$, i.e. $0 < a < 1$, when $F(T_j) \neq P(T_j)$ the price at time T_{j+1} does not attain $F(T_j)$ as it does for $\delta = 0$. For example, if $\delta = -1/2$, then

$$P(T_{j+1}) = \frac{1}{2}F(T_j) + \frac{1}{2}P(T_j)$$

so that $P(T_{j+1})$ is only the average of the old price, $P(T_j)$, and the ‘‘target equilibrium price,’’ $F(T_j)$.

Interpreting $\delta \neq 0$ within the context of experimental markets, one can consider $\tau = 3$ and $\delta T = 1$ so that $a = 1/3$ and $\delta = -2/3$. This means that the time scale of adjustment to equilibrium is 3 units while the trading time scale is 1. Hence by 2.6 the price adjustment at time T_{j+1} attains 1/3 of the deviation, $F(T_j) - P(T_j)$, from equilibrium.

Unless specified otherwise, we assume $\delta T = \tau$ below. For $\delta T \neq \tau$ the equations 2.7 and 2.8 below must be modified by replacing $P(T_{j+1})$ by $F(T_j)$.

The equation 2.2 remains valid if there is an influx or outflow of cash or shares during the time period between T_j and T_{j+1} with the assumption that this new cash or shares cannot be submitted until time T_{j+1} . At this point we distinguish between a totally conserved system in which no shares or cash are added or subtracted versus one in which there is an influx or outflow of shares or cash. In the latter case, we can define $m_*^{(i)}(T_j)\delta T$ and $n_*^{(i)}(T_j)\delta T$ as the net inflows of cash and shares during this time period between T_j and T_{j+1} . Note that negative values for either of these variables denote outflows. Thus we can write the basic conservation law as

$$\begin{aligned} M^{(i)}(T_{j+1}) - M^{(i)}(T_j) &= -k_*^{(i)}(\delta T) M^{(i)}(T_j) \\ &\quad + \tilde{k}_*^{(i)}(\delta T) N^{(i)}(T_j)P(T_{j+1}) + (\delta T) m_*^{(i)}(T_j). \end{aligned} \quad (2.7)$$

In other words, during the time period from T_j to T_{j+1} the cash decreases due to the fraction of existing cash, $k_*^{(i)}\delta T$, that is submitted for purchase of the equity, and increases due to the fraction of shares, $\tilde{k}_*^{(i)}\delta T$, that is submitted for sale. Setting $m_*^{(i)} = 0$ for all time and each group, i , specifies the special case in which we have

no additional cash entering or leaving the system. Similarly, for each group i , the change in the number of shares owned must satisfy:

$$\begin{aligned} P(T_{j+1}) \left\{ N^{(i)}(T_{j+1}) - N^{(i)}(T_j) \right\} &= k_*^{(i)} (\delta T) M^{(i)}(T_j) \\ &\quad - \tilde{k}_*^{(i)} (\delta T) N^{(i)}(T_j) P(T_{j+1}) \\ &\quad + (\delta T) n_*^{(i)}(T_j) P(T_{j+1}). \end{aligned} \quad (2.8)$$

For each group i , the change in the value of stock owned must be balanced by the flow plus the influx of cash or shares. In fact, comparing the right hand sides of 2.7 and 2.8 we observe:

$$\begin{aligned} P(T_{j+1}) \left\{ N^{(i)}(T_{j+1}) - N^{(i)}(T_j) \right\} &= - \left\{ M^{(i)}(T_{j+1}) - M^{(i)}(T_j) \right\} \\ &\quad + (\delta T) m_*^{(i)}(T_j) + (\delta T) n_*^{(i)}(T_j) P(T_{j+1}). \end{aligned} \quad (2.9)$$

The system of equations 2.2, 2.7, and 2.8 can be studied as a system of difference equations upon specifying initial conditions together with $k_*^{(i)}$ as a function of $(P(T_0), \dots, P(T_j))$ and possibly other variables. In other words, these equations can be solved algebraically for $(P(T_{j+1}), M^{(i)}(T_{j+1}), N^{(i)}(T_{j+1}))$ in terms of the variables at earlier times. The choice of $k_*^{(i)}$ involves assumptions on the motivations of the traders. The dependence of these functions on various factors characterizes the trader population.

Remark 1. In the case of no additional cash or asset, if we sum either 2.7 or 2.8 over all groups i , then we obtain from $\sum_i N^{(i)}(T_j) = \text{Constant}$ or $\sum_i M^{(i)}(T_j) = \text{Constant}$ that the left hand sides vanish. As a result, we obtain simply the definition of $F(T_j)$ so it is consistent with conservation. With non-zero $m_*^{(i)}$ or $n_*^{(i)}$ one has from 2.7

$$\begin{aligned} \sum_i \left\{ \frac{M^{(i)}(T_{j+1}) - M^{(i)}(T_j)}{\delta T} \right\} & \\ = \sum_i \left\{ -k_*^{(i)} M^{(i)}(T_j) + \tilde{k}_*^{(i)} N^{(i)}(T_j) P(T_{j+1}) + m_*^{(i)}(T_j) \right\}, & \end{aligned} \quad (2.10)$$

yielding

$$\frac{M^{(1)}(T_{j+1}) - M^{(1)}(T_j)}{\delta T} + \frac{M^{(2)}(T_{j+1}) - M^{(2)}(T_j)}{\delta T} = m_*^{(1)}(T_j) + m_*^{(2)}(T_j) \quad (2.11)$$

for $n = 2$, i.e. two investor groups, using $P(T_{j+1}) = F(T_j)$ and the definition of F .

Remark 2. The relationship between the discrete asset flow equations 2.2, 2.7, and 2.8 and the stochastic asset pricing equation 1.1 can be understood formally in the following sense. One considers two groups, one of which has the preponderance of assets and focuses on value, buying slightly below and selling slightly above the valuation, $P_a(t) = P_a(0) e^{\mu t}$, where μ is the risk-free rate. The other group employs a variety of strategies that cancel out one another. The additional funds entering into the system are a random process leading to a price dynamics that is stochastic and similar to 1.1. Further details are provided in Appendix A.

3. The continuum limit. In deriving the continuum limit it is essential to examine the roles of δT and τ . The relaxation parameter τ is intrinsic to the system and describes the time scale on which the traders react to the changes in price and other variables. On the other hand, δT is a computational time scale so that we can consider the limit $\delta T \rightarrow 0$. In practice, δT is small compared to the overall time period, e.g., daily prices within a year of trading.

We can regard the discrete equations (particularly with $\delta T = \tau$) as the exact equations, with the continuum as an approximation. Thus, we can approximate the ratio $\{P(T_{j+1}) - P(T_j)\} / \delta T$ by $P'(T_j)$. More precisely, for a smooth function, P , we have

$$P(T_{j+1}) = P(T_j) + (\delta T) P'(T_j) + \frac{(\delta T)^2}{2} P''(\zeta), \text{ for some } \zeta \in (T_j, T_{j+1}) \quad (3.1)$$

which we can write as

$$\{P(T_{j+1}) - P(T_j)\} / \delta T - P'(T_j) = O[\delta T]$$

where $f(x) = O[x]$ signifies

$$\limsup_{x \rightarrow 0} \frac{|f(x)|}{|x|} < \infty.$$

Hence we replace the discrete derivative on the left hand side of 2.2 with a very small error and obtain the continuum limit

$$\tau \frac{1}{P(t)} \frac{dP}{dt}(t) = \frac{F(t)}{P(t)} - 1 \quad (3.2)$$

where we have replaced the discrete variables T_j with the continuum variable t in F (see below) and P . Note that the discarded term is $\tau O[\delta T]$.

Another perspective on this is that in equation 2.2 the discrete derivative is $O[1]$ so that one must have $F/P - 1 = O[\tau]$ in order to maintain equality in this equation. Hence, in the discrete equation with $\delta T := \tau$ as a small parameter, we write 2.2 as

$$\tau \frac{1}{P(T_j)} \{P'(T_j) + O[\delta T]\} = \left\{ \frac{F(T_j)}{P(T_j)} - 1 \right\}.$$

Then each term is of order τ except for the second term $\tau O[\delta T]$ which is order τ^2 and can be neglected, leading to 3.2.

To obtain the continuum approximation for the remaining equations we first define

$$k^{(i)} := k_*^{(i)} \tau, \quad \tilde{k}^{(i)} := \tilde{k}_*^{(i)} \tau, \quad m^{(i)} := m_*^{(i)} \tau, \quad n^{(i)} := n_*^{(i)} \tau.$$

We treat the $m^{(i)}$ and $n^{(i)}$ as known functions that represent the inflow/outflow of cash and shares, respectively, due to macroscopic factors. Following a similar argument as above the equations describing the cash and stock position of each group can be written as

$$\tau \frac{dM^{(i)}(t)}{dt} = -k^{(i)}(t)M^{(i)}(t) + \tilde{k}^{(i)}(t)N^{(i)}(t)F(t) + m^{(i)}(t), \quad (3.3)$$

$$\tau \frac{dN^{(i)}(t)}{dt} = \frac{k^{(i)}(t)M^{(i)}(t)}{F(t)} - \tilde{k}^{(i)}(t)N^{(i)}(t) + n^{(i)}(t). \quad (3.4)$$

We define the continuous version of the function F as

$$F(t) := \frac{k_*^{(1)} M^{(1)}(t) + k_*^{(2)} M^{(2)}(t)}{\tilde{k}_*^{(1)} N^{(1)}(t) + \tilde{k}_*^{(2)} N^{(2)}(t)}.$$

Thus, equations 3.2 - 3.4 can be studied numerically subject to initial conditions upon specifying $k^{(i)}$. By rescaling time we can set τ to unity in each of the three equations.

Equilibrium Conditions. In the totally conserved case (no shares or cash added or withdrawn from the system) setting the time derivatives in 3.2 - 3.4 to zero yields

$$\begin{aligned} P &= F = \text{Constant} \\ 0 &= -k^{(i)} M^{(i)} + (1 - k^{(i)}) N^{(i)} P, \end{aligned} \quad (3.5)$$

provided we define $\tilde{k}^{(i)} := 1 - k^{(i)}$, so that a necessary and sufficient condition for equilibrium (which we denote now by P_{eq}) is

$$P_{eq} = \frac{k^{(i)} M^{(i)}}{1 - k^{(i)} N^{(i)}} \quad \text{for } i = 1, 2. \quad (3.6)$$

Note that 3.5 and 3.6 imply the identities

$$\frac{k^{(1)} M^{(1)} + k^{(2)} M^{(2)}}{(1 - k^{(1)}) N^{(1)} + (1 - k^{(2)}) N^{(2)}} = \frac{k^{(1)} M^{(1)}}{1 - k^{(1)} N^{(1)}} = \frac{k^{(2)} M^{(2)}}{1 - k^{(2)} N^{(2)}} \quad (3.7)$$

establishing a compatibility relationship between preferences (determined by $k^{(i)}$) and asset positions of the two groups. Thus, given a compatible set $(k^{(1)}, k^{(2)}, M^{(1)}, M^{(2)}, N^{(1)}, N^{(2)})$ there is a unique equilibrium price P_{eq} . This differs from classical theory where P_{eq} would be uniquely determined by P_a alone. If we regard the $k^{(i)}$ as known, then we have two nonlinear equations, i.e. equations 3.6, for three unknowns, namely $M^{(1)}$, $N^{(1)}$, and P .

Remark 3. The equations 3.2 - 3.4 reduce to the single group model of [4] when the assets of one group are reduced to zero. To verify this, let $M^{(2)} = N^{(2)} = 0$ and drop the subscript 1 on the first group. Thus, Group 1 controls the total amount of cash, M , and total number of shares, N , in the system; and any trading must be among traders within Group 1. Hence, the definition of F , i.e.

$$F = \frac{k^{(1)} M^{(1)}}{\tilde{k}^{(1)} N^{(1)}} = \frac{kM}{\tilde{k}N},$$

and the price equation 3.2 with $\tilde{k} = 1 - k$ imply the result

$$\frac{1}{P(t)} \frac{dP(t)}{dt} = \frac{1}{\tau} \left\{ \frac{k}{1 - k} \frac{M}{NP(t)} - 1 \right\}.$$

The concept of a liquidity value, $L := M/N$, was introduced in this paper. With this substitution we have the equation

$$\frac{1}{P(t)} \frac{dP(t)}{dt} = \frac{1}{\tau} \left\{ \frac{k}{1 - k} \frac{L}{P} - 1 \right\}.$$

In the limit of a single group without influx of additional shares or cash, the variables, M , N , and L are all constants in time. Nevertheless, the fraction of funds in the asset, B , defined by

$$B := \frac{NP}{NP + M},$$

varies in time and satisfies

$$\frac{B}{1-B} = \frac{N}{M}P = \frac{P}{L}$$

yielding the differential equation

$$\tau P^{-1} \frac{dP}{dt} = \frac{k(1-B)}{(1-k)B} - 1.$$

Hence, for any definition of $k^{(i)}$ the multi-group model defined above reduces to the single group model.

4. Analysis of equilibrium and stability with arbitrary parameter values.

As noted above, the transition rate functions, $k_*^{(i)}$, account for the different motivations investors have to buy or sell an asset. Through statistical modeling and regression analysis on market data ([5]), it has been shown that two such motivations are (i) the asset's recent trend in price and (ii) the asset price's deviation from its fundamental value. These motivations are modeled by equations 1.7. To account for multiple groups, these equations are modified as follows:

$$\zeta_1^{(i)}(t) := q_1^{(i)} c_1^{(i)} \int_{-\infty}^t e^{-c_1^{(i)}(t-\tau)} \frac{1}{P(\tau)} \frac{dP(\tau)}{d\tau} d\tau \quad (4.1)$$

$$\zeta_2^{(i)}(t) := q_2^{(i)} c_2^{(i)} \int_{-\infty}^t e^{-c_2^{(i)}(t-\tau)} \frac{P_a^{(i)}(\tau) - P(\tau)}{P_a^{(i)}(\tau)} d\tau \quad (4.2)$$

where the superscript $i = 1, 2, \dots, n$ represents the investor group. The parameters, now group-specific, are discussed in Section 1. For example, $\zeta_2^{(i)}(t)$ represents group i 's motivation arising from the deviation between the asset's price and group i 's assessment of the asset's fundamental value, $P_a^{(i)}(t)$, when trading ([7]). Thus, a quantification of an investor group's motivation to buy/sell is given by $\zeta^{(i)}(t) = \zeta_1^{(i)}(t) + \zeta_2^{(i)}(t)$.

Differentiating equations 4.1 and 4.2 yields the ordinary differential equations

$$\frac{d\zeta_1^{(i)}(t)}{dt} = q_1^{(i)} c_1^{(i)} \frac{1}{P(t)} \frac{dP(t)}{dt} - c_1^{(i)} \zeta_1^{(i)}(t) \quad (4.3)$$

$$\frac{d\zeta_2^{(i)}(t)}{dt} = q_2^{(i)} c_2^{(i)} \frac{P_a^{(i)}(t) - P(t)}{P_a^{(i)}(t)} - c_2^{(i)} \zeta_2^{(i)}(t). \quad (4.4)$$

Adding these to equations 3.2 - 3.4 gives the following system

$$\tau \frac{dN^{(i)}}{dt} = \frac{k^{(i)} M^{(i)}(t)}{F(t)} - \tilde{k}^{(i)} N^{(i)}(t) + n^{(i)}(t) \quad (3.3)$$

$$\tau \frac{dM^{(i)}}{dt} = -k^{(i)} M^{(i)}(t) + \tilde{k}^{(i)} N^{(i)}(t) F(t) + m^{(i)}(t) \quad (3.4)$$

$$\tau \frac{dP}{dt} = F(t) - P(t) \quad (3.2)$$

$$\frac{d\zeta_1^{(i)}(t)}{dt} = q_1^{(i)} c_1^{(i)} \frac{1}{P(t)} \frac{dP(t)}{dt} - c_1^{(i)} \zeta_1^{(i)}(t) \quad (4.3)$$

$$\frac{d\zeta_2^{(i)}(t)}{dt} = q_2^{(i)} c_2^{(i)} \frac{P_a^{(i)}(t) - P(t)}{P_a^{(i)}(t)} - c_2^{(i)} \zeta_2^{(i)}(t) \quad (4.4)$$

where $i = 1, 2, \dots, n$ and

$$F(t) := \frac{k^{(1)}(t)M^{(1)}(t) + k^{(2)}(t)M^{(2)}(t) + \dots + k^{(n)}(t)M^{(n)}(t)}{\tilde{k}^{(1)}(t)N^{(1)}(t) + \tilde{k}^{(2)}(t)N^{(2)}(t) + \dots + \tilde{k}^{(n)}(t)N^{(n)}(t)}.$$

Thus we have a system of $4n + 1$ equations with $5n + 1$ parameters.

For the remainder of Section 4 and Section 5 we consider the two group system with conserved cash and shares, i.e. $M^{(1)}(t) + M^{(2)}(t) = M_0$ and $N^{(1)}(t) + N^{(2)}(t) = N_0$, where M_0 and N_0 are fixed parameters representing the total amount of cash and the total number of shares in the system. In addition we make the following assumptions:

(A1) $\tilde{k}^{(i)} = 1 - k^{(i)}$

(A2) Group 1 is focused solely on the recent trend in price and Group 2 is focused solely on the deviation from its perception of the fundamental value, $P_a^{(2)}(t)$. Classical finance suggests that the price fluctuates randomly about this value ([30]).

(A3) To simplify calculations we use the Taylor series approximation $\tanh(x) \simeq x$. Reasonable realistic parameter values will typically result in the argument, $\zeta_i^{(i)}(t)$, $i = 1, 2$, lying in the interval $(-1, 1)$, so that differences between the two models, i.e. with or without this approximation, should be negligible. Assumptions (A2) and (A3) imply $k^{(i)} \simeq \frac{1}{2} \left(1 + \zeta_i^{(i)}\right)$, $i = 1, 2$.

(A4) $P_a^{(2)}(t) = P_a^{(2)}$ is constant.

(A5) $0 \leq N^{(1)} \leq N_0$ and $0 \leq M^{(1)} \leq M_0$.

(A6) As discussed in Sections 2 and 3, we set $\tau = 1$, i.e. one period. As noted in Section 3, through a rescaling of time we may set $\tau = 1$, i.e. one period, which implies $k^{(i)}(t) = k_*^{(i)}(t)$.

(A7) $M_0, N_0, q_i^{(i)}, c_i^{(i)},$ and $P_a^{(i)}$, $i = 1, 2$ are strictly positive constants.

With these assumptions the system corresponding to equations 3.2 - 3.4, 4.3, and 4.4 becomes

$$\frac{dN^{(1)}}{dt} = \frac{1}{2}(1 + \zeta_1^{(1)})M^{(1)} \frac{(-\zeta_1^{(1)} + \zeta_2^{(2)})N^{(1)} + (1 - \zeta_2^{(2)})N_0}{(\zeta_1^{(1)} - \zeta_2^{(2)})M^{(1)} + (1 + \zeta_2^{(2)})M_0} - \frac{1}{2}(1 - \zeta_1^{(1)})N^{(1)} \quad (3.3')$$

$$\frac{dM^{(1)}}{dt} = -\frac{1}{2}(1 + \zeta_1^{(1)})M^{(1)} + \frac{1}{2}(1 - \zeta_1^{(1)})N^{(1)} \frac{(\zeta_1^{(1)} - \zeta_2^{(2)})M^{(1)} + (1 + \zeta_2^{(2)})M_0}{(-\zeta_1^{(1)} + \zeta_2^{(2)})N^{(1)} + (1 - \zeta_2^{(2)})N_0} \quad (3.4')$$

$$\frac{dP}{dt} = \frac{(\zeta_1^{(1)} - \zeta_2^{(2)})M^{(1)} + (1 + \zeta_2^{(2)})M_0}{(-\zeta_1^{(1)} + \zeta_2^{(2)})N^{(1)} + (1 - \zeta_2^{(2)})N_0} - P \quad (3.2')$$

$$\frac{d\zeta_1^{(1)}}{dt} = q_1^{(1)} c_1^{(1)} \frac{1}{P} \frac{dP}{dt} - c_1^{(1)} \zeta_1^{(1)} \quad (4.3')$$

$$\frac{d\zeta_2^{(2)}}{dt} = q_2^{(2)} c_2^{(2)} \frac{P_a^{(2)} - P}{P_a^{(2)}} - c_2^{(2)} \zeta_2^{(2)} \quad (4.4')$$

where we have suppressed the dependence of $M^{(1)}$, $N^{(1)}$, P , $\zeta_1^{(1)}$, and $\zeta_2^{(2)}$ on t . Equations 3.3' and 3.4' give the following relationship

$$\frac{dN^{(1)}}{dt} = -\frac{1}{F(t)} \frac{dM^{(1)}}{dt}. \quad (4.5)$$

We define equilibrium in terms of all time derivatives set to zero. Note that equations 4.3' and 4.4' indicate that existence of equilibria is independent of the parameters $c_1^{(1)}$, $c_2^{(2)}$, and $q_1^{(1)}$. Setting the time derivatives in equations 3.2', 3.3', and 3.4' to zero yields the following relationship between $\hat{N}^{(1)}$ and $\hat{M}^{(1)}$, where the hat notation is used to denote the equilibrium value,

$$\hat{N}^{(1)} = \frac{\hat{M}^{(1)}}{P_{eq}}. \quad (4.6)$$

Thus, at equilibrium the algebraic system of equations admits a one-dimensional curve of equilibrium points that may be parameterized by one of the dynamic variables, i.e. $\hat{N}^{(1)}$, $\hat{M}^{(1)}$, P_{eq} , $\hat{\zeta}_1^{(1)}$, or $\hat{\zeta}_2^{(2)}$. Caginalp and Balenovich [4] proved that the equilibrium price for the one group model is between the liquidity value, L , and the fundamental value, P_a . We prove a similar result (see Theorem 4.1) for the two group model with conserved cash and shares.

As noted above, the parameter $P_a^{(2)}$ represents Group 2's assessment of the fundamental value of the asset; while the parameter $L = M_0/N_0$ equals the total amount of cash divided by the total number of shares in the system¹.

Theorem 4.1. *Consider the system 3.2' - 3.4', 4.3', and 4.4' and assume (A1)-(A7). Let L and P_a be arbitrary in \mathbb{R}^+ where $L = \frac{M_0}{N_0}$ is the liquidity value and P_a is Group 2's assessment of the fundamental value of the asset.*

(i) *Suppose $L \neq P_a$. Then for any $P_{eq} \in \mathbb{R}^+$, $P_{eq} \in (\min[P_a, L], \max[P_a, L])$ the system has a unique equilibrium point of the form*

$$\begin{aligned} \hat{x} &= (\hat{N}^{(1)}, \hat{M}^{(1)}, P_{eq}, \hat{\zeta}_1, \hat{\zeta}_2) \\ &= \left(\frac{[P_a - q_2(P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2(P_a - P_{eq})] M_0}{2P_{eq}q_2(P_{eq} - P_a)}, P_{eq}\hat{N}^{(1)}, P_{eq}, 0, \right. \\ &\quad \left. q_2 \frac{P_a - P_{eq}}{P_a} \right). \end{aligned}$$

Consider the limiting case, $L = P_{eq} \neq P_a$. For any $\hat{\zeta}_2 \in (-1, 1) \setminus \{0, q_2\}$ the system has a unique equilibrium point of the form

$$\hat{x} = (N_0, M_0, P_a(1 - \frac{\hat{\zeta}_2}{q_2}), 0, \hat{\zeta}_2).$$

(ii) *Suppose $P_{eq} = P_a$. Then $P_{eq} = P_a = L$, and for any $\hat{N}^{(1)} \in \mathbb{R}^+$, $0 \leq \hat{N}^{(1)} \leq N_0$, the system has a unique equilibrium point of the form*

$$\hat{x} = (\hat{N}^{(1)}, \hat{N}^{(1)}P_{eq}, P_{eq}, 0, 0).$$

Moreover, under the above assumptions any equilibrium point of this system must lie in the interval $[\min(P_a, L), \max(P_a, L)]$.

Remark 4. Classical finance suggests that, because (1) all traders have the same information and (2) any advantage is quickly exploited by completely rational, well informed traders with ample capital, there exists a unique equilibrium price for a given time t . For example, the basic Capital Asset Pricing Model (CAPM) assumes all traders are rational mean-variance optimizers and share a common economic view of the world, i.e. the investor population has homogeneous expectations ([2]). The

¹For notational convenience we set $P_a = P_a^{(2)}$, $q_2 = q_2^{(2)}$, $\hat{\zeta}_2 = \hat{\zeta}_2^{(2)}$, $\hat{\zeta}_1 = \hat{\zeta}_1^{(1)}$, $c_1 = c_1^{(1)}$, and $c_2 = c_2^{(2)}$ for the remainder of the paper.

CAPM theory then provides a method for calculating the unique equilibrium price, which corresponds to what we have labeled P_a , the fundamental value of the asset. As noted in Appendix A, if Group 2 has the vast majority of the assets, then the formal limit of the solution(s) of the difference equations, 2.2, 2.7, and 2.8, is the solution of the classical price equation 1.1. This suggests the price, P , is a small fluctuation about this unique equilibrium price. However, if neither group has an infinite amount of capital, then our model, as shown in Theorem 4.1, suggests that there exists a range of equilibrium prices. Thus, our model encompasses the classical theory as a subset, and one may determine from market data and optimization of parameters which assumptions are applicable at any given time.

Remark 5. It is appropriate to consider the scenario $L = P_{eq} \neq P_a$ as a distinguished limit of Case (i). Indeed, in Case (i) for given M_0 , N_0 , and P_a the variable $\hat{N}^{(1)}$ can vary, forming a one-dimensional curve of equilibrium points. However, in the limit $L \rightarrow P_{eq}$, i.e. the case $L = P_{eq} \neq P_a$, for given M_0 , N_0 , and P_a , the $\hat{\zeta}_2$ variable is uniquely determined.

Proof. Let $L, P_a \in \mathbb{R}^+$ be arbitrary.

(i) Suppose the system 3.2' - 3.4', 4.3', and 4.4' is at equilibrium, i.e. all time derivatives vanish. We solve the resulting algebraic system of equations and denote the solution, i.e. the equilibrium point, by $\hat{x} := (\hat{N}^{(1)}, \hat{M}^{(1)}, P_{eq}, \hat{\zeta}_1, \hat{\zeta}_2)$. Equation 4.3' yields:

$$\hat{\zeta}_1 = 0 \quad (4.7)$$

while equation 4.4' yields:

$$\hat{\zeta}_2 = q_2 \frac{P_a - P_{eq}}{P_a} \quad (4.8)$$

Note that $\hat{\zeta}_2$ is defined as $P_a > 0$ by assumption (A7), and $\hat{\zeta}_2 \neq 0$ as $P_{eq} \in (\min [P_a, L], \max [P_a, L])$.

Next, solve 3.2' for P_{eq} and then substitute this value along with $\hat{\zeta}_1 = 0$ into 3.3'. This yields

$$\hat{M}^{(1)} = \hat{N}^{(1)} P_{eq}.$$

Then substituting this result into 3.2' gives

$$\begin{aligned} P_{eq} &= \frac{-\hat{\zeta}_2 \hat{M}^{(1)} + (1 + \hat{\zeta}_2) M_0}{\hat{\zeta}_2 \hat{N}^{(1)} + (1 - \hat{\zeta}_2) N_0} \\ &= \frac{-\hat{\zeta}_2 \hat{N}^{(1)} P_{eq} + (1 + \hat{\zeta}_2) M_0}{\hat{\zeta}_2 \hat{N}^{(1)} + (1 - \hat{\zeta}_2) N_0}. \end{aligned}$$

Solving for $\hat{N}^{(1)}$ yields

$$\hat{N}^{(1)} = \frac{-(1 - \hat{\zeta}_2) N_0 P_{eq} + (1 + \hat{\zeta}_2) M_0}{2 P_{eq} \hat{\zeta}_2}.$$

Using equation 4.8 we obtain

$$\begin{aligned} \hat{N}^{(1)} &= \frac{-(1 - q_2 \frac{P_a - P_{eq}}{P_a}) N_0 P_{eq} + (1 + q_2 \frac{P_a - P_{eq}}{P_a}) M_0}{2 P_{eq} q_2 \frac{P_a - P_{eq}}{P_a}} \\ &= \frac{-[P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} + [P_a + q_2 (P_a - P_{eq})] M_0}{2 P_{eq} q_2 (P_a - P_{eq})}. \end{aligned} \quad (4.9)$$

Using equation 4.6 we obtain the form of an arbitrary equilibrium point

$$\hat{x} = \left(\frac{[P_a - q_2(P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2(P_a - P_{eq})] M_0}{2P_{eq}q_2(P_{eq} - P_a)}, P_{eq} \hat{N}^{(1)}, \right. \\ \left. P_{eq}, 0, q_2 \frac{P_a - P_{eq}}{P_a} \right) \quad (4.10)$$

for any $P_{eq} \in \mathbb{R}$, $P_{eq} \neq 0$ and $P_{eq} \neq P_a$. Substitution of the above point into system 3.2' - 3.4', 4.3', and 4.4' shows that the point given by equation 4.10 is indeed an equilibrium point of the system.

Although the equilibrium point given by equation 4.10 holds for all $P_{eq} \in \mathbb{R}$, $P_{eq} \neq 0$ and $P_{eq} \neq P_a$, we must restrict P_{eq} to the interval $[\min(P_a, L), \max(P_a, L)]$ in order to satisfy assumptions (A1)-(A7). Indeed, suppose the contrary, i.e. $P_{eq} > \max(P_a, L)$ and consider equation 4.9:

$$\hat{N}^{(1)} = \frac{\{P_a - q_2(P_a - P_{eq})\} N_0 P_{eq} - \{P_a + q_2(P_a - P_{eq})\} M_0}{2P_{eq}q_2(P_{eq} - P_a)} \\ = N_0 \frac{P_a(P_{eq} - L) + q_2(P_{eq} - P_a)(P_{eq} + L)}{2P_{eq}q_2(P_{eq} - P_a)}.$$

Notice that we have $\hat{N}^{(1)} \geq 0$ as needed. However, we also require $\hat{N}^{(1)} \leq N_0$ by assumption (A5). For this latter condition we require

$$0 \leq \frac{P_a(P_{eq} - L) + q_2(P_{eq} - P_a)(P_{eq} + L)}{2P_{eq}q_2(P_{eq} - P_a)} \leq 1$$

The left inequality is satisfied. Thus we verify the right inequality:

$$\frac{P_a(P_{eq} - L) + q_2(P_{eq} - P_a)(P_{eq} + L)}{2P_{eq}q_2(P_{eq} - P_a)} < 1 \Leftrightarrow \\ -1 > q_2 \left(1 - \frac{P_{eq}}{P_a}\right) = \hat{\zeta}_2.$$

Hence, we obtain a contradiction with assumption (A3) as we require $-1 < \hat{\zeta}_2 = q_2 \left(1 - \frac{P_{eq}}{P_a}\right) < 1$.

Similarly, one may show $P_{eq} < \min(P_a, L)$ is not permissible.

We have proven that we cannot have $P_{eq} > \max(P_a, L)$ or $P_{eq} < \min(P_a, L)$. Thus, $\min(P_a, L) \leq P_{eq} \leq \max(P_a, L)$. An arbitrary equilibrium point is given by equation 4.10 provided $\min(P_a, L) < P_{eq} < \max(P_a, L)$. The cases where P_{eq} is constant, $P_{eq} = P_a$ and $P_{eq} = L$, are considered in (ii) and (iii) below. These cases also exhibit a one dimensional curve of equilibrium points; however, as P_{eq} is constant, we parameterize this curve by variables $\hat{N}^{(1)}$ and $\hat{\zeta}_2$, respectively.

We next consider the limiting case $P_{eq} = L \neq P_a$ and show the existence of an equilibrium satisfying for any $\hat{\zeta}_2 \in (-1, 1) \setminus \{0, q_2\}$.

Suppose $P_{eq} = L$. Equation 3.2' with $\hat{\zeta}_1 = 0$ yields $P_{eq} = \frac{-\hat{\zeta}_2 \hat{M}^{(1)} + (1 + \hat{\zeta}_2) M_0}{\hat{\zeta}_2 \hat{N}^{(1)} + (1 - \hat{\zeta}_2) N_0} = L = \frac{M_0}{N_0}$. Cross multiplying gives:

$$\hat{\zeta}_2(2M_0N_0 - \hat{M}^{(1)}N_0 - \hat{N}^{(1)}M_0) = 0 \quad (4.11)$$

which implies (1) $\hat{\zeta}_2 = 0$ or (2) $\hat{M}^{(1)} = M_0$ and $\hat{N}^{(1)} = N_0$, i.e. group 1 has all of the system's cash and shares. Indeed, assuming $\hat{\zeta}_2 \neq 0$ and dividing both sides of

equation 4.11 by $\hat{\zeta}_2 M_0 N_0$ gives

$$2 - \frac{\hat{M}^{(1)}}{M_0} - \frac{\hat{N}^{(1)}}{N_0} = 0.$$

As $0 \leq \frac{\hat{M}^{(1)}}{M_0} \leq 1$ and $0 \leq \frac{\hat{N}^{(1)}}{N_0} \leq 1$ by assumption (A5), this holds if and only if $\hat{M}^{(1)} = M_0$ and $\hat{N}^{(1)} = N_0$.

If $\hat{\zeta}_2 = 0$, then $P_{eq} = P_a = L$. However, if $\hat{\zeta}_2 \neq 0$, then we must have $\frac{\hat{M}^{(1)}}{M_0} = \frac{\hat{N}^{(1)}}{N_0} = 1$. Thus, equilibrium may exist where $P_{eq} = L \neq P_a^{(2)}$, and an arbitrary equilibrium point has the form

$$\hat{x} = (N_0, M_0, P_a(1 - \frac{\hat{\zeta}_2}{q_2}), 0, \hat{\zeta}_2) \quad (4.12)$$

where $\hat{\zeta}_2 \in (-1, 1) \setminus \{0, q_2\}$ is arbitrary. A simple calculation shows that \hat{x} is an equilibrium point of the system 3.2' - 3.4', 4.3', and 4.4'.

Note this result may also be obtained by taking the limit as $P_{eq} \rightarrow L$ of the arbitrary equilibrium point given by 4.10.

(ii) We show that if the equilibrium price equals group 2's estimation of the fundamental value of the asset, i.e. $P_{eq} = P_a$, then it must also equal the liquidity value, i.e. $P_{eq} = P_a = L = \frac{M_0}{N_0}$.

Indeed, suppose $P_{eq} = P_a$. Then equation 4.8 yields $\hat{\zeta}_2 = 0$ which implies $P_{eq} = \frac{M_0}{N_0} = L$ by considering equation 3.2'. Thus, if $P_{eq} = P_a$, then we must have $P_{eq} = P_a = L$. By assumption (A4) P_a and thus P_{eq} is a constant. Therefore, we parameterize the curve of equilibrium points by $\hat{N}^{(1)}$. It is clear from equations 4.3' and 4.4' that $\hat{\zeta}_1 = \hat{\zeta}_2 = 0$. Using equation 4.6 an arbitrary equilibrium point is of the form

$$\hat{x} = (\hat{N}^{(1)}, \hat{N}^{(1)} P_{eq}, P_{eq}, 0, 0) \quad (4.13)$$

for any $\hat{N}^{(1)} \in \mathbb{R}$, $0 \leq \hat{N}^{(1)} \leq N_0$ and $N_0 > 0$.

As in the above case, a simple calculation shows that the point \hat{x} given by equation 4.13 is an equilibrium point of the system 3.2' - 3.4', 4.3', and 4.4'. \square

Combining assumptions (A1)-(A7) with the results from Theorem 4.1 allows us to rewrite the criteria for equilibrium in Corollary 1. As noted above the parameters c_1 , c_2 , and q_1 do not affect the existence of equilibrium points.

Corollary 1. *Consider the system 3.2' - 3.4', 4.3', and 4.4' under assumptions (A1)-(A7). Then any equilibrium point(s) must satisfy the following criteria:*

Case (1) $L < P_{eq} < P_a$

$$(a) P_a > P_{eq} > 0, (b) 0 < q_2 < \frac{P_a}{P_a - P_{eq}}, \text{ and}$$

$$(c) \frac{M_0}{P_{eq}} < N_0 \leq \frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]}.$$

In the limit $P_{eq} \rightarrow L$ we have the subcase $P_a > P_{eq} = L$.

$$(a) 0 < \hat{\zeta}_2 < 1, (b) q_2 > \hat{\zeta}_2, \text{ and } (c) L = P_a \frac{q_2 - \hat{\zeta}_2}{q_2}.$$

Case (2) $P_a < P_{eq} < L$

$$(a) 0 < P_a < P_{eq}, (b) 0 < q_2 < -\frac{P_a}{P_a - P_{eq}}, \text{ and}$$

$$(c) \frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} \leq N_0 < \frac{M_0}{P_{eq}}.$$

In the limit $P_{eq} \rightarrow L$ we have the subcase $P_a < P_{eq} = L$.

$$(a) -1 < \hat{\zeta}_2 < 0 \text{ and } (b) L = P_a \frac{q_2 - \hat{\zeta}_2}{q_2}.$$

Case (3) $P_a = P_{eq} = L$

$$(a) 0 \leq \hat{N}^{(1)} \leq N_0 \text{ with } N_0 > 0, (b) P_a = P_{eq} = L, \text{ and } (c) \hat{\zeta}_1 = \hat{\zeta}_2 = 0.$$

Proof. Case (1) $L < P_{eq} < P_a$

By assumption (A3) and the condition $L < P_{eq} < P_a$, we have $0 < \hat{\zeta}_2 < 1$. As $\hat{\zeta}_2 = q_2 \frac{P_a - P_{eq}}{P_a}$ by equation 4.10 we have

$$\frac{P_a}{P_a - P_{eq}} > q_2 > 0, \quad (4.14)$$

which is (b). Also, $P_a > P_{eq}$ and $N_0 > \frac{M_0}{P_{eq}}$ imply (a) and the left inequality in (c). Thus, it remains to show:

$$\frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} \geq N_0.$$

From assumption (A5) we have

$$0 \leq \hat{N}^{(1)} \leq N_0.$$

Using the definition of $\hat{N}^{(1)}$ from equation 4.10 yields

$$0 \leq \frac{[P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2 (P_a - P_{eq})] M_0}{2P_{eq} q_2 (P_{eq} - P_a)} \leq N_0.$$

Multiplying the inequality by $2P_{eq} q_2 (P_{eq} - P_a)$ gives

$$0 \geq [P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2 (P_a - P_{eq})] M_0 \geq 2N_0 P_{eq} q_2 (P_{eq} - P_a),$$

which is equivalent to

$$\begin{aligned} -[P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} &\geq -[P_a + q_2 (P_a - P_{eq})] M_0 \\ &\geq N_0 P_{eq} [-q_2 (P_a - P_{eq}) - P_a]. \end{aligned}$$

Dividing by $-[P_a - q_2 (P_a - P_{eq})] P_{eq}$ gives the desired result

$$N_0 \leq \frac{[P_a + q_2 (P_a - P_{eq})] M_0}{[P_a - q_2 (P_a - P_{eq})] P_{eq}} \leq \frac{N_0 [P_a + q_2 (P_a - P_{eq})]}{[P_a - q_2 (P_a - P_{eq})]}.$$

Note that from equation 4.14 we have

$$0 < P_a - q_2 (P_a - P_{eq}) < P_a.$$

This justifies reversing the inequalities when we divided by $-[P_a - q_2 (P_a - P_{eq})] P_{eq}$ above.

We next consider the limiting case $P_a > P_{eq} = L$. The criterion $P_a > P_{eq} = L$ along with assumption (A3) implies (a). Condition (c) holds as equation 4.12 gives

$L = P_{eq} = P_a(1 - \frac{\hat{\zeta}_2}{q_2})$. Condition (b) is necessary to ensure $P_{eq} > 0$. Indeed, by equation 4.12

$$P_{eq} = P_a(1 - \frac{\hat{\zeta}_2}{q_2}).$$

The inequality $q_2 \leq \hat{\zeta}_2$ implies $P_{eq} \leq 0$, which is not feasible. Thus, $q_2 > \hat{\zeta}_2$ is necessary.

Case (2) $P_a < P_{eq} < L$

By assumption (A3) and the condition $P_a < P_{eq} < L$ we have $-1 < \hat{\zeta}_2 < 0$. Again, equation 4.10 implies $\hat{\zeta}_2 = q_2 \frac{P_a - P_{eq}}{P_a}$ which leads to

$$-\frac{P_a}{P_a - P_{eq}} > q_2 > 0, \quad (4.15)$$

namely, (b). Also, $P_a < P_{eq}$ and $N_0 < \frac{M_0}{P_{eq}}$ verify (a) and the right inequality in (c). Thus, it remains to show:

$$\frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} \leq N_0.$$

By assumption (A5) we have

$$0 \leq \hat{N}^{(1)} \leq N_0.$$

Using the definition of $\hat{N}^{(1)}$ from equation 4.10 yields

$$0 \leq \frac{[P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2 (P_a - P_{eq})] M_0}{2P_{eq} q_2 (P_{eq} - P_a)} \leq N_0.$$

Multiplying the inequality by $2P_{eq} q_2 (P_{eq} - P_a)$ gives

$$0 \leq [P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2 (P_a - P_{eq})] M_0 \leq 2N_0 P_{eq} q_2 (P_{eq} - P_a),$$

which is equivalent to

$$\begin{aligned} -[P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} &\leq -[P_a + q_2 (P_a - P_{eq})] M_0 \\ &\leq N_0 P_{eq} [-q_2 (P_a - P_{eq}) - P_a]. \end{aligned}$$

Dividing by $-[P_a - q_2 (P_a - P_{eq})] P_{eq}$ gives the desired result

$$N_0 \geq \frac{[P_a + q_2 (P_a - P_{eq})] M_0}{[P_a - q_2 (P_a - P_{eq})] P_{eq}} \geq \frac{N_0 [P_a + q_2 (P_a - P_{eq})]}{[P_a - q_2 (P_a - P_{eq})]}.$$

Note that from equation 4.15 we have

$$2P_a > P_a - q_2 (P_a - P_{eq}) > P_a > 0$$

This justifies reversing the inequalities when we divided by $-[P_a - q_2 (P_a - P_{eq})] P_{eq}$ above.

We next consider the limiting case $P_a < P_{eq} = L$. It is easily seen that the criterion $P_a < P_{eq} = L$ along with (A3) implies (a). Condition (b) holds because in this case $L = P_{eq} = P_a(1 - \frac{\hat{\zeta}_2}{q_2})$ by equation 4.12.

Case (3) $P_a = P_{eq} = L$

It is clear from the arbitrary form of an equilibrium point, equation 4.13, that criteria (a)-(c) hold. \square

Now that criteria for equilibria have been established, we consider the stability of an arbitrary equilibrium point for each of the four cases identified in Corollary 1. For stability, we linearize about an arbitrary equilibrium point and consider the eigenvalues of the corresponding Jacobian matrix evaluated at the equilibrium point. As noted above, this system admits a curve of equilibria. As such, this system will always have a zero eigenvalue with corresponding eigenvector tangent to this curve of equilibria. In addition, we will see that -1 is also a root of the characteristic polynomial in all four cases. Thus, the equilibrium point is considered stable if the remaining three eigenvalues have real parts strictly less than zero and unstable if any one of these eigenvalues has a real part greater than zero.

Theorem 4.2 utilizes the Routh-Hurwitz criterion (see Appendix B) to prove the existence of stable and unstable regions of equilibrium points within the appropriate (parameter) spaces for each of the four cases. As 0 and -1 are always roots, we focus on the cubic factor of the characteristic polynomial. Routh-Hurwitz applied to a cubic polynomial ($\alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$) can be stated as: The real parts of all roots are strictly negative if and only if $\alpha_i > 0$ for $i = 0, 1, 2, 3$ and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ ([10]).

We define the following constants which will be used in the statement and proof of Theorem 4.2:

$$\begin{aligned} A &:= M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] \\ B &:= M_0 [P_{eq}^2 q_2 + P_a^2 (1 + q_2) - P_a P_{eq} (3 + 2q_2)] \\ &\quad + N_0 P_{eq} [-P_a^2 (-1 + q_2) - P_{eq}^2 q_2 + P_a (P_{eq} + 2P_{eq} q_2)] \\ C &:= M_0 [P_a + q_2 (P_a - P_{eq})] [-P_{eq} q_2 + P_a (-2q_1 + q_2)] \\ &\quad - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] [P_{eq} q_2 - P_a (2q_1 + q_2)] \\ D &:= [(1 + c_2) (P_a - P_{eq}) q_2 A + c_1 C] [c_2 q_2 B + c_1 \{(P_a - P_{eq}) q_2 A + c_2 C\}] \end{aligned}$$

Recall that $q_1 > 0$ by assumption (A7). This agrees with empirical studies on both experimental ([6]) and world market ([5]) data which show that if the trend is positive (alternatively negative) then the price continues to rise (alternatively fall) due to momentum traders.

Theorem 4.2. *Assume (A1)-(A7). (i) There exists a stable region of equilibrium points, \hat{x} , for the system 3.2' - 3.4', 4.3', and 4.4'. For each of the following cases, one has the following sufficient criteria for the stability of the equilibrium point:*

(1) $L < P_{eq} < P_a$:

$$(q_1 < \frac{1}{2}) \text{ and } (c_1 < 1 + c_2);$$

(2) $P_a < P_{eq} < L$:

$$(q_1 < \frac{1}{2}) \text{ and } (c_1 < 1 + c_2);$$

(3) $P_a = P_{eq} = L$:

$$q_1^{(1)} < \min \left[\frac{1}{2} \frac{N_0}{\hat{N}(1)}, \frac{1 + c_2}{2c_1} \frac{N_0}{\hat{N}(1)} \right].$$

(ii) *There exists an unstable region of equilibrium points. In each of the three cases one has the following sufficient criteria for instability:*

(1) $L < P_{eq} < P_a$:

$$\frac{-M_0 + N_0 P_{eq}}{M_0 + N_0 P_{eq}} \frac{P_a}{P_a - P_{eq}} < q_2 < \frac{P_a}{P_a - P_{eq}} \text{ and}$$

$$q_1 > \frac{A}{-M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]} \frac{1}{2} \frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}} \text{ and}$$

$$c_1 > \frac{-(1 + c_2) (P_a - P_{eq}) q_2 A}{C};$$

(2) $P_a < P_{eq} < L$:

$$\frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}} \frac{-P_a}{P_a - P_{eq}} < q_2 < \frac{-P_a}{P_a - P_{eq}} \text{ and}$$

$$q_1 > \frac{A}{-M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]} \frac{1}{2} \frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}} \text{ and}$$

$$c_1 > \frac{-(1 + c_2) (P_a - P_{eq}) q_2 A}{C};$$

(3) $P_a = P_{eq} = L$:

$$q_1^{(1)} > \frac{(1 + c_1 + c_2) N_0}{2c_1 \hat{N}^{(1)}}.$$

In addition, for $L = P_{eq} \neq P_a$, a distinguished limit of cases (1) and (2), there exists a stable region of equilibrium points for

$$q_1 < \frac{1 + c_1}{2c_1}$$

and an unstable region of equilibrium points for

$$q_1 > \frac{1 + c_1}{2c_1}.$$

Remark 6. The conditions for stability are consistent with the intuition of the model, as a small trend coefficient, q_1 , and a large time scale for trend, $1/c_1$, favor stability (see case (1), (2) of (i)). The stable condition for the distinguished limit case ($L = P_{eq} \neq P_a$) shows that one can have stability for large values of q_1 provided the trend time scale is large. Furthermore, in case (3) stability can also be maintained for larger values of q_1 if the number of shares of the trend-based group is small compared with the total number of shares in the system.

Proof. Let $\hat{x} = (\hat{N}^{(1)}, \hat{M}^{(1)}, P_{eq}, \hat{\zeta}_1, \hat{\zeta}_2)$ be an arbitrary equilibrium point.

Case (1) $L < P_{eq} < P_a$

(i) We first prove existence of a stable region of equilibrium points. Linearizing about

$$\hat{x} = \left(\frac{[P_a^{(2)} - q_2(P_a^{(2)} - P_{eq})]N_0 P_{eq} - [P_a^{(2)} + q_2(P_a^{(2)} - P_{eq})]M_0}{2P_{eq}q_2(P_{eq} - P_a^{(2)})}, P_{eq}, \hat{N}^{(1)}, \right.$$

$$\left. P_{eq}, 0, q_2 \frac{P_a^{(2)} - P_{eq}}{P_a^{(2)}} \right)$$

yields the following characteristic polynomial

$$\lambda(\lambda + 1) (\alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0) \quad (4.16)$$

where

$$\begin{aligned}\alpha_3 &= 1 \\ \alpha_2 &= \frac{(1+c_2)(P_a - P_{eq})q_2A + c_1C}{(P_a - P_{eq})q_2A} \\ \alpha_1 &= \frac{c_2q_2B + c_1[(P_a - P_{eq})q_2A + c_2C]}{(P_a - P_{eq})q_2A} \\ \alpha_0 &= \frac{c_1c_2B}{(P_a - P_{eq})A}.\end{aligned}$$

The zero eigenvalue is due to the curve of equilibria. As such, it does not impact the stability. Thus, \hat{x} is stable if the roots of the cubic factor of the characteristic polynomial have strictly negative real parts. By the Routh-Hurwitz criterion these roots lie in the left half of the complex plane if and only if $\alpha_i > 0$ for $i = 0, 1, 2, 3$ and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$. Note that

$$\begin{aligned}\alpha_2\alpha_1 - \alpha_3\alpha_0 &= \frac{-c_1c_2(P_a - P_{eq})q_2^2AB}{(P_a - P_{eq})^2q_2^2A^2} \\ &+ \frac{[(1+c_2)(P_a - P_{eq})q_2A + c_1C][c_2q_2B + c_1\{(P_a - P_{eq})q_2A + c_2C\}]}{(P_a - P_{eq})^2q_2^2A^2}.\end{aligned}$$

Any equilibrium point must satisfy the criteria given by Corollary 1, which for this case is

$$\begin{aligned}\text{(a) } P_a &> P_{eq} > 0, \text{ (b) } 0 < q_2 < \frac{P_a}{P_a - P_{eq}}, \\ \text{and (c) } \frac{M_0}{P_{eq}} &< N_0 \leq \frac{M_0[P_a + q_2(P_a - P_{eq})]}{P_{eq}[P_a - q_2(P_a - P_{eq})]}.\end{aligned}$$

Using these conditions in conjunction with $L < P_{eq} < P_a$ we show that $A, B,$ and C are positive, which, in turn, implies the existence of a region in which $\alpha_i > 0$ for $i = 1, 2, 3, 4$ and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ (i.e. \hat{x} is stable). In fact, $A > 0$ by (a) and (b). Next, show B is positive. Indeed,

$$\begin{aligned}B &= M_0[P_{eq}^2q_2 + P_a^2(1 + q_2) - P_aP_{eq}(3 + 2q_2)] \\ &+ N_0P_{eq}[-P_a^2(-1 + q_2) - P_{eq}^2q_2 + P_a(P_{eq} + 2P_eq_2)] \\ &= M_0[q_2(P_a - P_{eq})^2 + P_a(P_a - 3P_{eq})] + N_0P_{eq}[-q_2(P_a - P_{eq})^2 + P_a^2 + P_aP_{eq}] \\ &> M_0[q_2(P_a - P_{eq})^2 + P_a(P_a - 3P_{eq})] \\ &+ N_0P_{eq}\left[-\frac{P_a}{P_a - P_{eq}}(P_a - P_{eq})^2 + P_a^2 + P_aP_{eq}\right] \text{ by (b)} \\ &= M_0[q_2(P_a - P_{eq})^2 + P_a(P_a - 3P_{eq})] + 2N_0P_{eq}P_aP_{eq} \\ &> M_0[P_a(P_a - 3P_{eq})] + 2N_0P_{eq}P_aP_{eq} = M_0[P_a^2 - 3P_aP_{eq}] + 2N_0P_{eq}P_aP_{eq} \\ &> M_0[P_a^2 - 3P_aP_{eq}] + 2M_0P_aP_{eq} \text{ by (c)} \\ &= M_0[P_a^2 - P_aP_{eq}] > 0 \text{ by (a)}.\end{aligned}$$

We claim there exists a region in parameter space where C is also greater than zero. Based upon the current case and assumptions we have $M_0[P_a + q_2(P_a - P_{eq})] > 0$ and $N_0P_{eq}[P_a - q_2(P_a - P_{eq})] > 0$, while $[P_{eq}q_2 - P_a(2q_1 + q_2)] < 0$. Thus, if

$[-P_{eq}q_2 + P_a(-2q_1 + q_2)] \geq 0$ (i.e. $q_1 \leq \frac{q_2(P_a - P_{eq})}{2P_a}$), then $C > 0$. Alternatively, suppose $[-P_{eq}q_2 + P_a(-2q_1 + q_2)] \leq 0$ (i.e. $q_1 \geq \frac{q_2(P_a - P_{eq})}{2P_a}$). Then $C > 0$ provided $\frac{M_0}{P_{eq}} < N_0$ and $q_1 < \frac{1}{2}$. Indeed, one has

$$\begin{aligned} C &= M_0 [P_a + q_2 (P_a - P_{eq})] [-P_{eq}q_2 + P_a (-2q_1 + q_2)] \\ &\quad - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] [P_{eq}q_2 - P_a (2q_1 + q_2)] \\ &> M_0 [P_a + q_2 (P_a - P_{eq})] [-P_{eq}q_2 + P_a (-2q_1 + q_2)] \\ &\quad - M_0 [P_a - q_2 (P_a - P_{eq})] [P_{eq}q_2 - P_a (2q_1 + q_2)] \\ &= M_0 [P_a + q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) - 2P_a q_1] \\ &\quad + M_0 [P_a - q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) + 2P_a q_1] \\ &= 2M_0 P_a q_2 (P_a - P_{eq}) - 4M_0 q_2 q_1 P_a (P_a - P_{eq}) \\ &= 2M_0 P_a q_2 (P_a - P_{eq}) (1 - 2q_1) \\ &> 0 \text{ provided } 0 < q_1 < \frac{1}{2}. \end{aligned}$$

So, if $[-P_{eq}q_2 + P_a(-2q_1 + q_2)] \leq 0$, then $\frac{q_2(P_a - P_{eq})}{2P_a} \leq q_1 < \frac{1}{2}$ ensures $C > 0$.

So, $A > 0$ and $B > 0$ for this case and the above assumptions. One also has $C > 0$ provided

$$q_1 < \frac{1}{2}.$$

Furthermore, A and B positive implies $\alpha_0 > 0$; A and C positive implies $\alpha_2 > 0$; and A , B , and C positive give α_1 positive. Finally, consider $\alpha_2\alpha_1 - \alpha_3\alpha_0$. With A , B , and C positive, the terms

$$\begin{aligned} &[(1 + c_2) (P_a - P_{eq}) q_2 A + c_1 C] \\ &\text{and } [c_2 q_2 B + c_1 \{(P_a - P_{eq}) q_2 A + c_2 C\}] \end{aligned}$$

are positive. Thus, $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ if

$$\begin{aligned} &|-c_1 c_2 (P_a - P_{eq}) q_2^2 AB| < D \Leftrightarrow \\ &c_1 c_2 (P_a - P_{eq}) q_2^2 AB < D \end{aligned}$$

where

$$D = [(1 + c_2) (P_a - P_{eq}) q_2 A + c_1 C] [c_2 q_2 B + c_1 \{(P_a - P_{eq}) q_2 A + c_2 C\}].$$

The product of the first terms in each factor D is

$$(1 + c_2) (P_a - P_{eq}) c_2 q_2^2 AB.$$

Thus, this inequality holds and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ provided $c_1 < 1 + c_2$.

Therefore, if $q_1 < \frac{1}{2}$ and $(c_1 < 1 + c_2)$, then the equilibrium is stable.

(ii) Using the Routh-Hurwitz criterion we show that an unstable region exists by showing $\alpha_2 < 0$. Indeed, one has

$$\alpha_2 = \frac{(1 + c_2) (P_a - P_{eq}) q_2 A + c_1 C}{(P_a - P_{eq}) q_2 A}$$

where

$$\begin{aligned} A &:= M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] \\ C &:= M_0 [P_a + q_2 (P_a - P_{eq})] [-P_{eq}q_2 + P_a (-2q_1 + q_2)] \\ &\quad - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] [P_{eq}q_2 - P_a (2q_1 + q_2)]. \end{aligned}$$

A is positive. This implies that both the denominator and the first term in the numerator are positive. Thus C must be negative.

$$C < 0 \Leftrightarrow$$

$$\frac{(P_a - P_{eq}) q_2 A}{2P_a \{M_0 [P_a + q_2 (P_a - P_{eq})] - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]\}} < q_1$$

provided

$$2P_a \{M_0 [P_a + q_2 (P_a - P_{eq})] - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]\} > 0 \quad (4.17)$$

which holds for

$$\frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} > N_0. \quad (4.18)$$

Recall equilibrium conditions (b) and (c) for this case

$$(b) \ 0 < q_2 < \frac{P_a}{P_a - P_{eq}} \text{ and (c) } \frac{M_0}{P_{eq}} < N_0 \leq \frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]}.$$

Note equation 4.18 is the strict version of the right inequality in (c). For equation 4.18 and subsequently 4.17 to hold we require $q_2 \neq \frac{(-M_0 + N_0 P_{eq})}{(M_0 + N_0 P_{eq})} \frac{P_a}{P_a - P_{eq}}$. Thus, using condition (b), for $C < 0$ we restrict q_2 to the interval

$$\frac{-M_0 + N_0 P_{eq}}{M_0 + N_0 P_{eq}} \frac{P_a}{P_a - P_{eq}} < q_2 < \frac{P_a}{P_a - P_{eq}}$$

and require

$$\begin{aligned} q_1 &> \frac{(P_a - P_{eq}) q_2 A}{2P_a \{M_0 [P_a + q_2 (P_a - P_{eq})] - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]\}} \\ &> \frac{A}{-M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]} \frac{1}{2} \frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}}. \end{aligned}$$

Choosing $c_1 > \frac{-(1+c_2)(P_a - P_{eq})q_2 A}{C}$ gives $\alpha_2 < 0$. Since $\alpha_3 = 1 > 0$ there is at least one sign change in the first column of the Routh array. By the Routh-Hurwitz criterion there is at least one eigenvalue with positive real part. Thus, the equilibrium is unstable.

Case (2) $P_a < P_{eq} < L$

(i) We prove existence of a stable region of equilibrium points. The critical point, \hat{x} , and characteristic polynomial both have the same form as in Case (1). Similar to the proof of Case (1), we utilize the equilibrium conditions given by Corollary 1 to show that $A > 0$, $B < 0$, and $C < 0$. These criteria are:

$$\begin{aligned} (a) \ 0 < P_a < P_{eq}, \quad (b) \ 0 < q_2 < -\frac{P_a}{P_a - P_{eq}}, \\ \text{and (c) } \frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} \leq N_0 < \frac{M_0}{P_{eq}}. \end{aligned}$$

Note that $\frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} > 0$. Indeed, $[P_a - q_2 (P_a - P_{eq})] > 0$ by (a). In addition, $[P_a + q_2 (P_a - P_{eq})] > 0$ by (b).

Using these equilibrium conditions and assumption (A7) we show that $\alpha_i > 0$ for $i = 0, 1, 2, 3$ and $\alpha_2 \alpha_1 - \alpha_3 \alpha_0 > 0$ (i.e. \hat{x} is stable). Indeed, $A > 0$ by (a) and

(b). Next, we show B is negative. Indeed,

$$\begin{aligned}
B &= M_0 [P_{eq}^2 q_2 + P_a^2 (1 + q_2) - P_a P_{eq} (3 + 2q_2)] \\
&\quad + N_0 P_{eq} [-P_a^2 (-1 + q_2) - P_{eq}^2 q_2 + P_a (P_{eq} + 2P_e q_2)] \\
&= M_0 [q_2 (P_a - P_{eq})^2 + P_a^2 - 3P_a P_{eq}] + N_0 P_{eq} [-q_2 (P_a - P_{eq})^2 + P_a^2 + P_a P_{eq}] \\
&< M_0 [-\frac{P_a}{P_a - P_{eq}} (P_a - P_{eq})^2 + P_a^2 - 3P_a P_{eq}] + N_0 P_{eq} [0 + P_a^2 + P_a P_{eq}] \text{ by (b)} \\
&= M_0 [-P_a (P_a - P_{eq}) + P_a^2 - 3P_a P_{eq}] + N_0 P_{eq} [P_a^2 + P_a P_{eq}] \\
&< M_0 [-2P_a P_{eq}] + M_0 [P_a^2 + P_a P_{eq}] \text{ by (c)} \\
&= M_0 [-P_a P_{eq} + P_a^2] \\
&< 0 \text{ by (a)}.
\end{aligned}$$

We will also show that there exists a region in parameter space where C is also negative. Based upon (a) and (b) and assumption (A7) we have $M_0 [P_a + q_2 (P_a - P_{eq})] > 0$ and $N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] > 0$, while $[-P_{eq} q_2 + P_a (-2q_1 + q_2)] < 0$. Thus, if $[P_{eq} q_2 - P_a (2q_1 + q_2)] \geq 0$ (i.e. $q_1 \leq \frac{-q_2(P_a - P_{eq})}{2P_a}$), then $C < 0$. Alternatively, suppose $[P_{eq} q_2 - P_a (2q_1 + q_2)] \leq 0$ (i.e. $q_1 \geq \frac{-q_2(P_a - P_{eq})}{2P_a}$). Then $C < 0$ provided $N_0 < \frac{M_0}{P_{eq}}$ and $q_1 < \frac{1}{2}$. Indeed,

$$\begin{aligned}
C &= M_0 [P_a + q_2 (P_a - P_{eq})] [-P_{eq} q_2 + P_a (-2q_1 + q_2)] \\
&\quad - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] [P_{eq} q_2 - P_a (2q_1 + q_2)] \\
&= M_0 [P_a + q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) - 2q_1 P_a] \\
&\quad + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) + 2q_1 P_a] \\
&< M_0 [P_a + q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) - 2q_1 P_a] \\
&\quad + M_0 [P_a - q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) + 2q_1 P_a] \text{ by (c)} \\
&= 2M_0 P_a q_2 (P_a - P_{eq}) - 4M_0 P_a q_2 (P_a - P_{eq}) q_1 \\
&= 2M_0 P_a q_2 (P_a - P_{eq}) [1 - 2q_1] \\
&< 0 \text{ provided } 0 < q_1 < \frac{1}{2}.
\end{aligned}$$

So, if $[P_{eq} q_2 - P_a (2q_1 + q_2)] \leq 0$, then $\frac{-q_2(P_a - P_{eq})}{2P_a} \leq q_1 < \frac{1}{2}$ ensures $C < 0$.

So, $A > 0$ and $B < 0$ for this case and the above assumptions. $C < 0$ provided

$$q_1 < \frac{1}{2}.$$

$A > 0$ and $B < 0$ imply $\alpha_0 > 0$; $A > 0$ and $C < 0$ imply $\alpha_2 > 0$; and $A > 0$, $B < 0$, and $C < 0$ give α_1 positive. Finally, consider $\alpha_2 \alpha_1 - \alpha_3 \alpha_0$. With $A > 0$, $B < 0$, and $C < 0$ the terms $[(1 + c_2) (P_a - P_{eq}) q_2 A + c_1 C]$ and $[c_2 q_2 B + c_1 \{(P_a - P_{eq}) q_2 A + c_2 C\}]$ are negative. Thus, $\alpha_2 \alpha_1 - \alpha_3 \alpha_0 > 0$ if

$$\begin{aligned}
&|-c_1 c_2 (P_a - P_{eq}) q_2^2 AB| < D \Leftrightarrow \\
&c_1 c_2 (P_a - P_{eq}) q_2^2 AB < D \\
&= (1 + c_2) (P_a - P_{eq}) c_2 q_2^2 AB \\
&\quad + (1 + c_2) (P_a - P_{eq}) c_1 q_2 A \{(P_a - P_{eq}) q_2 A + c_2 C\} \\
&\quad + c_1 c_2 q_2 BC + c_1^2 C \{(P_a - P_{eq}) q_2 A + c_2 C\}.
\end{aligned}$$

where

$$D = [(1 + c_2)(P_a - P_{eq})q_2A + c_1C][c_2q_2B + c_1\{(P_a - P_{eq})q_2A + c_2C\}].$$

Notice that each term on the right side of this inequality is positive for this case. Further, the first term on the right side of the inequality is greater than the term on the left side of the inequality provided $c_1 < 1 + c_2$. Thus, if $c_1 < 1 + c_2$, then $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$.

Therefore, if $q_1 < \frac{1}{2}$ and $c_1 < 1 + c_2$, then the equilibrium is stable.

(ii) Using the Routh-Hurwitz criterion we show that an unstable region exists by showing $\alpha_2 < 0$. Indeed,

$$\alpha_2 = \frac{(1 + c_2)(P_a - P_{eq})q_2A + c_1C}{(P_a - P_{eq})q_2A}$$

where

$$\begin{aligned} A &:= M_0[P_a + q_2(P_a - P_{eq})] + N_0P_{eq}[P_a - q_2(P_a - P_{eq})] \\ C &:= M_0[P_a + q_2(P_a - P_{eq})][-P_{eq}q_2 + P_a(-2q_1 + q_2)] \\ &\quad - N_0P_{eq}[P_a - q_2(P_a - P_{eq})][P_{eq}q_2 - P_a(2q_1 + q_2)]. \end{aligned}$$

A is positive. This implies that both the denominator and the first term in the numerator are negative. Thus, we need C to be positive. Consider

$$C > 0 \Leftrightarrow$$

$$q_1 > \frac{(P_a - P_{eq})q_2A}{2P_a\{M_0[P_a + q_2(P_a - P_{eq})] - N_0P_{eq}[P_a - q_2(P_a - P_{eq})]\}}$$

provided

$$2P_a\{M_0[P_a + q_2(P_a - P_{eq})] - N_0P_{eq}[P_a - q_2(P_a - P_{eq})]\} > 0 \quad (4.19)$$

which holds for

$$\frac{M_0[P_a + q_2(P_a - P_{eq})]}{P_{eq}[P_a - q_2(P_a - P_{eq})]} < N_0. \quad (4.20)$$

Recall equilibrium conditions (b) and (c) for this case

$$(b) \ 0 < q_2 < -\frac{P_a}{P_a - P_{eq}} \quad \text{and} \quad (c) \ \frac{M_0[P_a + q_2(P_a - P_{eq})]}{P_{eq}[P_a - q_2(P_a - P_{eq})]} \leq N_0 < \frac{M_0}{P_{eq}}.$$

Note equation 4.20 is the strict version of the left inequality in (c). For equation 4.20 and subsequently 4.19 to hold we require $q_2 \neq \frac{M_0 - N_0P_{eq}}{M_0 + N_0P_{eq}} \frac{-P_a}{P_a - P_{eq}}$. Thus, using condition (b), for $C > 0$ we restrict q_2 to the interval

$$\frac{M_0 - N_0P_{eq}}{M_0 + N_0P_{eq}} \frac{-P_a}{P_a - P_{eq}} < q_2 < \frac{-P_a}{P_a - P_{eq}}$$

and require

$$\begin{aligned} q_1 &> \frac{(P_a - P_{eq})q_2A}{2P_a\{M_0[P_a + q_2(P_a - P_{eq})] - N_0P_{eq}[P_a - q_2(P_a - P_{eq})]\}} \\ &> \frac{A}{-M_0[P_a + q_2(P_a - P_{eq})] + N_0P_{eq}[P_a - q_2(P_a - P_{eq})]} \frac{1}{2} \frac{M_0 - N_0P_{eq}}{M_0 + N_0P_{eq}}. \end{aligned}$$

Choosing $c_1 > \frac{-(1+c_2)(P_a - P_{eq})q_2A}{C}$ gives $\alpha_2 < 0$. Since $\alpha_3 = 1 > 0$ there is at least one sign change in the first column of the Routh array. By the Routh-Hurwitz criterion there is at least one eigenvalue with positive real part. Thus, the equilibrium is unstable.

Case (3) $L = P_{eq} = P_a$

(i) Linearizing about $\hat{x} = (\hat{N}^{(1)}, \hat{N}^{(1)}P_{eq}, P_{eq}, 0, 0)$ yields the following characteristic polynomial

$$-\lambda(1+\lambda) \left\{ \begin{array}{l} \lambda^3 + \lambda^2[(1+c_1+c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1] \\ + \lambda[c_1 + c_2 + c_1 c_2(1 - 2\frac{\hat{N}^{(1)}}{N_0} q_1) + 2c_2 q_2(1 - \frac{\hat{N}^{(1)}}{N_0})] \\ + c_1 c_2[1 + 2q_2(1 - \frac{\hat{N}^{(1)}}{N_0})] \end{array} \right\} \quad (4.21)$$

where

$$\begin{aligned} \alpha_3 &= 1 \\ \alpha_2 &= (1 + c_1 + c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 \\ \alpha_1 &= c_1 + c_2 + c_1 c_2(1 - 2\frac{\hat{N}^{(1)}}{N_0} q_1) + 2c_2 q_2(1 - \frac{\hat{N}^{(1)}}{N_0}) \\ \alpha_0 &= c_1 c_2[1 + 2q_2(1 - \frac{\hat{N}^{(1)}}{N_0})] \end{aligned}$$

The zero eigenvalue is due to the curve of equilibria. As such, it does not impact the stability. Thus, \hat{x} is stable if the roots of the cubic factor of the characteristic polynomial have strictly negative real parts. By the Routh-Hurwitz criterion these roots lie in the left half of the complex plane if and only if $\alpha_i > 0$ for $i = 0, 1, 2, 3$ and $\alpha_2 \alpha_1 - \alpha_3 \alpha_0 > 0$. Note that $0 \leq \frac{\hat{N}^{(1)}}{N_0} \leq 1$ by assumption (A5).

Consider $\alpha_2 > 0$, i.e.

$$\begin{aligned} (1 + c_1 + c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 &> 0 \Leftrightarrow \\ q_1 &< \frac{(1 + c_1 + c_2) N_0}{2c_1 \hat{N}^{(1)}}. \end{aligned}$$

Next, determine a region where $\alpha_1 > 0$. Indeed, if $q_1 < \frac{1}{2} \frac{N_0}{\hat{N}^{(1)}}$, then

$$\begin{aligned} \alpha_1 &= c_1 + c_2 + c_1 c_2(1 - 2\frac{\hat{N}^{(1)}}{N_0} q_1) + 2c_2 q_2(1 - \frac{\hat{N}^{(1)}}{N_0}) \\ &> 0 \text{ (Note this is independent of } q_2\text{.)} \end{aligned}$$

Note that $\alpha_0 > 0$ by assumptions (A5) and (A7). Finally, consider $\alpha_2\alpha_1 - \alpha_3\alpha_0$.

$$\begin{aligned}
& \alpha_2\alpha_1 - \alpha_3\alpha_0 \\
&= \left[(1 + c_1 + c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 \right] \left[c_1 + c_2 + c_1 c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) + 2c_2 q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right] \\
&\quad - c_1 c_2 \left[1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right] \\
&= \left[(1 + c_1 + c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 \right] \left[c_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} + c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} \right] \\
&\quad - c_1 c_2 \left[1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right] \\
&= (1 + c_1 + c_2) c_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} + (1 + c_1 + c_2) c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} \\
&\quad - 2c_1^2 \frac{\hat{N}^{(1)}}{N_0} q_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} \\
&\quad - c_1 c_2 \left[1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right]
\end{aligned}$$

If $q_1 < \frac{(1+c_1+c_2)N_0}{2c_1\hat{N}^{(1)}}$, which ensures $\alpha_2 > 0$, and $q_1 < \frac{1}{2} \frac{N_0}{\hat{N}^{(1)}}$, which ensures $\alpha_1 > 0$, then

$$(1 + c_1 + c_2) c_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} - 2c_1^2 \frac{\hat{N}^{(1)}}{N_0} q_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} > 0.$$

Thus, we obtain

$$\begin{aligned}
& (1 + c_1 + c_2) c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} \\
&\quad - c_1 c_2 \left[1 + 2q_2 - 2 \frac{\hat{N}^{(1)}}{N_0} q_2 \right] \\
&= \left[(1 + c_2) c_2 - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 c_2 \right] \left[1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right] \\
&> 0
\end{aligned}$$

provided

$$\begin{aligned}
(1 + c_2) c_2 - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 c_2 > 0 &\Leftrightarrow \\
q_1 < \frac{1 + c_2}{2c_1} \frac{N_0}{\hat{N}^{(1)}}.
\end{aligned}$$

Thus, the equilibrium point \hat{x} is stable if

$$0 < q_1 < \min \left[\frac{(1 + c_1 + c_2) N_0}{2c_1 \hat{N}^{(1)}}, \frac{1}{2} \frac{N_0}{\hat{N}^{(1)}}, \frac{1 + c_2}{2c_1} \frac{N_0}{\hat{N}^{(1)}} \right],$$

i.e.

$$0 < q_1 < \min \left[\frac{1}{2} \frac{N_0}{\hat{N}^{(1)}}, \frac{1 + c_2}{2c_1} \frac{N_0}{\hat{N}^{(1)}} \right].$$

as $c_1 > 0$ by assumption (A7).

(ii) With respect to instability, if $q_1 > \frac{(1+c_1+c_2)N_0}{2c_1\hat{N}^{(1)}}$, then $\alpha_2 < 0$. Since $\alpha_3 = 1 > 0$ there is at least one sign change in the first column of the Routh array.

By the Routh-Hurwitz criterion there is at least one eigenvalue with positive real part. Thus, the equilibrium is unstable.

Finally, we consider the limiting case of cases (1) and (2): $L = P_{eq} \neq P_a$. Linearizing about

$$\hat{x} = (N_0, M_0, P_a(1 - \frac{\hat{\zeta}_2}{q_2}), 0, \hat{\zeta}_2)$$

yields the following characteristic polynomial

$$-\lambda(1+\lambda)(c_2+\lambda)[\lambda^2 + \{1+c_1(1-2q_1)\}\lambda + c_1]. \quad (4.22)$$

The zero eigenvalue is due to the curve of equilibria. As such, it does not impact the stability. Thus, \hat{x} is stable if the roots of the quadratic factor of the characteristic polynomial have strictly negative real parts. Note that stability only depends on the q_1 and c_1 parameters.

The quadratic formula yields

$$\lambda_{+,-} = \frac{-\{1+c_1(1-2q_1)\} \pm \sqrt{\{1+c_1(1-2q_1)\}^2 - 4c_1}}{2}.$$

Note that

$$\sqrt{\{1+c_1(1-2q_1)\}^2 - 4c_1} < \sqrt{\{1+c_1(1-2q_1)\}^2} = |-\{1+c_1(1-2q_1)\}|.$$

As such, there are two cases to consider.

Case 1. $\{1+c_1(1-2q_1)\}^2 - 4c_1 > 0$ and $-\{1+c_1(1-2q_1)\} < 0$. If these criteria are met then we have $\lambda_{+,-} \in \mathbb{R}$ and $\lambda_{+,-} < 0$. These criteria are met when $q_1 < \frac{-2\sqrt{c_1+1+c_1}}{2c_1}$. Indeed, consider

$$\begin{aligned} \{1+c_1(1-2q_1)\}^2 - 4c_1 > 0 &\Leftrightarrow \\ \{1+c_1(1-2q_1)\}^2 > 4c_1 &\Leftrightarrow \\ q_1 < \frac{-2\sqrt{c_1+1+c_1}}{2c_1}. \end{aligned}$$

Note that $-2\sqrt{c_1+1+c_1} > 0 \Leftrightarrow 1+c_1 > 2\sqrt{c_1} \Leftrightarrow (1+c_1)^2 > 4c_1 \Leftrightarrow (1-c_1)^2 > 0 \Leftrightarrow c_1 \neq 1$. Next, consider

$$\begin{aligned} -\{1+c_1(1-2q_1)\} < 0 &\Leftrightarrow \\ q_1 < \frac{1+c_1}{2c_1}. \end{aligned}$$

So, for the above criteria to be met we need $q_1 < \min\left[\frac{-2\sqrt{c_1+1+c_1}}{2c_1}, \frac{1+c_1}{2c_1}\right] = \frac{-2\sqrt{c_1+1+c_1}}{2c_1}$ with $c_1 \neq 1$.

Case 2. $\{1+c_1(1-2q_1)\}^2 - 4c_1 \leq 0$ and $-\{1+c_1(1-2q_1)\} < 0$. If these criteria are met then we have $\text{Im}(\lambda_{+,-}) \geq 0$. The second criteria ensures $\text{Re}(\lambda_{+,-}) < 0$. These criteria are met when $\frac{-2\sqrt{c_1+1+c_1}}{2c_1} < q_1 < \frac{1+c_1}{2c_1}$. Indeed, consider

$$\begin{aligned} \{1+c_1(1-2q_1)\}^2 - 4c_1 \leq 0 &\Leftrightarrow \\ q_1 \geq \frac{-2\sqrt{c_1+1+c_1}}{2c_1} \end{aligned}$$

And the second criterion gives

$$\begin{aligned}\Re(\lambda_{+,-}) &< 0 \Leftrightarrow \\ -\{1 + c_1(1 - 2q_1)\} &< 0 \Leftrightarrow \\ q_1 &< \frac{1 + c_1}{2c_1}.\end{aligned}$$

Thus, $\Re(\lambda_{+,-}) < 0$ provided $\frac{-2\sqrt{c_1+1+c_1}}{2c_1} \leq q_1 < \frac{1+c_1}{2c_1}$.

Note that since

$$\begin{aligned}\sqrt{\{1 + c_1(1 - 2q_1)\}^2 - 4c_1} &< \sqrt{\{1 + c_1(1 - 2q_1)\}^2} \\ &= |-\{1 + c_1(1 - 2q_1)\}|,\end{aligned}$$

$\Re(\lambda_{+,-}) < 0$ if $0 < q_1 < \frac{1+c_1}{2c_1}$. Further, this also indicates that if $q_1 > \frac{1+c_1}{2c_1}$, then the equilibrium is unstable. \square

5. Equilibrium and Stability for Specific Parameter Values. The system 3.2' - 3.4', 4.3', and 4.4' has eight parameters ($\tau, c_1, c_2, q_1, q_2, M_0, N_0, P_a$) which may assume any real and positive value. Now that the existence of stable and unstable regions has been established, we compute specific regions of stability and instability. To this end we set some of these parameters to specific values in order to facilitate the analysis of equilibrium and stability. We set $c_1 = \frac{1}{10}$ and $c_2 = 1$ indicating that Group 1 focuses on the price trend over the past two weeks, while Group 2 considers the current day's deviation from its assessment of the asset's fundamental value. The values for M_0 and N_0 are set at 30,000 and 3,000 so that $L = 10$. P_a is set to 12, 8, or 10 depending upon the case. These are typical values for L and P_a in that the fundamental value of the asset is set to 120% or 80% of the liquidity value. Note that for L, P_a , and P it is their relative magnitudes rather than absolute values that are essential.

We consider the effect of the q_2 parameter on the existence of equilibrium points, and the effects of the q_1 and q_2 parameters on the stability of these equilibrium points. It is clear from equations 4.3' and 4.4' that the existence of equilibria is independent of the values for the q_1, c_1 , and c_2 parameters. Given Theorem 4.1, we determine the conditions for the existence and stability of equilibria for the following four cases: (1) $L < P_{eq} < P_a$, (2) $P_a < P_{eq} < L$, (3) $P_{eq} = P_a = L$, and (4) $L = P_{eq} \neq P_a$, a distinguished limit of cases (1) and (2). Note that assumptions (A1)-(A7) are still applicable to all results in this section.

Case 1. $10 = L < P_{eq} < P_a = 12$

Equilibrium. Let $P_{eq} \in (10, 12)$ be arbitrary. Then equation 4.10 gives the form of an equilibrium point dependent upon the value of q_2 . All P_{eq} between L and P_a yield an equilibrium point of the system 3.2' - 3.4', 4.3', and 4.4'; however, only some (P_{eq}, q_2) pairs yield an equilibrium satisfying assumptions (A1)-(A7). The region of (P_{eq}, q_2) pairs that yield permissible equilibrium points is derived from Corollary 1 and given by

$$L < P_{eq} < P_a \text{ and } \frac{(P_{eq} - L)P_a}{(L + P_{eq})(P_a - P_{eq})} \leq q_2 < \frac{P_a}{P_a - P_{eq}}.$$

For the specific parameter values given above the region is described by

$$10 < P_{eq} < 12 \text{ and } \frac{(P_{eq} - 10)12}{(10 + P_{eq})(12 - P_{eq})} \leq q_2 < \frac{12}{12 - P_{eq}}$$

and shown in Figure 1.

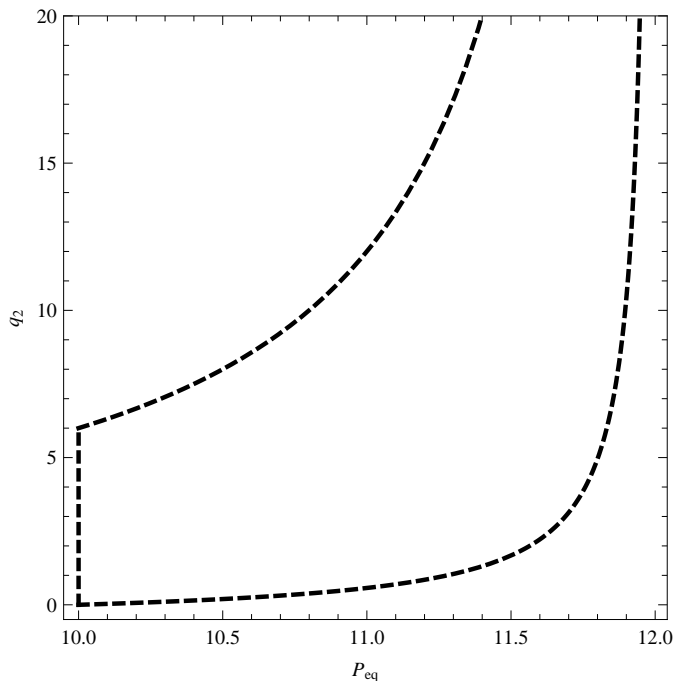


FIGURE 1. Region of (P_{eq}, q_2) -pairs that yield permissible, i.e. satisfy assumptions (A1)-(A7), equilibrium points ($10 = L < P_a = 12$).

The shape of this region is analogous to the Caginalp and Balenovich [4] finding for the single group model. Specifically, a large value for q_2 implies P_{eq} is close to P_a , while a small q_2 means that P_{eq} must be close to L . Indeed, suppose Group 2 is strongly influenced by the price’s deviation from P_a . If the price is above (below) this value, then Group 2 will sell (buy) the asset; thereby, lowering (raising) the price. Thus, it is reasonable to expect the equilibrium price, P_{eq} , to be close to P_a .

Stability. Linearization about an arbitrary equilibrium point yields a Jacobian matrix, which is included in Appendix C. The characteristic polynomial can be factored into the form:

$$\lambda(\lambda + 1)(\text{cubic polynomial in } \lambda).$$

Thus, as noted above, two of the jacobian’s five eigenvalues are 0 and -1 . We are interested in the signs of the real parts of the remaining three eigenvalues. These eigenvalues correspond to the roots of the cubic factor of the characteristic polynomial which is given by equation 4.16.

We utilize the Routh-Hurwitz criterion to determine the regions of stability in the (P_{eq}, q_2) -plane. To determine the effect q_1 has on stability, we set² $q_1 = 0.447$,

²Using linear regressions on experimental data, Caginalp and Ilieva [6] found a q_1 of 0.447 and a $q_2^{(2)}$ of 0.073 for their closed book experiment. Thus, we utilize these values in Cases 1-4 as appropriate.

10, 20. These regions, which are nested subsets (the larger subset corresponding to the smaller q_1 value), are shown in Figure 2.

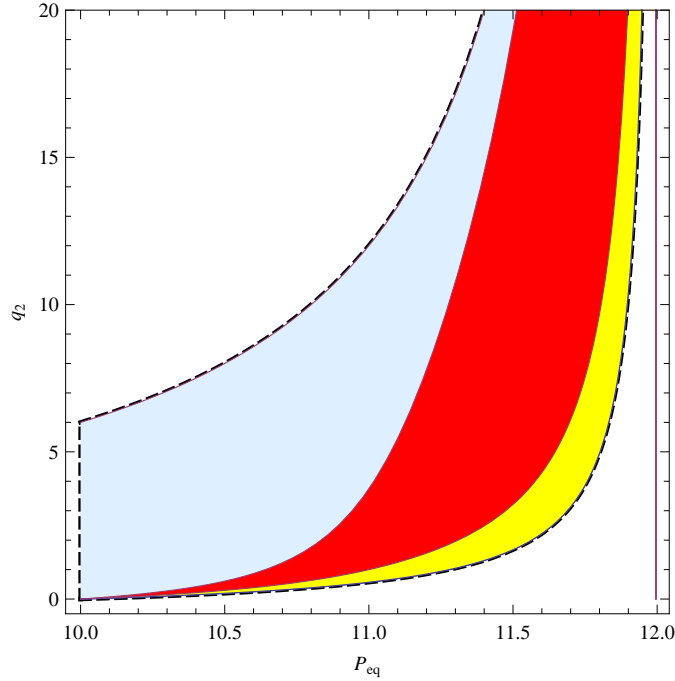


FIGURE 2. Region of (P_{eq}, q_2) pairs for $L = 10$ and $P_a = 12$ corresponding to $q_1 = 0.447$ (Blue, Red, and Yellow), 10 (Red and Yellow), and 20 (Yellow) that yield stability. Areas outside of the designated colored regions but inside the dashed lines correspond to non-stable equilibria.

The union of the Blue, Red, and Yellow regions, which corresponds to $q_1 = 0.447$, is identical to the equilibrium region plotted in Figure 1. Thus, we see that all equilibrium points are stable for $q_1 = 0.447$. As previously noted, the region of equilibria does not change as the value of q_1 changes. However, the value of q_1 does affect the regions of stable and non-stable equilibria. For example, for $q_1 = 20$ all equilibrium points inside the Yellow region in Figure 2 are stable, while the equilibrium points in the Blue and Red regions are not stable.

The q_2 parameter controls the existence of equilibria, while the q_1 parameter is a key factor in the stability of the equilibria (at least for this parameter region). For example, if q_2 is large, then the equilibrium price, P_{eq} , must be close to P_a for stability. The magnitude of the q_1 parameter determines how close P_{eq} must be to P_a for this equilibrium to be stable. Alternatively, if Group 1 is very strongly focused on the price trend while Group 2 is not strongly affected by the asset price's deviation from the fundamental value (i.e., large q_1 and small q_2), then P_{eq} must be very close to L for stability. Indeed, from Figure 2 it is clear in this scenario that the equilibrium is most likely not stable.

Case 2. $8 = P_a < P_{eq} < L = 10$

Equilibrium. This case contains the same curve of equilibrium points (prior to substituting any parameter values) as in Case 1 where $P_{eq} \neq P_a$ may assume any value between P_a and L . Requiring these equilibrium points to satisfy assumptions (A1)-(A7) restricts the set of (P_{eq}, q_2) -pairs that yield equilibrium to the following region

$$P_a < P_{eq} < L \text{ and } \frac{(P_{eq} - L) P_a}{(L + P_{eq})(P_a - P_{eq})} \leq q_2 < \frac{-P_a}{P_a - P_{eq}}$$

which is derived from Corollary 1. For the specific parameter values given above the region is described by

$$8 < P_{eq} < 10 \text{ and } \frac{(P_{eq} - 10) 8}{(10 + P_{eq})(8 - P_{eq})} \leq q_2 < \frac{-8}{8 - P_{eq}}$$

and shown in Figure 3.

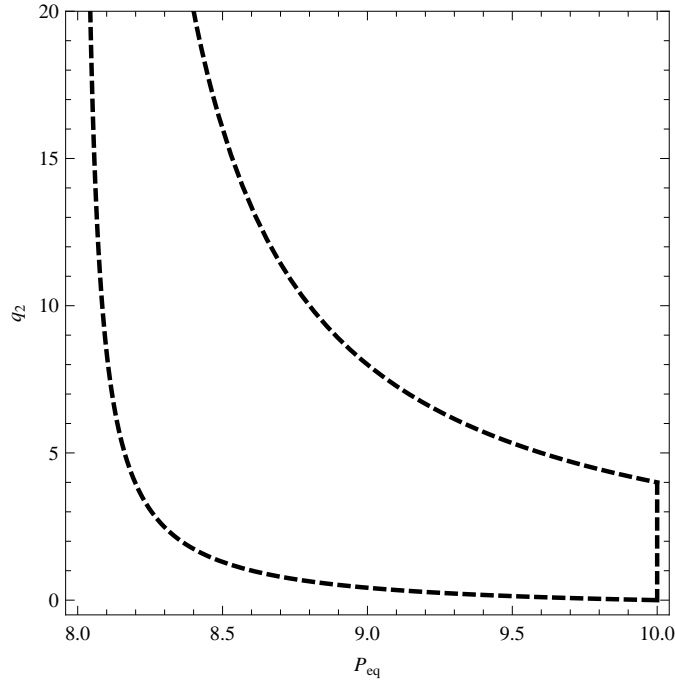


FIGURE 3. Region of (P_{eq}, q_2) -pairs that yield permissible, i.e. satisfy assumptions (A1)-(A7), equilibrium points ($8 = P_a < L = 10$).

Large q_2 implies P_{eq} is close to P_a , while small values for q_2 mean P_{eq} must be close to L . Again, this result is analogous to that of [4] for the single group model.

Stability. The Jacobian matrix and characteristic polynomial for this case are identical (prior to setting parameter values) to those of Case 1. Refer to Appendix C for the Jacobian matrix and equation 4.16 for the characteristic polynomial.

Using the Routh-Hurwitz criterion, we determine regions in the (P_{eq}, q_2) -plane that yield stable equilibrium points. To understand the effect of the q_1 parameter on stability, we produce plots for $q_1 = 0.447, 10, \text{ and } 20$. These plots are displayed

in Figure 4. As q_1 increases, each corresponding region is nested within the prior region.

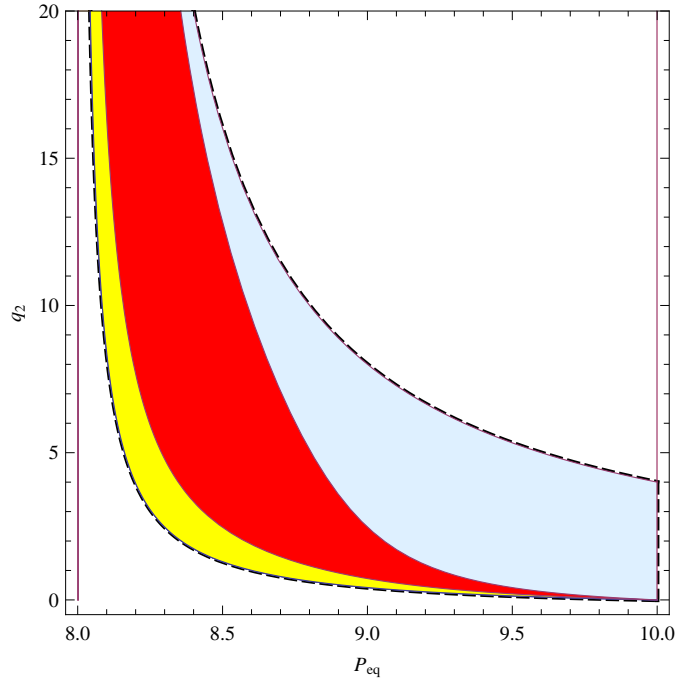


FIGURE 4. Region of (P_{eq}, q_2) pairs for $P_a = 8$ and $L = 10$ corresponding to $q_1 = 0.447$ (Blue, Red, and Yellow), 10 (Red and Yellow), and 20 (Yellow) that yield stability. Areas outside of the designated colored regions but inside the dashed lines correspond to non-stable equilibria.

The union of the Blue, Red, and Yellow regions corresponds to $q_1 = 0.447$ and is identical to the region of permissible equilibrium points in Figure 3. Thus, we see that all equilibrium points are stable for $q_1 = 0.447$. As in Case 1, the q_2 parameter controls the existence of equilibria, while q_1 is a key factor in determining stability. As the value of the q_1 parameter increases, the region of equilibria remains unchanged, while the region of (P_{eq}, q_2) -pairs that yield stable equilibria gets smaller. Thus, if $q_1 = 20$, then the Yellow region corresponds to stable equilibria, while the Blue and Red regions correspond to non-stable equilibria. Note that if Group 2 is strongly focused on the asset price's deviation from the fundamental value and Group 1 is not strongly motivated by the recent price trend (i.e., q_1 is small and q_2 is large), then P_{eq} must remain close to P_a for stability.

It has been noted previously through experimental ([8]), theoretical ([4]), and empirical ([5]) studies that an increase in the amount of cash available to purchase an asset boosts the asset's price. We increase the value of L (by increasing M_0 while holding N_0 fixed) to study the effect of an increase of cash on the system. Figure 5 contains the plots of the stability regions corresponding to $L = 20$ and 30, respectively.

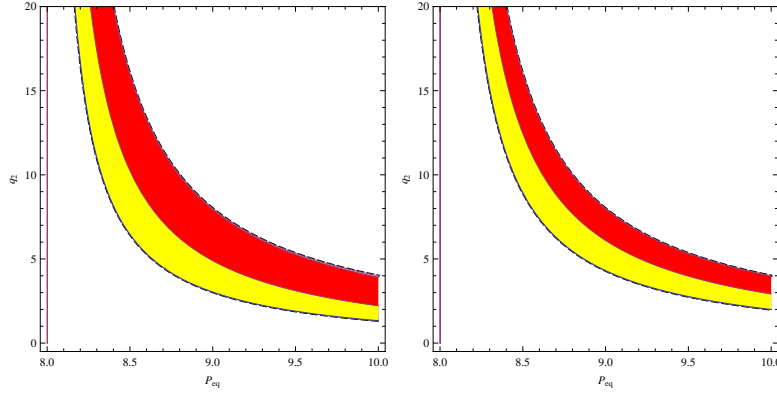


FIGURE 5. Region of (P_{eq}, q_2) pairs for $P_a = 8$ corresponding to $q_1 = 0.447$ (Blue, Red, and Yellow), 10 (Red and Yellow), and 20 (Yellow) that yield stability. The left plot corresponds to $L = 20$ and the right $L = 30$. Compare these with Figure 4.

As shown in Figure 5, the region of permissible equilibrium points (dashed lines) becomes smaller as L increases, though the overall shape is similar. Note that a similar phenomenon occurs as L is reduced with P_a fixed.

Case 3. $P_a = P_{eq} = L = 10$

Equilibrium. For any $\hat{N}^{(1)}$ such that $0 \leq \hat{N}^{(1)} \leq N_0$ and $N_0 > 0$ equation 4.13 yields an equilibrium point of the system 3.2' - 3.4', 4.3', and 4.4'. In contrast with Cases 1 and 2, any $(\hat{N}^{(1)}, q_2)$ -pair yields an equilibrium point satisfying assumptions (A1)-(A7).

Stability. The Jacobian matrix evaluated at an arbitrary equilibrium point is given in Appendix C while the characteristic polynomial is given by equation 4.21.

Letting q_1 and q_2 assume any real and positive number, the Routh-Hurwitz criterion yields the stable region $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ contained in $(N^{(1)}, q_1, q_2)$ space where

$$\Omega_1 = \left[0 < q_1 < \frac{11}{2} \text{ and } 0 \leq \hat{N}^{(1)} \leq 3,000 \text{ and } q_2 > 0 \right],$$

$$\Omega_2 = \left[q_1 = \frac{11}{2} \text{ and } 0 \leq \hat{N}^{(1)} < 3,000 \text{ and } q_2 > 0 \right],$$

$$\Omega_3 = \frac{11}{2} < q_1 \leq 10$$

$$\text{and } \left\{ \begin{array}{l} \left(0 \leq \hat{N}^{(1)} \leq \frac{16,500}{q_1} \text{ and } q_2 > 0 \right) \\ \left(\frac{16,500}{q_1} < \hat{N}^{(1)} < 3,000 \right) \\ \text{or } \left(\text{and } q_2 > \frac{-544,500,000 + 49,500\hat{N}^{(1)}q_1 - (\hat{N}^{(1)}q_1)^2}{900,000,000 - 300,000\hat{N}^{(1)} - 30,000\hat{N}^{(1)}q_1 + 10(\hat{N}^{(1)})^2q_1} \right) \end{array} \right\},$$

and

$$\Omega_4 = q_1 > 10$$

$$\text{and } \left\{ \begin{array}{l} \left(0 \leq \hat{N}^{(1)} \leq \frac{16,500}{q_1} \text{ and } q_2 > 0 \right) \\ \frac{16,500}{q_1} < \hat{N}^{(1)} < \frac{30,000}{q_1} \\ \text{and } q_2 > \frac{-544,500,000 + 49,500\hat{N}^{(1)}q_1 - (\hat{N}^{(1)}q_1)^2}{900,000,000 - 300,000\hat{N}^{(1)} - 30,000\hat{N}^{(1)}q_1 + 10(\hat{N}^{(1)})^2q_1} \end{array} \right\}.$$

This region is displayed in Figure 6 for various q_1 and q_2 values. Note that larger q_1 values yield smaller stable regions while larger q_2 values correspond to larger stable regions.

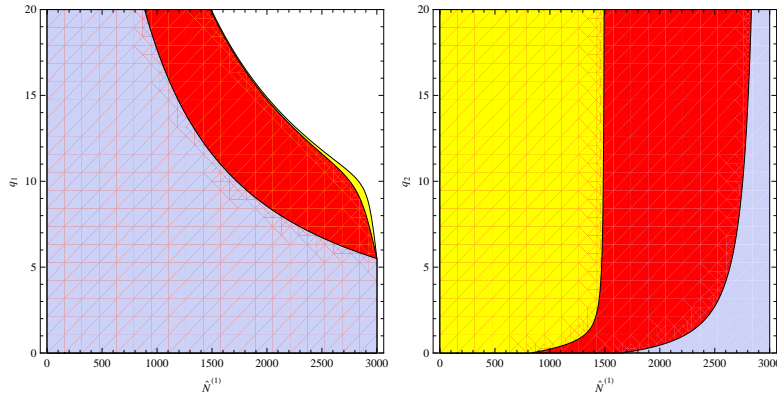


FIGURE 6. Regions in $(\hat{N}^{(1)}, q_1)$ and $(\hat{N}^{(1)}, q_2)$ space for $L = P_{eq} = P_a = 10$ corresponding to stability. Diagram on left shows $(\hat{N}^{(1)}, q_1)$ pairs for $q_2 = 0.073$ (Blue), 10 (Blue and Red), and 20 (Blue, Red, and Yellow) that yield stability. The diagram on the right displays $(\hat{N}^{(1)}, q_1)$ pairs for $q_1 = 0.447$ (Blue, Red, and Yellow), 10 (Red and Yellow), and 20 (Yellow) that yield stability. Note that any point within the plotted region yields an equilibrium point of the system. Areas outside of the designated colored regions correspond to non-stable equilibria.

From the diagram on the left in Figure 6 we see that the region of stable equilibrium points increases in size as the value of the q_2 parameter increases. Note that from this diagram we see that for large q_1 if $\hat{N}^{(1)} \approx N_0$, then the equilibrium point is not stable. The diagram on the right in Figure 6 shows that for small q_1 all equilibrium points are stable and as q_1 increases the region of stable equilibrium points gets smaller.

Case 4. $P_{eq} = L = 10 \neq P_a$

Equilibrium. For any $\hat{\zeta}_2 \in (-1, 1) \setminus \{0, q_2\}$ an equilibrium point is given by equation 4.12. Note that P_a may be less than or greater than $P_{eq} = L$. As such, we consider two scenarios: $0 < P_a < L$ and $P_a > L$.

If $0 < P_a < L$, then the surface of equilibrium points in (ζ_2, q_2, P_a) - space is given by³

$$-1 < \hat{\zeta}_2 < 0 \text{ and } q_2 > 0 \text{ and } P_a = \frac{10q_2}{q_2 - \hat{\zeta}_2},$$

and if $P_a > L$, then this surface is described by

$$0 < \hat{\zeta}_2 < q_2 \text{ and } P_a = \frac{10q_2}{q_2 - \hat{\zeta}_2}.$$

In both cases we have ensured these equilibrium points satisfy assumptions (A1)-(A7). These regions are depicted graphically in Figure 7.

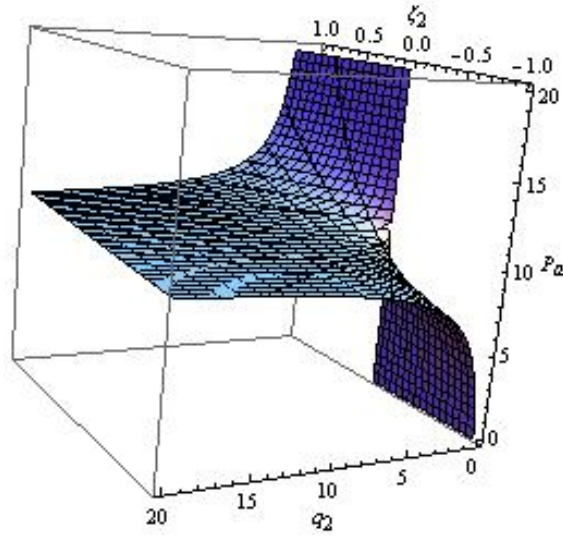


FIGURE 7. Surface of $(\hat{\zeta}_2, q_2, P_a)$ - triples that yield permissible, i.e. satisfy assumptions (A1)-(A7), equilibrium points ($10 = P_{eq} = L \neq P_a$).

Note that Figure 7 shows that equilibrium may exist for values of P_a that are drastically different from P_{eq} .

Stability. With $P_a = \frac{10q_2}{q_2 - \hat{\zeta}_2}$ (found by solving the P_{eq} critical point equation for P_a) the Jacobian matrix evaluated at an arbitrary equilibrium point is given by:

$$\begin{bmatrix} \frac{1}{2}(-1 + \hat{\zeta}_2) & \frac{1 + \hat{\zeta}_2}{20} & 0 & 0 & 0 \\ 5 - 5\hat{\zeta}_2 & \frac{1}{2}(-1 - \hat{\zeta}_2) & 0 & 0 & 0 \\ -\frac{\hat{\zeta}_2}{300} & -\frac{\hat{\zeta}_2}{3000} & -1 & 20 & 0 \\ -\frac{q_1 \hat{\zeta}_2}{30000} & -\frac{q_1 \hat{\zeta}_2}{300000} & -\frac{q_1}{100} & \frac{1}{10}(-1 + 2q_1) & 0 \\ 0 & 0 & -\frac{q_2}{P_a} & 0 & -1 \end{bmatrix}$$

³The actual criteria necessary to ensure a valid equilibrium is $\hat{\zeta}_2$ less than zero. However, as noted above we assume $|\hat{\zeta}_2| < 1$ which gives the first inequality.

with characteristic equation:

$$-\frac{1}{10}\lambda(1+\lambda)^2\{1+(11-2q_1)\lambda+10\lambda^2\}=0.$$

This system has a zero eigenvalue and a negative one eigenvalue with algebraic multiplicity two. Thus, the sign of the real parts of the remaining two eigenvalues will determine the stability of the equilibrium point. Note that the stability is independent of q_2 and only depends upon the value of parameter q_1 .

The remaining eigenvalues are of the form:

$$\lambda_{4,5} = \frac{-11 + 2q_1 \pm \sqrt{81 - 44q_1 + 4q_1^2}}{20}.$$

The real parts of these eigenvalues are negative for $q_1 < \frac{11}{2}$, equal to zero for $q_1 = \frac{11}{2}$, and positive for $q_1 > \frac{11}{2}$. So, the equilibrium point is stable for $q_1 \in (-\infty, \frac{11}{2})$ and unstable for $q_1 \in (\frac{11}{2}, \infty)$. It is interesting to note that the point of transition from stability to instability, $q_1 = \frac{11}{2}$, is also significant in Case 3, $P_{eq} = P_a = L$, in that this is the point where parameter q_1 begins to influence the stability.

6. Conclusion. The stability of financial markets is of crucial practical importance; however, studying stability is not feasible within classical models that are idealizations based on near equilibrium conditions. A key assumption in classical finance is the existence of an infinite amount of capital controlled by experts, free of bias, and unanimous in their assessment of true value of the asset. Thus any study of price dynamics within this setting leads to prices that evolve quickly and smoothly to the equilibrium price with some random fluctuations, or noise, along the way. The equations developed by Caginalp and collaborators since 1990 model asset price dynamics by utilizing basic microeconomics of supply and demand but also allowing these functions to depend on motivations beyond valuation. Also inherent in this approach is the finiteness of assets and the possibility that different investor groups (controlling parts of the total asset pool) may also differ in their assessments of the value of an asset. We sketch an argument in Appendix A that the formal limit of solutions of this model corresponds to solutions of the classical price equation 1.1. Thus, this model encompasses the classical model upon removal of the generalized features.

After a precise derivation of the link between the discrete (difference equations) and the continuum (differential equations) for the multi-group models, we study the stability properties with the objective of determining the parameter regions for equilibrium, stability, and instability. An important difference between the neo-classical dynamics and our asset flow models is that equilibrium is not uniquely determined by the valuation in the latter. Instead it is determined by a set of algebraic equations incorporating the characteristics of each group. In other words, given a particular valuation (held by Group 2 that focuses on value) there are a spectrum of equilibrium prices. One has a unique price given the other parameters such as the cash and share position of each group, etc.

Specifically, for the two group system with conserved cash and shares we prove (Section 4) the existence of a 1-dimensional curve of equilibrium points under various parameter regimes. In addition, this equilibrium price, P_{eq} , must lie between the liquidity value,

$$L := (\text{total amount of cash}) / (\text{total number of shares}),$$

and Group 2's assessment of the fundamental value, P_a . In Theorem 4.2 the existence of local stable and unstable regions for an arbitrary equilibrium point is established. We note that existence of equilibrium is independent of the q_1 parameter, i.e. the magnitude of trend based investing does not impact the existence of equilibrium.

Numerical computations (Section 5) with specific parameter values establish the exact stable regions for an equilibrium point. In general, we showed that if Group 2 is strongly focused on valuation, i.e. q_2 is large, then the equilibrium price is close to P_a . The strength of Group 1's motivation due to the recent price trend determines how close P_{eq} must be to either P_a or L to ensure stability. For instance, if q_1 is large, i.e. Group 1 is very strongly focused on the recent trend in price, and q_2 is small, then P_{eq} must be close to L for stability (see Figure 2).

The results show that in Cases 1, 2, and 4 the parameter q_2 controls the existence of equilibria, while in Case 3 existence of equilibria is independent of the q_2 parameter. In all four cases, the q_1 parameter plays a key role in determining the stability. In Cases 1, 2, and 3 as q_1 increases, the region of (P_{eq}, q_2) pairs that yield stable equilibria gets smaller (see Figures 2, 4, and 6). In Case 4 the equilibrium point is stable for $q_1 < 11/2$ and unstable for $q_1 > 11/2$. In Case 3 as the value for q_2 increases, the stable region also grows larger. Figure 6 also shows that if Group 1 is strongly focused on the trend, i.e. q_1 large, and owns a vast majority of the shares, then the equilibrium is most likely not stable. Our results regarding the influence of the q_1 parameter are analogous to those of Duran [11] where he finds that the market price is more sensitive to small changes in the q_1 parameter than the q_2 parameter.

Thus the asset flow models are capable of addressing stability issues. The remaining challenge is to establish a stronger connection with world markets by estimating the assets controlled by different groups characterized by their trading strategy. This requires not only estimation of the q_1 and q_2 parameters but also an estimate of the magnitudes of the assets of each group. With these estimates one can develop criteria that lead to instability in markets.

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Appendix A. The classical asset price equation as a limit of the asset flow differential equations. We show that solutions of the discrete asset flow equations 2.2, 2.7, and 2.8 have a formal limit as solutions to the classical stochastic asset price equation 1.1. This is accomplished by utilizing the basic assumptions of classical finance within the context of our equations. The key assumption needed to attain the classical limit is that Group 2 is focused solely on the value of the asset, $P_a(t)$, while Group 1 consists of “noise traders,” a common assumption in classical economics and finance. In other words, these participants hope to make a profit, but without additional information, they try a variety of strategies that amount to nothing more than noise in the aggregate. While individual traders may have strategies, there are many disparate ideas, and so there is no net dependence on either price or value. This summarizes the viewpoint of classical finance.

Classical finance stipulates the existence of a risk-free interest rate (per year), which we will call μ , and assume that it is constant within the time period we are

considering. Since all investors have the same publicly available information, the expectation for the return of any asset is given by

$$P_a(t) = P_a(0) e^{\mu t}. \quad (\text{A.1})$$

We consider a small time frame (compared to one year), and denote by P_a the fundamental value of the stock during that time. With the assumption of non-zero trading costs we assume that $k^{(2)}$ is given by

$$k^{(2)} := \begin{cases} 1 & \text{if } P < P_a - \delta \\ 1/2 & \text{if } P_a - \delta < P < P_a + \delta \\ 0 & \text{if } P > P_a + \delta \end{cases} \quad (\text{A.2})$$

for some small, positive δ . We can also consider the smoothed version of this function, given by

$$k^{(2)}(P) := \frac{1}{2} \left[1 + \tanh \left(\frac{P_a - P}{\varepsilon} \right) \right] \quad (\text{A.3})$$

for some small, positive ε .

We make the classical assumption that the knowledgeable investors, namely Group 2, represent the vast majority of wealth, expressed by the following.

Assumption A. At the initial time, the endowments to the two groups satisfy the following:

$$M^{(1)}/M^{(2)} \ll 1 \quad \text{and} \quad N^{(1)}/N^{(2)} \ll 1. \quad (\text{A.4})$$

Assumption B. We assume that the available cash and asset are balanced so that the cash supply of Group 2 satisfies

$$L_2(t) := M^{(2)}(t)/N^{(2)}(t) = P_a(t). \quad (\text{A.5})$$

The assumption that Group 1 consists of “noise traders” implies that the net flow of orders by this group does not depend on P_a or P .

Assumption C. The transition rate of Group 1, $k^{(1)}$, is a constant independent of P_a and P and $0 < k^{(1)} < 1$.

Under Assumptions A and B, for $P < P_a - \delta$, one has that $k^{(2)} = 1$ from A.2, so we can write 2.1b as

$$F(T_j) = \frac{k^{(1)}M^{(1)} + M^{(2)}}{(1 - k^{(1)})N^{(1)}} \doteq \frac{1}{(1 - k^{(1)})} \frac{M^{(2)}}{N^{(1)}} = \frac{P_a}{(1 - k^{(1)})} \frac{N^{(2)}}{N^{(1)}} = O\left(\frac{N^{(2)}}{N^{(1)}}\right) \quad (\text{A.6})$$

which is large and positive. Hence, by 2.2 the price quickly moves into the range $P > P_a - \delta$. Similarly, when the stock is overvalued, i.e., $P > P_a + \delta$, one has $k^{(2)} = 0$, yielding

$$F(T_j) = \frac{k^{(1)}M^{(1)}}{(1 - k^{(1)})N^{(1)} + N^{(2)}} \doteq \frac{k^{(1)}M^{(1)}}{N^{(2)}} = k^{(1)} \frac{M^{(1)}}{M^{(2)}} P_a \ll P_a.$$

Hence,

$$\frac{F(T_j)}{P(T_j)} - 1 \doteq -1, \quad (\text{A.7})$$

so that the price falls back into the range $P < P_a + \delta$ where $k^{(2)}$ no longer vanishes.

Finally, we consider the range $P_a - \delta < P < P_a + \delta$ where the knowledgeable investors are essentially inactive since it is within their trading costs. In this region

of price, the “noise traders” dominate due to the absence of Group 2 investors. Using Assumptions A and B for prices $P(T_j)$ near P_a , we have, from 2.1b and A.2,

$$F(T_j) = \frac{M^{(2)}}{N^{(2)}} \frac{1 + 2k^{(1)}M^{(1)}/M^{(2)}}{1 + 2(1 - k^{(1)})N^{(1)}/N^{(2)}}$$

yielding, from 2.2, the identity,

$$\begin{aligned} \frac{P(T_{j+1}) - P(T_j)}{P(T_j)} &= \frac{P_a}{P(T_j)} \left\{ \frac{1 + 2k^{(1)}M^{(1)}/M^{(2)}}{1 + 2(1 - k^{(1)})N^{(1)}/N^{(2)}} \right\} - 1 \\ &\doteq 2k^{(1)}\frac{M^{(1)}}{M^{(2)}} - 2(1 - k^{(1)})\frac{N^{(1)}}{N^{(2)}}. \end{aligned} \quad (\text{A.8})$$

Indeed, as $P_a - \delta < P < P_a + \delta$, we approximate $P_a/P(T_j) \doteq 1$. In addition, we use a geometric series argument and drop the $4k^{(1)}M^{(1)}/M^{(2)}(1 - k^{(1)})N^{(1)}/N^{(2)}$ as it is much smaller in magnitude than the other terms due to Assumption A.

The system A.8, 2.7, and 2.8 is not yet close to 1.1, since $M^{(1)}$ and $N^{(1)}$ are not constant.

We now examine $M^{(1)}(t)$ and assume that there is a normal random variable, $X(T)$, governing the influx/outflow of investors and their cash. The parameter M_0 represents the total amount of cash in the system, i.e. $M^{(1)} + M^{(2)} = M_0$.

Assumption E. The cash supply $M^{(1)}(t)$ in 2.7 includes an inflow/outflow term $m^{(1)}(T_j) := \bar{\sigma}X(T_j)M_0$ that dominates the right hand side of A.8.

Thus, we obtain

$$\begin{aligned} \frac{P(T_{j+1}) - P(T_j)}{P(T_j)} &= 2k^{(1)}\frac{m^{(1)}(T_j)}{M^{(2)}} - 2(1 - k^{(1)})\frac{N^{(1)}}{N^{(2)}} \\ &\doteq \sigma X(T_j) \end{aligned} \quad (\text{A.9})$$

where σ incorporates the additional constants multiplying $\bar{\sigma}$. In other words, the existing “noise traders” effectively cancel out one another as they employ a variety of strategies. More significant, however, is the stochastic influx and exit of new investors that becomes the main stochastic contribution to price change. For example, if there is a net inflow of investor cash (as in the late 1990’s) due to macroscopic or demographic changes, then the larger supply of cash leads to higher prices. This holds even assuming that investors, on average, are just as likely to sell as they are to buy. Formally taking the continuum limit of A.9 yields

$$dP = \sigma P dX. \quad (\text{A.10})$$

Since, $P(t)$ and $P_a(t)$ are essentially identical in classical finance, we assume $P(t)$ satisfies the same differential equation, namely

$$dP = \mu P dt. \quad (\text{A.11})$$

Combining A.10 and A.11 yields the stochastic equation 1.1, i.e.

$$\frac{dP}{P} = \sigma dX + \mu dt. \quad (1.1)$$

The main difference is that we are assuming a deterministic motivation once the price has moved beyond δ away from the fundamental value.

Appendix B. Summary of Routh-Hurwitz Criterion. Two necessary conditions for the real parts of all roots of a polynomial (with real coefficients) to lie in the left half of the complex plane are: (i) all coefficients must be nonzero and (ii) all coefficients must have the same sign. Thus, if either condition (i) or (ii) is not satisfied, then the equation has a root(s) with real part greater than or equal to zero. The Routh-Hurwitz criterion, however, gives a necessary and sufficient condition for the real parts of the roots of a polynomial to be negative. As such, this method is utilized to determine the stability, i.e. real parts of all eigenvalues strictly negative, of the equilibrium points considered in Sections 4 and 5.

We consider the specific case of a cubic polynomial $p(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. The Routh Array is:

$$\begin{array}{c|cc} \lambda^3 & a_3 & a_1 \\ \lambda^2 & a_2 & a_0 \\ \lambda^1 & \frac{a_2a_1 - a_3a_0}{a_2} & 0 \\ \lambda^0 & a_0 & 0 \end{array} .$$

The Routh-Hurwitz criterion states that the number of roots with positive real parts is given by the number of sign changes in the middle column. In general there are four cases to consider:

- (i) There are no zero elements in the middle column.
 - (ii) A zero appears in the middle column but the other element in the same row (right-most column) is non zero, i.e. $a_3 = 0$ or $a_2 = 0$.
 - (iii) There is a row of zeros in the array.
 - (iv) Same as (iii) but in this case there are repeated roots on the imaginary axis.
- This cannot occur for the cubic polynomial case because it only has three roots.

For a cubic polynomial the following is a necessary and sufficient condition for stability: all roots of the polynomial have negative real parts if and only if the coefficients are positive and $a_2a_1 - a_3a_0 > 0$ ([10]).

If $a_2a_1 = a_3a_0$, then case (iii) applies, and the equilibrium point is marginally stable. This means that the polynomial has roots that are located symmetrically about the origin of the complex plane, i.e., the roots are of the form $(\lambda - a)(\lambda + a)$ or $(\lambda - ia)(\lambda + ia)$, $a \in \mathbb{R}$. These roots are solutions of the auxiliary polynomial, $a(\lambda) = a_2\lambda^2 + a_0 = 0$ ([10]). Solving $a(\lambda) = 0$ yields $\lambda_{1,2} = \pm\sqrt{-\frac{a_0}{a_2}}$. So, if a_2 and a_0 are of the same sign (which we assume for stability), then the roots will be pure imaginary. Otherwise, they will be real roots with opposite sign.

Using this criterion for the cubic polynomial, critical points may be classified as one of the following types:

- (i) **Stable:** $a_3 > 0, a_2 > 0, a_1 > 0, a_0 > 0$, and $a_2a_1 - a_3a_0 > 0$. Solutions go exponentially fast to critical point.
- (ii) **Marginally Stable:** $a_3 > 0, a_2 > 0, a_0 > 0$, and $a_2a_1 - a_3a_0 = 0$. Sustained oscillations; solution trajectories will be ellipses in the plane determined by the eigenvectors corresponding to the pure imaginary eigenvalues.
- (iii) **Unstable:** All other scenarios.

Appendix C. Jacobian Matrices for Cases 1-3 in Section 5. Linearization about the arbitrary equilibrium point identified by equation 4.10 for Cases 1 and 2, i.e. $L < P_{eq} < P_a$ and $P_a < P_{eq} < L$, yields a Jacobian matrix that, when evaluated

at this equilibrium point, is given by

$$\begin{pmatrix} -\frac{N_0 P_{eq} C1}{C4} & \frac{M_0 C2}{P_{eq} C4} & 0 & \frac{(M_0 - N_0 P_{eq}) C3 C5}{2C0^2 P_{eq} q_2^2 C4} & \frac{P_a^2 C6 C3}{2C0^2 P_{eq} q_2^2 C4} \\ \frac{N_0 P_{eq}^2 C1}{C4} & -\frac{M_0 C2}{C4} & 0 & \frac{(M_0 - N_0 P_{eq})(-C3) C5}{2C0^2 q_2^2 C4} & -\frac{P_a^2 C6 C3}{2C0^2 q_2^2 C4} \\ -\frac{2P_{eq}^2 C0 q_2}{C4} & -\frac{2P_{eq} C0 q_2}{C4} & -1 & \frac{2P_a P_{eq} C3}{C0 q_2 C4} & -\frac{2P_a^2 P_{eq} C6}{C0 q_2 C4} \\ -\frac{2c_1 P_{eq} C0 q_1 q_2}{C4} & -\frac{2c_1 C0 q_1 q_2}{C4} & -\frac{c_1 q_1}{P_{eq}} & \frac{c_1 (M_0 C2 C8 + N_0 P_{eq} C1 C7)}{C0 q_2 C4} & -\frac{2c_1 P_a^2 C6 q_1}{C0 q_2 C4} \\ 0 & 0 & -\frac{c_2 q_2}{P_a} & 0 & -c_2 \end{pmatrix}$$

where

$$\begin{aligned} C0 &:= P_a - P_{eq}, \\ C1 &:= P_a - q_2 C0, \\ C2 &:= P_a + q_2 C0, \\ C3 &:= M_0 C2 - N_0 P_{eq} C1, \\ C4 &:= M_0 C2 + N_0 P_{eq} C1, \\ C5 &:= -2P_a P_{eq} q_2^2 + P_{eq}^2 q_2^2 + P_a^2 (-1 + q_2^2), \\ C6 &:= M_0 - N_0 P_{eq}, \\ C7 &:= P_{eq} q_2 - P_a (2q_1 + q_2), \text{ and} \\ C8 &:= 2P_a q_1 + C1. \end{aligned}$$

For Case 3 in Section 4 where $P_{eq} = P_a = L$ an arbitrary equilibrium point is given by equation 4.13. Linearization about this equilibrium point yields the following Jacobian matrix (evaluated at the equilibrium point):

$$\begin{pmatrix} -\frac{1}{2} & \frac{N_0}{2M_0} & 0 & \hat{N}(1) - \frac{\hat{N}(1)^2}{N_0} & \frac{\hat{N}(1)(-N_0 + \hat{N}(1))}{N_0} \\ \frac{M_0}{2N_0} & -\frac{1}{2} & 0 & \frac{M_0 \hat{N}(1)(-N_0 + \hat{N}(1))}{N_0^2} & \frac{M_0 (N_0 - \hat{N}(1)) \hat{N}(1)}{N_0^2} \\ 0 & 0 & -1 & \frac{2M_0 \hat{N}(1)}{N_0^2} & \frac{2M_0 (N_0 - \hat{N}(1))}{N_0^2} \\ 0 & 0 & -\frac{c_1 N_0 q_1}{M_0} & c_1 \left(-1 + \frac{2\hat{N}(1) q_1}{N_0} \right) & \frac{2c_1 (N_0 - \hat{N}(1)) q_1}{N_0} \\ 0 & 0 & -\frac{c_2 N_0 q_2}{M_0} & 0 & -c_2 \end{pmatrix}.$$

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