

RENORMALIZATION GROUP CALCULATION OF LATE STAGE INTERFACE DYNAMICS*

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Abstract. The temporal evolution of an interface separating two phases is studied for its large time behavior. The calculation involves adaptation of methodology known in the physics literature as renormalization group and scaling theory. We consider the fully dynamic set of equations, i.e., the sharp interface model incorporating surface tension and kinetic undercooling. This extends earlier results and those of Jasnow and Vinals that were obtained for the quasi-static regime. The characteristic length, $R(t)$, governing the morphology of late stage growth is found to vary as $t^{1/2}$, while the total surface area of the interface, $S(t)$, varies as $t^{(d-1)/2}$. Thus the growth exponents in this fully dynamic analysis differ from the $R(t) \sim t$ found for the quasi-static regime by Jasnow and Vinals. In both cases, however, the capillarity length arising from surface tension is irrelevant for large time behavior, even though it is a crucial factor for the initial growth velocity and the linear stability of the interface.

Key words. renormalization group, interface dynamics, scaling theory, surface tension

AMS subject classifications. 82C24, 82B24, 35K55

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1. Introduction. In recent years a number of mathematical methods have been used to study pattern formation. These include large scale computations, linear perturbation, and analytical methods. Analytical methods have been useful in addressing some key questions such as the existence and nature of steady states. Large scale computations, which have provided details of interface patterns, appear to resolve many questions related to the evolution of the pattern. However, there is a stochastic aspect of the interface in that two computer runs will exhibit two different patterns, though some key features will be identical in both, particularly in the initial and late stages. Linear stability has provided a valuable tool for studying the early stages of the development of the interface. An equally systematic approach to the late stage growth is desirable in order to understand interfacial phenomena.

In particular, if the growth is self-similar, one would like to be able to describe it by calculating the characteristic length, $R(t)$, as a function of time, t . Significant progress toward this end was made by Jasnow and Vinals (1989, 1990; see also references contained therein) who implemented a renormalization group (RG) approach to study approximations to a one-phase interface problem. A related work focuses on the Hele–Shaw problem (Jasnow and Yeung (1993)).

Using a quasi-static approximation (i.e., the heat equation $u_t = \nabla^2 u$ was replaced by Laplace’s equation, $\nabla^2 u = 0$, in one of the phases), they found that $R(t)$ of a self-similar system arising from a plane wave evolves linearly in time so that $R(t) \sim t$. In a previous paper (Caginalp (1999)) we considered the evolution of a perturbed plane wave modelled within the full two-phase and dynamical problem in arbitrary spatial dimension in a highly supercooled environment so that rapid solidification occurs.

The results also confirmed, as in the quasi-static case, that the capillarity length, d_0 , which is the length scale associated with surface tension, is not relevant to the

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scaling of the large scale behavior. This is an intriguing consequence, particularly since we know that the surface tension has a critical influence upon the early stage growth. Indeed, small d_0 permits instabilities, while large d_0 tends to suppress them.

In this paper we examine the late stage growth issue in the context of a general geometry and more general conditions on the degree of undercooling. The main result of this work is that the characteristic length in the system, $R(t)$, evolves as $t^{1/2}$ within this rapid solidification regime. This suggests that the important difference is between the dynamic and the quasi-static states.

The methodology in Caginalp (1999) and the current paper involves rewriting the basic equations in terms of a Green's function identity after introducing a phase function. The RG analysis then proceeds in several steps as the equations are transformed and then converted back into their original form with renormalized physical parameters.

The analysis suggests that the important connection between the dynamic and static renormalization methods can be studied through a Green's function approach and that the transition between $R(t) \sim t^{1/2}$ and $R(t) \sim t$ can be viewed in terms of the elliptic limit of the parabolic Green's function.

While RG methodology was originally used for calculations in quantum field theory and statistical mechanics, its application to a broad array of problems in applied mathematics provides an important challenge for future work (see, for example, the texts by Goldenfeld (1992) and Creswick, Farach, and Poole (1992)). A number of recent papers have utilized some aspects of this philosophy to understand physical problems described by differential equations, including Goldenfeld et al. (1990); Goldstein, Pesci, and Shelley (1993); Bertozzi et al. (1994); Zhang and Graham (1997); Moise and Ziane (1999); and Moise and Temam (2000).

2. The model and Green's function transformation. We consider a homogeneous material that can be in either of two phases, which we assume are liquid and solid, occupying a spatial region, Ω , in d -dimensional space. The mathematical model consists of determining a temperature, $T(x, t)$, and an interface, $\Gamma(t)$, satisfying the system of equations

$$(2.1) \quad CT_t = K\Delta T \quad \text{in } \Omega,$$

$$(2.2) \quad lv_n = -K [\nabla T \cdot \hat{n}]_{\pm}^+ \quad \text{on } \Gamma(t),$$

$$(2.3) \quad T = \frac{-\sigma}{[s]_{eq}} (\kappa + \alpha v_n) \quad \text{on } \Gamma(t).$$

Here, C is the specific heat per unit volume, K is the thermal conductivity, l is the latent heat per unit volume, σ is the surface tension, $[s]_{eq}$ is the entropy difference per unit volume between phases, α is the relaxation time coefficient, v_n is the (normal) velocity of the interface, and $[\cdot \cdot]_{\pm}^{\pm}$ is the difference between the two sides of the interface. We will let $+$ denote the phase with the higher internal energy, and refer to it as the liquid, and call the $-$ phase the solid. Mechanical properties such as stress will not be considered. We consider (2.1)–(2.3) subject to initial conditions $T(\vec{x}, t_0) = g(\vec{x})$ as described below.

Both (2.1) and (2.2) can be reformulated and incorporated into a single equation by introducing a “phase” variable, $\varphi(\vec{x}, t)$, that has the value $+1$ in the liquid phase

and -1 in the solid phase. This is known as the Oleinik formulation and is closely related to a continuously varying φ in the phase field equations. Thus, by writing

$$(2.4) \quad CT_t - K\Delta T = -\frac{l}{2}\varphi_t$$

(see Oleinik (1960)), one can treat the phase change as source term with support along the interface, $\Gamma(t)$. At this point, in order to utilize the Green's formulation, we consider an infinite domain, $\Omega := \mathbf{R}^d$, so that the simplest Green's function can be implemented. For very large but finite domains, the Green's function would be approximated by the one for the infinite domain. We assume the initial condition $g(\vec{x})$ is continuous and bounded in Ω , though one can allow g to grow exponentially as $|\vec{x}| \rightarrow \infty$ if the time interval under consideration is constrained. The main restriction on further generalization is in terms of satisfying the hypotheses required for the Green's function. Then solutions of (2.4) can be expressed as

$$(2.5) \quad T(x, t) = \int_0^t \int_{\Omega} d^d y G(\vec{x} - \vec{y}, t - s) \left(-\frac{l}{2C} \varphi_s(\vec{y}, s) \right) \\ + \int_{\Omega} d^d y G(\vec{x} - \vec{y}, t) g(\vec{y}),$$

where we use the Green's function, with $D := K/C$,

$$(2.6) \quad G(\vec{y}, t) := (4\pi Dt)^{-d/2} \exp(-\{4Dt\}^{-1} |y|^2).$$

In order to perform part of the integration in (2.5) we examine the region near the interface. Let $h(\vec{X}, t)$ denote the distance between a point on the interface and its projection onto the point $\vec{X} \in \mathbf{R}^{d-1}$ on the plane perpendicular to the z direction with unit normal \hat{k} . With $z = h(\vec{X}, t)$ and the assumption that the interface is sufficiently smooth, we compute the normal velocity as

$$(2.7) \quad v_n = \hat{k} \cdot \hat{n} \frac{dh}{dt},$$

where dh/dt is the velocity in the \hat{k} direction.

Let $\tilde{\phi}(x, t)$ be a smoothing of the step function $\varphi(x, t)$, where the transition from -1 to $+1$ occurs on a small distance scale denoted by the parameter ϵ . This is similar to the phase field approach in which φ is a variable that is coupled with T through an additional equation.

To perform the integration, we define a coordinate system in which \vec{r} is the signed normal to the interface. (We consider a suitable local neighborhood of the interface in order to avoid problems with uniqueness of the signed normal.)

To leading order, the function $\tilde{\phi}$ and its derivatives are functions of $r - v_n t$. Hence one can define a variable ϕ as a function of $r - v_n t$, so that to leading order one has

$$(2.8) \quad \varphi(\vec{x}, t) = \tilde{\phi}(x, t) = \phi(r - v_n t) = \phi \left(r - \left\{ \hat{k} \cdot \hat{n} \frac{dh}{dt} \right\} t \right).$$

For a smooth interface and a sufficiently small value of ϵ , this transition region is within the local neighborhood, so that we can utilize this approximation to compute the integration across the interface in (2.5). In particular, one has

$$(2.9) \quad \varphi_t(\vec{x}, t) = \phi_t(r - v_n t) = - \left(\hat{k} \cdot \hat{n} \frac{dh}{dt} \right) \phi_r \left(r - \left\{ \hat{k} \cdot \hat{n} \frac{dh}{dt} \right\} t \right).$$

Note that upon utilizing (2.9) one observes that (2.1) and (2.2) are consequences of (2.4). In particular, (2.1) is clearly valid in either phase since the derivative of ϕ vanishes. Also, integrating (2.4) across the interface, noting for small δ the identity

$$(2.10) \quad \int_{-\delta}^{\delta} \phi_r \left(r - \left\{ \hat{k} \cdot \hat{n} \frac{dh}{dt} \right\} t \right) dr = 2,$$

and utilizing (2.9) and (2.7), one establishes (2.2).

Using the new definitions, we rewrite the integral (2.5) as

$$(2.11) \quad T(\vec{x}, t) = \int_0^t ds \int_{\Omega} d^d y G(\vec{x} - \vec{y}, t - s) \left(\frac{-l}{2C} \right) \left(-\hat{k} \cdot \hat{n} \frac{dh}{dt} \right) \\ \cdot \phi_{r_y} \left(r_y - \left\{ \hat{k} \cdot \hat{n} \frac{dh}{dt} \right\} s \right) + IC,$$

where IC denotes the initial condition portion of the Green's formulation.

Since the derivatives of ϕ vanish in the region immediately outside the interfacial region, one can perform the integral in the normal direction so that the integral over Ω reduces to an integral over the interfacial region, $\Gamma(t)$, as follows:

$$(2.12) \quad T(\vec{x}, t) = \int_0^t ds \int_{\Gamma(s)} d^{d-1} \sigma_y G(\vec{x} - \vec{y}, t - s) \left(\frac{l}{C} \right) \left(\hat{k} \cdot \hat{n} \frac{dh}{dt} \right) + IC.$$

Using (2.12) in conjunction with (2.3) we can write for points (\vec{x}, t) on the interface the identity

$$(2.13) \quad \frac{-\sigma}{[s]_{eq}} \left\{ \kappa(\vec{x}, t) + \alpha \hat{k} \cdot \hat{n}(\vec{x}, t) \frac{dh(\vec{x}, t)}{dt} \right\} \\ = \frac{l}{C} \int_0^t ds \int_{\Gamma(s)} d^{d-1} \sigma_y G(\vec{x} - \vec{y}, t - s) \left(\frac{l}{C} \right) \left(\hat{k} \cdot \hat{n}(\vec{y}, t) \frac{dh(\vec{y}, t)}{dt} \right) \\ + \int_{\Omega} d^d y G(\vec{x} - \vec{y}, t) g(\vec{y}).$$

Next we convert the variables into their dimensionless counterparts. This is done not for convenience, but in order to compare pure numbers as the renormalization procedure is implemented.

We let L_0 be some reference length scale in the problem, so that $\xi := h/L_0$ is the dimensionless displacement from the reference hyperplane ($\vec{z} = 0$). Similarly, we let T_0 denote a reference time scale, so that velocity is scaled with respect to a reference velocity L_0/T_0 and denoted \tilde{v}_n . Similarly, one obtains a dimensionless curvature, $\tilde{\kappa}$, and \tilde{u}_0 as the dimensionless initial temperature g , etc. We define the standard capillarity length to be

$$d_0 := \frac{\sigma/[s]_{eq}}{l/C},$$

so that \tilde{d}_0 is the dimensionless capillarity length. Using these definitions, and dropping the tildes, we now rewrite (2.13) for any $\vec{\zeta} \in \Gamma(t)$ as

$$d_0 \left\{ \kappa(\vec{\zeta}, t) + \alpha \left[\frac{d\xi(\vec{\zeta}, \tilde{t})}{d\tilde{t}} \right]_{\tilde{t}=t} \hat{k} \cdot \hat{n}(\vec{\zeta}, t) \right\}$$

$$\begin{aligned}
&= \int_0^t ds \int_{\Gamma(s)} d^{d-1} \sigma_y G(\vec{\zeta} - \vec{y}, t - s) \left[\frac{d\xi(\vec{\sigma}, \check{t})}{d\check{t}} \right]_{\check{t}=s} \hat{k} \cdot \hat{n}(\vec{y}, s) \\
(2.13') \quad &+ \int_{\Omega} d^d y G(\vec{\zeta} - \vec{y}, t) u_0(\vec{y}).
\end{aligned}$$

We now proceed with the renormalization analysis of (2.13').

Stage 1. For any $b > 0$ and any real valued λ , we first make a purely algebraic substitution

$$(2.14) \quad b\vec{\zeta} \text{ for } \vec{\zeta}, \quad b^{-\lambda}t \text{ for } t,$$

in (2.13'). This leads to the equation

$$\begin{aligned}
&d_0 \left\{ \kappa(b\vec{\zeta}, b^{-\lambda}t) + \left[\frac{d\xi(b\vec{\zeta}, \check{t})}{d\check{t}} \right]_{\check{t}=b^{-\lambda}t} \hat{k} \cdot \hat{n}(b\vec{\zeta}, b^{-\lambda}t) \right\} \\
(2.15) \quad &= \int_0^{b^{-\lambda}t} ds \int_{\Gamma(s)} d^{d-1} \sigma_y G(b\vec{\zeta} - \vec{\sigma}, b^{-\lambda}t - s) \left[\frac{d\xi(\vec{\sigma}, \check{t})}{d\check{t}} \right]_{\check{t}=s} \hat{k} \cdot \hat{n}(\vec{\sigma}, s) \\
&+ \int_{\Omega} d^d y G(b\vec{\zeta} - \vec{y}, b^{-\lambda}t) u_0(\vec{y}).
\end{aligned}$$

Next, we define the new variables, $s' = s/b^{-\lambda}$, $\vec{\sigma}' = \vec{\sigma}/b$, and $\vec{y}' = \vec{y}/b$ and rewrite the right-hand side of (2.15) to yield

$$\begin{aligned}
&d_0 \left\{ \kappa(b\vec{\zeta}, b^{-\lambda}t) + \left[\frac{d\xi(b\vec{\zeta}, \check{t})}{d\check{t}} \right]_{\check{t}=b^{-\lambda}t} \hat{k} \cdot \hat{n}(b\vec{\zeta}, b^{-\lambda}t) \right\} \\
(2.16) \quad &= \int_{t_0}^t b^{-\lambda} ds' \int_{\Gamma(s')} b^{d-1} d^{d-1} \sigma_y G(b\vec{\zeta} - b\vec{\sigma}', b^{-\lambda}t - b^{-\lambda}s') \\
&\quad \cdot \left[\frac{d\xi(b\vec{\sigma}', \check{t})}{d\check{t}} \right]_{\check{t}=b^{-\lambda}s'} \hat{k} \cdot \hat{n}(b\vec{\sigma}', b^{-\lambda}s') \\
&+ \int_{\Omega} b^d d^d y' G(b\vec{\zeta} - b\vec{y}', b^{-\lambda}t) u_0(b\vec{y}', t_0).
\end{aligned}$$

Stage 2. We now examine the scaling of individual terms. For the Green's function we have the purely algebraic transformation

$$\begin{aligned}
(2.17) \quad G\left(b(\vec{\zeta} - \vec{y}), b^{-\lambda}t; D\right) &= (4\pi D b^{-\lambda}t)^{-d/2} \exp\left\{- (4D b^{-\lambda}t)^{-1} |b\vec{y}'|^2\right\} \\
&= (4\pi Dt)^{-d/2} b^{\lambda d/2} \exp\left\{\frac{-b^2 |\vec{y}'|^2}{4b^{-\lambda}Dt}\right\} \\
&= b^{-d} (4\pi Dt/b^{\lambda+2})^{-d/2} \exp\left(-\frac{|\vec{y}'|^2}{4Dt/b^{\lambda+2}}\right) \\
&= b^{-d} G\left(\vec{\zeta} - \vec{y}, t; D/b^{2+\lambda}\right).
\end{aligned}$$

Note that this identity is valid for arbitrary b and λ , and remains valid when t is replaced by $t - s$.

The scaling of the physical quantities involving length is made with the following assumption.

Single scale self-similarity. We assume that physical lengths and times scale in accordance with (2.18) and (2.19) below. From those two assumptions one obtains the analogous relations for inverse lengths and velocities which scale in accordance with (2.20) and (2.21) below.

All physical lengths in the problem scale as

$$(2.18) \quad \xi(b\vec{\sigma}, b^{-\lambda}t) = b\xi(\vec{\sigma}, t),$$

while all physical time measurements in the problem scale as

$$(2.19) \quad \tau(b\vec{\sigma}, b^{-\lambda}t) = b^{-\lambda}\tau(\vec{\sigma}, t).$$

As a consequence of these assumptions one has the following scaling relations.

An inverse length (e.g., curvature κ) scales as

$$(2.20) \quad b\kappa(b\vec{\sigma}, b^{-\lambda}t) = \kappa(\vec{\sigma}, t).$$

Velocities, including the normal velocity, v_n , scale as

$$(2.21) \quad v_n(b\vec{\zeta}, b^{-\lambda}t) = b^{1+\lambda}v_n(\vec{\zeta}, t).$$

This follows from the scaling of space and time (2.18) and (2.19):

$$(2.22) \quad \begin{aligned} v_n(b\vec{\zeta}, b^{-\lambda}t) &= \frac{\text{distance}}{\text{time}}(b\vec{\zeta}, b^{-\lambda}t) = \frac{b \cdot \text{distance}}{b^{-\lambda} \cdot \text{time}}(\vec{\zeta}, t) \\ &= b^{1+\lambda}v_n(\vec{\zeta}, t). \end{aligned}$$

The initial condition satisfies the scaling relation

$$(2.23) \quad u_0(b\vec{y}, b^{-\lambda}t_0) = u_0(\vec{y}, t_0).$$

One interpretation of this relation is that the system is at a sufficiently late stage that the “initial condition” is in the self-similar state.

Stage 3. We now use the relations of Stage 2 in order to transform the key interface equation (2.16) into one that can be compared with the original equation once parameters are renormalized. Utilizing (2.18), (2.20), and (2.21), one writes (2.16) as

$$(2.24) \quad \begin{aligned} & d_0 \left\{ b^{-1}\kappa(\vec{\zeta}, t) + \alpha b^{1+\lambda} \left[\frac{d\xi(\vec{\zeta}, \check{t})}{d\check{t}} \right]_{\check{t}=t} \hat{k} \cdot \hat{n}(\vec{\zeta}, t) \right\} \\ &= b^{-\lambda+d-1} \int_{t_0}^t ds' \int_{\Gamma(s')} d^{d-1}\sigma_y b^{-d} G(\vec{\zeta} - \vec{\sigma}, t - s'; D/b^{2+\lambda}) \\ & \quad \cdot \left[\frac{d\xi(\vec{\sigma}, \check{t})}{d\check{t}} \right]_{\check{t}=s'} \hat{k} \cdot \hat{n}(\vec{\sigma}, s') \\ & \quad + \int_{\Omega} b^d d^d y' b^{-d} G(\vec{\zeta} - \vec{y}', t; D/b^{2+\lambda}) u_0(b\vec{y}', t_0). \end{aligned}$$

Using the scaling of the u_0 term and simplifying terms we obtain

$$\begin{aligned}
 & \frac{d_0}{b} \left\{ \kappa(\vec{\zeta}, t) + \alpha b^{2+\lambda} \left[\frac{d\xi(\vec{\zeta}, \check{t})}{d\check{t}} \right]_{\check{t}=t} \hat{k} \cdot \hat{n}(\vec{\zeta}, t) \right\} \\
 (2.25) \quad &= \int_{t_0}^t ds' \int_{\Gamma(s')} d^{d-1} \sigma_y G(\vec{\zeta} - \vec{\sigma}, t - s'; D/b^{2+\lambda}) \\
 & \quad \cdot \left[\frac{d\xi(\vec{\sigma}, \check{t})}{d\check{t}} \right]_{\check{t}=s'} \hat{k} \cdot \hat{n}(\vec{\sigma}, s') \\
 & \quad + \int_{\Omega} d^d y' G(\vec{\zeta} - \vec{y}', t; D/b^{2+\lambda}) u_0(y', b^{-1}t_0).
 \end{aligned}$$

We now observe that (2.25) is identical to the original (2.13) if we were to replace the three physical parameters, d_0 , α , and D , with the following values:

$$(2.26) \quad d_0 \rightarrow d_0/b, \quad \alpha \rightarrow \frac{\alpha}{b^{-2-\lambda}}, \quad D \rightarrow \frac{D}{b^{2+\lambda}}.$$

At this point we recapitulate the transformations that led from the original interface equation, (2.13), to (2.25) above. First a set of algebraic substitutions were used, followed by a rescaling of all physical lengths and time intervals in accordance with (2.18) and (2.19). In order to render (2.13) and (2.25) identical, one needs to also rescale the parameters in accordance with (2.26).

Another perspective into this is that the purely algebraic transformation resulted in (2.16) and contained terms such as $\xi(b\vec{\zeta}, b^{-\lambda}t)$, which we call the ‘‘Old.’’ The rescaling of the physical length and time scales into the ‘‘New’’ transforms the equation back into the original form. Hence one has the following:

	Old	Replaced by	New
(2.27)	$\xi(b\vec{\zeta}, b^{-\lambda}t)$		$b\xi(\vec{\zeta}, t)$
	d_0		d_0/b
	D		$D/b^{2+\lambda}$
	α		$\alpha/b^{-2-\lambda}$

Under the assumption that the system evolves self-similarly with a single length scale, all physical quantities that have units of length must grow at a rate proportional to a characteristic length R that depends on (t, d_0, α, D) . Since R must satisfy the same relationship as the length ξ above, one must also have the following:

	Old	Replaced by	New
(2.28)	$R(b^{-\lambda}t, d_0, \alpha, D)$		$bR(t, d_0/b, \alpha/b^{-2-\lambda}, D/b^{2+\lambda})$

This key relationship for the characteristic length, R , expresses the change in parameters (d_0, α, D) that is required in order to compensate for the rescaling of space (i.e., R) and time, t . In other words the characteristic lengths of two systems with different parameters and at different times can be related with the equality

$$(2.29) \quad R(b^{-\lambda}t, d_0, \alpha, D) = bR(t, d_0/b, \alpha/b^{-2-\lambda}, D/b^{2+\lambda}).$$

Using the algebraic substitution $t = b^\lambda t_1$, one has the relation

$$(2.30) \quad R(t_1, d_0, \alpha, D) = bR(b^\lambda t_1, d_0/b, \alpha/b^{-2-\lambda}, D/b^{2+\lambda}).$$

Stage 4. Since (2.30) is valid for any positive b , we choose $b = t_1^{-1/\lambda}$ so that $b^\lambda t_1 = 1$. With that choice for b , and omitting the subscript 1 on t_1 (since (2.30) is valid for any positive t_1), we write the identity

$$(2.31) \quad R(t, d_0, \alpha, D) = t^{-1/\lambda} R(1, d_0/t^{-1/\lambda}, \alpha/t^{(2+\lambda)/\lambda}, D/t^{-(2+\lambda)/\lambda}).$$

Recalling that λ is a parameter that was to be determined, we examine (2.31). One can implement the standard RG procedure of examining the implications for different values of λ , and thereby eliminate trivial fixed points.

We note that for $\lambda = -2$ the coefficients of α and D are unity. We first consider the alternatives. If $0 > \lambda > -2$, then $D/t^{-(2+\lambda)/\lambda}$ approaches 0 as $t \rightarrow \infty$, while $\alpha/t^{(2+\lambda)/\lambda}$ approaches ∞ . This is a physically irrelevant fixed point in which there is no diffusion. Similarly, if $\lambda < -2$ or $\lambda > 0$, then D approaches ∞ while α approaches 0, one attains the quasi-static limit, $\Delta u = 0$, and the interface relation (2.3) ignores the velocity. In this case the relation (2.31) expresses the relationship between the characteristic length of a system with finite α and D and that of one with nonfinite α and D . Hence, choosing λ as any number other than -2 does not yield an illuminating relation (2.31). Focusing now on $\lambda = -2$, we obtain the result

$$(2.32) \quad R(t, d_0, \alpha, D) = t^{1/2} R(1, d_0/t^{1/2}, \alpha, D).$$

This means that the characteristic length evolves as $R(t) \sim t^{1/2}$, and thus the surface area behaves as $S(t) \sim t^{(d-1)/2}$. Furthermore, it means that the capillarity length, d_0 , that is associated with the surface tension is essentially irrelevant in terms of large time behavior. This is in sharp contrast to its role during the initial stages, where a nonzero d_0 is a stabilizing force (Mullins and Sekerka (1963), Ockendon (1980)).

3. Conclusion. We have considered solidification in the fully dynamic context of the surface tension and kinetics model (2.1)–(2.3) above. The calculations are based upon very general assumptions on the initial conditions. The main physical assumption is that the growth is self-similar with a single scale. The asymptotic large scale growth is within the context of a statistical set of interfaces that evolve from a set of initial conditions. The key result is that the characteristic length, $R(t)$, increases as $t^{1/2}$, so that the surface area, $S(t)$, increases as $t^{(d-1)/2}$ within this regime. This differs from the quasi-static results in which $R(t) \sim t$. The difference can be traced to the use of a parabolic Green's function in place of the elliptic.

Both the fully dynamic and the quasi-static approaches, however, yield the result that the capillarity length, d_0 , associated with the surface tension is not relevant to the large time behavior of the interface. The diminished role for the surface tension is in sharp contrast to its role in the onset of instability. It would be interesting to investigate quantitatively how the role of the surface tension evolves from a crucial role at the initial stage to an irrelevant one at the late stage.

The methodology utilizes renormalization group transformations that are identities involving Green's representations for parabolic differential equations. Although the methodology is more complicated, this calculation of large time behavior is analogous to the systematic approach provided by linear stability analysis. The basic ideas are in a framework that can be applied systematically to other physical problems, thereby providing a tool for the calculation of large time behavior.

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