

# RENORMALIZATION AND SCALING METHODS FOR NONLINEAR PARABOLIC SYSTEMS

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Abstract: We consider systems of nonlinear parabolic differential equations of the form

$$u_{it'} = D_i u_{ixx} + \epsilon N_i(x, u_i, u_{ix}, u_{ixx}) \quad i = 1, \dots, M$$

where  $\epsilon$  is a small positive number and  $N_i$  is dimensionally consistent without additional dimensional constants. A systematic renormalization group calculation is developed based upon asymptotic analysis and used to compute the large time and space behavior that is characterized the dominant part of  $u_i$  as

$$u_i(x, t) \sim t^{-\frac{1}{2} - \alpha_i} u_i^*(xt^{-\frac{1}{2}}).$$

The anomalous exponent  $\alpha_i$  is given as a simple function of the exponents in  $N_i$ , and an explicit description of  $u_i^*$  is also obtained.

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## 1. INTRODUCTION.

The renormalization group (RG) methods of Wilson (see for example, Wilson and Kogut [1974], Fisher [1974] and references contained therein) that were very successful in resolving delicate issues of statistical mechanics, such as critical exponents, have been applied to a broad spectrum of problems such as self-avoiding walk, fractals, etc. (see Creswick, Farach and Poole [1992]). More recently, the fundamental philosophy of RG has been directed toward understanding some basic aspects of nonlinear differential equations. As with critical exponents, the potential of this research direction lies in the capability to determine a characteristic scaling exponent with a relatively simple calculation upon understanding a transformation that relates two parameters. In the case of critical exponents the dependent parameter is the thermodynamic quantity which diverges while the independent parameter, e.g. temperature, is a measure of the distance from the singularity at  $T_c$ . In the case of blow-up in differential equations (see Berger and Kohn [1988], Giga and Kohn [1985], Galaktionov and Posashkov [1986] and references within ) the solution  $u(x, t)$  diverges as  $t$  approaches the critical value  $t_c$ . Bricmont and Kupiainen [1994] have provided rigorous proofs of the existence of infinitely many profiles around the blow-up point using related methods. Bertozzi [1994] et al have applied the concept of similarity solutions for the onset of singularities in problems involving flow through thin films. Topological transitions and singularities in viscous flows have been studied by Goldstein, Pesci and Shelley [1993].

The justification for the renormalization group method in both problems can be made in terms of the asymptotic self-similarity of the solution (or thermodynamic variable) as the critical value is approached. More explicitly, the profile of the physical quantity  $u$  appears to be nearly identical as one zooms in on the critical value  $t_c$ , provided that  $u$  (and  $x$  in the blow-up case) is scaled appropriately. In the case of critical phenomena, real-space renormalization has its origin in the intuition of the underlying physical interactions (Wilson and Kogut [1974]). However, in the case of parabolic differential equations, one can obtain the leading behavior of the singularity by observing simple scaling rules that govern the key transformations, and then applying a methodology similar to asymptotic analysis. In other words, the differential equations, by virtue of their scaling properties, already incorporate the essential information on the cooperative behavior in the physical system.

A problem that is seemingly unrelated to blow-up is the large time delay in a nonlinear parabolic equation and the associated nonclassical exponents. However, the renormalization group techniques apply in a similar way to these problems due to the asymptotic self-similarity as the horizontal axis is approached. In a sense this problem is analogous to the inverse (in terms of analytic geometry) of the blow-up problem. An asymptotic delay of the form

$$u(x, t) \sim t^{-(1+\alpha)/2} u^*(xt^{-\frac{1}{2}}, 1) \quad (1.1)$$

for large  $t$  and  $x$ , can be regarded as an expression for  $t$  in terms of  $u$  and  $x/\sqrt{t}$ ,

$$t \sim \{u(x, t)/u^*(xt^{-\frac{1}{2}}, 1)\}^{-2/(1+\alpha)}, \quad (1.2)$$

so that  $t$  exhibits a divergence as  $u$  approaches zero with  $x$  scaled appropriately. Thus the self-similarity arises in much the same way as in blow-up problems. Given a profile of  $u$  as a function of  $t$  with  $x$  scaled appropriately, one can rescale  $t$  and the profile would look almost identical provided the size of  $u$  is reduced by the appropriate factor. The self-similarity is asymptotic in that the transformation is only approximate for any finite  $t$ , but the error vanishes in the limit as  $t \rightarrow \infty$ . It is in this sense that the classical applied mathematical techniques of asymptotic methods (involving small  $1/t$ ) can be used in conjunction with modern physical methods of renormalization to provide a powerful tool for analytic computation.

Calculation of nonclassical exponents in the absence of stochastics using renormalization methods was first done for the porous medium equation

$$u_t - \frac{1}{2}u_{xx} = \frac{\epsilon}{2}H(-u_{xx})u_{xx} \quad (1.3)$$

$$H(z) := \frac{1}{2}\{z/|z| + 1\}, \quad \epsilon > 0,$$

by Goldenfeld, Martin, Oono and Liu [1990]. These methods were extended in Caginalp [1996] to study a class of parabolic equations

$$u_t - \frac{1}{2}u_{xx} = \epsilon N(x, u_x, u_{xx}) \quad (1.4)$$

with dimensional consistency constraints imposed on  $N$ . In addition the methodology was rendered systematic within the context of asymptotic analysis. The close link between blow-up and asymptotic decay was also discussed in the context of scaling and renormalization. The ideas of renormalization have also been useful in obtaining existence proofs for nonlinear parabolic equations (see Bricmont, Kupianen and Lin [1994]).

Renormalization and scaling techniques have also been applied to stochastic differential equations. For example, see Glimm, Zhang and Sharp [1991] for chaotic mixing of interfaces, Zhang [1995] for random velocity fields and Avallaneda and Majda [1994] for stochastic and turbulent transport, and references contained therein.

Structural stability problems of propagating fronts have been investigated by Paquette et al [1994] and Paquette and Oono [1994].

In this paper we discuss a large class of systems of nonlinear parabolic equations with (possibly) different diffusion time scales. Specifically, we consider the vector generalization (2.1) of (1.4) for  $u_i (i = 1, \dots, M)$  and compute the exponent and scaling form of the leading term of the decay arising from a narrowly peaked Gaussian (of width  $l$ ) as an initial condition. The methodology consists of two basic parts. In Section 2 we obtain an asymptotic expression for the leading order behavior in  $l^{-1}$  of the form

$$u_i(x, t'; l, \epsilon) \sim u_i^{(0)}(x, t'; l) + \epsilon u_i^{(1)}(x, t'; l) + \dots \quad (1.5)$$

where  $t'$  is the original time variable (which is rescaled in the analysis). Here the term on the right hand side is not an approximation to the entire function but rather to the most singular term in  $l$ . Thus  $u_i^{(1)}$  is the part of the  $O(\epsilon)$  term that will dominate simultaneous expansions involving small values of  $\epsilon, l, 1/t$  and  $1/x$ .

In Section 3, the information in (1.5) is utilized along the lines of renormalization group ideas by writing  $u_i(x, t, l, \epsilon)$  in the form (suppressing  $l$  and  $\epsilon$  dependence)

$$u(x, t') = Z(b)^{-1}u(b^{1/2}x, bt') =: \mathcal{R}_{b,1/2}u(x, t') \quad (1.6)$$

for a suitable function  $Z$ , for all  $b > 1$ .

Upon iterating this transformation  $k$  times and taking the limit as  $k \rightarrow \infty$ , one finds that a fixed point will exist only if

$$u_i^*(x, t') = \lim_{k \rightarrow \infty} Z(b)^{-k}u_i(b^{k/2}x, b^k t') \quad (1.7)$$

is well defined. This relation then leads to the anomalous exponent  $\alpha$  through the relation

$$\lim_{k \rightarrow \infty} \left\{ Z \left\{ \left( \frac{b^k t'}{Q_1^2/D} \right)^{1/k} \right\} \right\}^k \sim (Const)(b^k t')^{-\frac{1}{2}-\alpha} \quad (1.8)$$

and  $u_i^*$  is obtained through (1.7).

This procedure for obtaining the exponent and scaling relation is thereby established within a standard applied mathematical context, and could be used in other systems of equations.

**2. ASYMPTOTICS OF NONLINEAR PARABOLIC EQUATIONS.** Let  $\epsilon$  be a small, positive, dimensionless number and consider the following system of equations for  $u_i$  ( $i = 1, \dots, M$ ) in which subscripts  $t'$  (which is the original time variable) and  $x$  denote differentiation:

$$C_i u_{i t'} = K_i \{ u_{i x x} + \epsilon N_i [x, u_i, u_{i x}, u_{i x x}] \} \quad i = 1, \dots, M \quad (2.1)$$

where  $C_i$  and  $K_i$  are constant and  $N_i$  is of the form

$$N_i [x, u_i, u_{i x}, u_{i x x}] = \sum_{\text{ind}} B_i(m_i, n_{ij}, p_{ij}, q_{ij}) x^{m_i} \prod_j u_j^{n_{ij}} u_{j x}^{p_{ij}} u_{j x x}^{q_{ij}}$$

where the sum is over the values of the indices  $m_i, n_{ij}$ , etc. We restrict our attention to those values of these indices that preserve dimensional consistency of (2.1) in terms of  $x$  and  $u_i$ . With the notation  $\sum_j n_{ij} =: n_i$ ,  $\sum_j p_{ij} =: p_i$ ,  $\sum_j q_{ij} =: q_i$  we impose the following.

**Condition A. Consistency in  $x$ :** For each  $i$ , we require

$$-m_i + p_i + 2q_i = 2 \quad (2.2)$$

**Condition B. Consistency in dimension of  $u_i$ :** For each  $i$ , we require

$$n_i + p_i + q_i = 1. \quad (2.3)$$

Note that if the  $u_i$  are not of the same dimensions then the required consistency is a subset of Condition B, namely

$$n_{ij} + p_{ij} + q_{ij} = \delta_{ij}$$

We also impose the following conditions which guarantee the boundedness of a key integral.

**Condition C:** For each  $i$ , we require  $q_{ij} \geq 0$  and

$$p_i + q_i - 1 \geq 0. \quad (2.4)$$

**Condition D:** for each  $i$ , we require

$$n_{i1} + p_{i1} + q_{i1} > 0. \quad (2.5)$$

Notation: We label the  $u_i$  so that  $D_1 := K_1/C_1$  is the minimum of the  $D_i := K_i/C_i$  and let  $\gamma_i := D_1/D_i$ . Then rescaling time by defining  $t := 2D_1t'$  (so that  $t$  has units of  $[length]^2$ ), we write (2.1) as

$$\frac{\partial u_i}{\partial(t/\gamma_i)} = \frac{1}{2} \frac{\partial^2 u_i}{\partial x^2} + \epsilon N_i \quad i = 1, \dots, M \quad (2.6)$$

with  $\gamma_i \in (0, 1]$  and  $\gamma_1 = 1$ .

We will carry out the full details of the calculation for the case  $M = 2$  and  $q_{ik(i)} > 0$ , for an arbitrary  $k(i)$ , with all other  $q_{ij} = 0$ . The general case is similar. The calculations concern the decay of a sharply peaked Gaussian so that the anomalous exponent and the leading term and scaling can be obtained.

We impose the initial conditions

$$u_i(x, 0) := g(x, l) := \frac{Q_0}{(2\pi l^2)^{1/2}} \exp\left(\frac{-x^2}{2l^2}\right) \quad (2.7)$$

where  $Q_0 := T_0 Q_1$  with  $T_0$  and  $Q_1$  having units of temperature and length, respectively. Our interest is in sharply peaked Gaussian initial profiles so that the length scale  $l =: L^{1/2}$  will be small, in addition to  $\epsilon$ . The subtleties of this double expansion will be discussed in later in the section.

Using the Green's function

$$G(x, t) := (2\pi t)^{-1/2} \exp\left(\frac{-x^2}{2t}\right) \quad (2.8)$$

and treating the nonlinearity as a source term we can express the solution of  $u_i$  in terms of  $\hat{t}_i := t/\gamma_i$  and  $\hat{u}_i(x, \hat{t}_i) := u_i(x, t)$  as

$$\hat{u}_i(x, \hat{t}_i) = \int_{-\infty}^{\infty} dy G(x-y, \hat{t}_i) g(y) + \epsilon \int_0^{\hat{t}_i} ds \int_{-\infty}^{\infty} dy G(x-y, \hat{t}_i - s) N_i[y, u_i(y, s), \dots] \quad (2.9)$$

The first integral in (2.9), namely the  $O(1)$  term in expansion in  $\epsilon$ , is computed as

$$u_i^{(0)}(x, t) = Q_0 \left(\frac{\gamma_i}{2\pi}\right)^{1/2} (t + \gamma_i l^2)^{-1/2} \exp\left(\frac{-\gamma_i x^2}{2(t + \gamma_i l^2)}\right) \quad (2.10)$$

in the rescaled  $t$  variable and has derivatives

$$u_{ix}^{(0)}(x, t) = \left( \frac{-x}{t/\gamma_i + l^2} \right) u_i^{(0)}(x, t)$$

$$u_{ixx}^{(0)}(x, t) = \frac{1}{t/\gamma_i + l^2} \left\{ \frac{x^2}{t/\gamma_i + l^2} - 1 \right\} u_i^{(0)}(x, t). \quad (2.11)$$

One then has a relation that can be used to compute the effects of  $u_{ixx}$  terms as a combination of  $u_i$  and  $u_{ix}$ , namely,

$$u_{ixx}^{(0)} = -\frac{(u_i^{(0)} + x u_{ix}^{(0)})}{t/\gamma_i + l^2}. \quad (2.12)$$

The second term in (2.9) can be written as

$$\epsilon u_i^{(1)}(x, t) \sim \epsilon \sum_{\text{ind}} B_i(\text{ind}) \int_0^{t/\gamma_i} ds \int_{-\infty}^{\infty} G(x-y, t/\gamma_i - s) dy$$

$$\cdot y^{m_i} u_1^{n_{i1}} u_2^{n_{i2}} \dots u_{1y}^{p_{i1}} u_{2y}^{p_{i2}} \dots u_{1yy}^{q_{i1}} u_{2yy}^{q_{i2}} \dots \quad (2.13)$$

where the  $u_j$  are evaluated at  $(s, y)$ .

We employ the standard asymptotic procedure of using the  $O(1)$  order terms,  $u_i^{(0)}$ , etc., to generate the  $O(\epsilon)$  terms, since the neglected terms are thereby formally  $O(\epsilon^2)$ . Hence, the latter part of the integrand of (2.13) can be written as:

$$y^{m_i} u_1^{n_{i1}} u_2^{n_{i2}} \dots u_{1y}^{p_{i1}} u_{2y}^{p_{i2}} \dots u_{1yy}^{q_{i1}} u_{2yy}^{q_{i2}}$$

$$\cong \frac{Q_0}{(2\pi)^{1/2}} (-1)^{p_i} y^{m_i + p_i} \prod_{\tau=1}^M \exp \left\{ -\frac{(n_{i\tau} + p_{i\tau} + q_{i\tau}) y^2}{2(s/\gamma_\tau + l^2)} \right\}. \quad (2.14)$$

$$\cdot \left( \frac{1}{s/\gamma_\tau + l^2} \right)^{\frac{1}{2}(n_{i\tau} + 3p_{i\tau} + 3q_{i\tau})} \left( \frac{y^2}{s/\gamma_\tau + l^2} - 1 \right)^{q_{i\tau}}$$

Since we are interested in the singular behavior at large times as  $l$  approaches zero we can approximate  $t/\gamma_i - s$  by  $t/\gamma_i$  and  $x-y$  by  $x$  in (2.13), to obtain

$$u_i^{(1)}(x, t) \sim \sum_{\text{ind}} B_i(\text{ind}) \int_0^{t/\gamma_i} ds \int_{-\infty}^{\infty} dy (2\pi t/\gamma_i)^{-1/2} \exp \left( \frac{-x^2}{2t/\gamma_i} \right) (-1)^{p_i} \frac{Q_0}{(2\pi)^{1/2}} y^{m_i + p_i}$$

$$\prod_{\tau=1}^M \exp \left\{ \frac{-(n_{i\tau} + p_{i\tau} + q_{i\tau}) y^2}{2(s/\gamma_\tau + l^2)} \right\} \left( \frac{1}{s/\gamma_\tau + l^2} \right)^{(n_{i\tau} + 3p_{i\tau} + 3q_{i\tau})/2} \left( \frac{y^2}{s/\gamma_\tau + l^2} - 1 \right)^{q_{i\tau}} \quad (2.15)$$

The justification for the approximation

$$\exp\left(\frac{-(x-y)^2}{2(t/\gamma_i - s)}\right) \sim \exp\left(\frac{-\gamma_i x^2}{2t}\right)$$

is based upon Laplace's method for integrals (see Erdelyi [1956] p. 36). Since the main contribution to the integral must arise from the regions near  $y = 0$  and  $s = 0$  for small  $l$ , the two expressions lead to the same coefficient of the singularity in  $l^{-1}$ .

Using  $I_1$  to denote the integrand of the  $s$  integral and  $I_{11}$  to denote the  $y$  integral, we write

$$I_1 := \prod_{j=1}^M \left( \frac{1}{s/\gamma_j + l^2} \right)^{\frac{1}{2}(n_{ij} + 3p_{ij} + 3q_{ij})} I_{11},$$

$$I_{11} := \int_{-\infty}^{\infty} dy y^{m_i + p_i} \prod_{\tau=1}^M \exp\left\{ \frac{-(n_{i\tau} + p_{i\tau} + q_{i\tau})y^2}{2(s/\gamma_\tau + l^2)} \right\} \left( \frac{y^2}{s/\gamma_\tau + l^2} - 1 \right)^{q_{i\tau}} \quad (2.16)$$

so that  $u_i^{(1)}(x, t)$  can be expressed as

$$u_i^{(1)}(x, t) \sim \sum_{\text{ind}} B_i (-1)^{p_i} \frac{Q_0}{2\pi} \gamma_i^{1/2} t^{-1/2} \exp\left(\frac{-\gamma_i x^2}{2t}\right) \int_0^{t/\gamma_i} I_1(s, x) ds \quad (2.17)$$

**Evaluation of  $I_{11}$**  : The exponential product of (2.16) can be written as

$$\prod_{\tau=1}^M \exp\left\{ \frac{-(n_{i\tau} + p_{i\tau} + q_{i\tau})}{2(l s/\gamma_\tau + l^2)} \right\} = \exp(-a^2 y/2)$$

$$a := \sum_{\tau} \left( \frac{n_{i\tau} + p_{i\tau} + q_{i\tau}}{s/\gamma_\tau + l^2} \right) \quad (2.18)$$

Using the transformation  $w := a^2 y$  we write  $I_{11}$  as

$$I_{11} = a^{-m_i - p_i - 1} \int_{-\infty}^{\infty} dw w^{m_i + p_i} e^{-w^2/2} \prod_{r=1}^M \left( \frac{w^2}{a^2(s/\gamma_r + l^2)} - 1 \right)^{q_{ir}} \quad (2.19)$$

In the general case one can expand  $\left( \frac{w^2}{a^2(s/\gamma_r + l^2)} - 1 \right)^{q_{ir}}$  in a Taylor series and multiply out. To avoid tediously long expressions, we first restrict attention to the case in which  $q_{ir} = 0$  except for  $q_{ik(i)}$  which may be any integer. In that case (2.19) becomes

$$I_{11} = a^{-m_i - p_i - 1} \sum_{z=0}^{q_{ik(i)}} \binom{q_{ik(i)}}{z} (-1)^{q_{ik(i)} - z} \frac{1}{a^{2z}(s/\gamma_{k(i)} + l^2)^z} \cdot \int_{-\infty}^{\infty} dw e^{-w^2/2} w^{m_i + p_i + 2z} \quad (2.20)$$

The integral can be expressed in terms of the Gamma function as

$$\mathcal{N}(j) := \int_{-\infty}^{\infty} dw \cdot w^j e^{-w^2/2} = 2^{-(j+1)/2} \Gamma \left[ \frac{j+1}{2} \right] = 1 \cdot 3 \cdot 5 \cdots (j-1) \sqrt{2\pi} \quad j \in \mathbf{Z}^+, \text{ even} \quad (2.21)$$

As mentioned earlier, we now let  $M := 2$  to allow for less cumbersome formulae. Then using (2.16) and (2.20) we write  $I_1$  as

$$I_1 = \sum_{z=0}^{q_{ik(i)}} ((-1)^{q_{ik(i)}-z} \binom{q_{ik(i)}}{z}) \mathcal{N}(m_i + p_i + 2z) a^{-m_i - p_i - 2z - 1} \cdot \left( \frac{1}{s/\gamma_{k(i)} + l^2} \right)^z \left( \frac{1}{s/\gamma_1 + l^2} \right)^{\frac{1}{2}(n_{i1} + 3p_{i1} + 3q_{i1})} \left( \frac{1}{s/\gamma_2 + l^2} \right)^{\frac{1}{2}(n_{i2} + 3p_{i2} + 3q_{i2})} \quad (2.22)$$

The key step at this stage is to understand how the  $s$  integral of  $I_1$  leads to the singular behavior in  $l$ . For brevity we define (using Condition B)

$$b := n_{i1} + p_{i1} + q_{i1} \quad \text{so} \quad 1 - b = n_{i2} + p_{i2} + q_{i2} \\ c^{-1} := b(1/\gamma_2 - 1) + 1 \quad (2.23)$$

so that  $a^2$ , defined by (2.18), can be written as

$$a^2 = \frac{s/c + L}{(s/\gamma_1 + L)(s/\gamma_2 + L)} \quad (2.24)$$

(Recall that  $L := l^2$  and  $\gamma_1 := 1$ ).

Note that we can write

$$\frac{1}{s/\gamma_{k(i)} + l^2} = \left( \frac{1}{s/\gamma_1 + l^2} \right)^{z\delta(k(i)-1)} \left( \frac{1}{\gamma_2 + l^2} \right)^{z\delta(k(i)-2)} \quad (2.25)$$

so the  $s$ -dependent part of  $I_1$ , which we denote by  $I$ , is

$$I = \left( \frac{c}{s + cL} \right)^{\frac{1}{2}(n_{i1} + 3p_{i1} + 3q_{i1})} \left( \frac{1}{s + L} \right)^{\frac{1}{2}(n_{i1} + 3p_{i1} + 3q_{i1}) + z\delta(k(i)-1) - \frac{1}{2}(m_i + p_i + 2z + 1)} \cdot \left( \frac{\gamma_2}{s + \gamma_2 L} \right)^{\frac{1}{2}(n_{i2} + 3p_{i2} + q_{i2}) + z\delta(k(i)-2) - \frac{1}{2}(m_i + p_i + 2z + 1)} \quad (2.26)$$

We expand the first and third terms in a Taylor series

$$\frac{\gamma_2}{s + \gamma_2 L} = \frac{\gamma_2}{s + L} \left\{ 1 - \frac{(\gamma_2 - 1)L}{s + L} + \frac{(\gamma_2 - 1)^2 L}{(s + L)^2} \cdots \right\} \quad (2.27)$$

Since  $s > 0$  and  $\gamma_2 \in (0, 1)$  this series converges as  $|(\gamma_2 - 1)L/(s + L)| < 1$ . Similarly

$$\frac{c}{s + cL} = \frac{c}{s + L} \left\{ 1 - \frac{(c-1)L}{s+L} + \frac{(c-1)^2}{(s+L)^2} \dots \right\} \quad (2.28)$$

provided  $|c - 1| < 1$  which is equivalent to  $b > 0$ , i.e. Condition D.

As a consequence of Conditions A and B the expression (2.26) will lead to  $(s + L)^{-1}$  multiplied by powers of the terms  $\{\dots\}$  in (2.27) and (2.28). Only the leading term in these series makes a contribution to the leading order singularity for small  $l$ , as one has

$$\begin{aligned} \int_0^t \frac{ds}{s+L} \left\{ 1 + \left( \frac{L}{s+L} \right)^\beta \right\} &= \left[ \ln(s+L) + \frac{(1-\beta)L^\beta}{(s+L)^{\beta-1}} \right]_0^t \\ &= \ln(t+L) - \ln L + (1-\beta)L^\beta \left\{ \left( \frac{1}{t+L} \right)^{\beta-1} - \left( \frac{1}{L} \right)^{\beta-1} \right\} \end{aligned} \quad (2.29)$$

for  $\beta \geq 1$ .

Consequently, we can ignore the remaining terms in these series since our aim is to calculate the coefficient of the singular term. The terms that contribute to the leading order in  $I$  are

$$I \sim \left( \frac{c}{\gamma_2} \right)^{\frac{1}{2}(m_i+p_i)+z+\frac{1}{2}} \left( \frac{1}{s+L} \right)^{\frac{1}{2}(n_i+3p_i+3q_i)+z-\frac{1}{2}(m_i+p_i+1)-z} \quad (2.30)$$

The exponent of  $1/(s + L)$  above can be simplified using Conditions A and B so that

$$\frac{1}{2}(n_i + p_i + q_i) - \frac{1}{2} + \frac{1}{2}(-m_i + p_i + 2q_i) = \frac{1}{2}(1) - \frac{1}{2} + \frac{1}{2}(2) = 1.$$

Then one has the following expressions for  $I$  and  $I_1$  from (2.30) and (2.22):

$$I \sim \left( \frac{c}{\gamma_2} \right)^{\frac{1}{2}(m_i+p_i)+z-\frac{1}{2}} \frac{1}{s+L} \quad (2.31)$$

$$I_1 \sim \sum_{z=0}^{q_{ik(i)}} (-1)^{q_{ik(i)}} (-z) \binom{q_{ik(i)}}{z} \mathcal{N}(m_i + p_i + 2z) \left( \frac{c}{\gamma_2} \right)^{\frac{1}{2}(m_i+p_i)+z+\frac{1}{2}} \frac{1}{s+L}$$

Upon performing the  $s$ -integral, which now has just one term  $(s + L)^{-1}$ , and substituting for  $I_1$  in (2.17) we obtain the expression

$$u_i^{(1)}(x, t) \sim \sum_{\text{ind}} B^i(\text{ind}) (-1)^{p_i} \frac{Q_0}{2\pi} \gamma_i^{-\frac{1}{2}} t^{-\frac{1}{2}} \exp\left( \frac{-\gamma_i x^2}{2t} \right) \quad (2.32)$$

$$\sum_{z=0}^{q_{ik(i)}} (-1)^{q_{ik(i)}-z} \binom{q_{ik(i)}}{z} \mathcal{N}(m_i + p_i + 2z) \left( \frac{c}{\gamma_2} \right)^{\frac{m_i}{2} + \frac{p_i}{2} + z + \frac{1}{2}} \ln\left( \frac{t/\gamma_i + l^2}{l^2} \right)$$

where  $\mathcal{N}(m_i + p_i + 2z)$  is defined by (2.21).

Using the expression (2.10) along with (2.32) and  $u_i \simeq u_i^{(0)} + \epsilon u_i^{(1)}$  one obtains an asymptotic expression for the leading order in  $l^{-1}$ .

We now take an example of mixed second derivatives so that the restriction on  $q_{ij}$  after (2.19) is eliminated.

**Cross terms such as  $u_{1xx} u_{2xx}$ :** We take as an example

$$q_{11} = q_{12} = q_{21} = q_{22} = 1 \quad (2.33)$$

In this case the term  $I_{11}$  becomes

$$I_{11} \sim a^{-1} \int_{-\infty}^{\infty} dw \cdot w^{m_i+p_i} a^{-m_i-p_i} e^{w^2/2} \prod_{j=1}^2 \left( \frac{w^2}{a^2(s/\gamma_i + l^2)} - 1 \right) \quad (2.34)$$

Again we can write the terms  $(s/\gamma_i + l^2)^{-1} = \gamma_i(s/\gamma_i + L)^{-1}$  as  $\gamma_i(s + L)^{-1}$  which results in the same singular behavior in  $L$ , to obtain the following expression for  $I_{11}$ :

$$\begin{aligned} I_{11} &\sim a^{-m_i-p_i-1} \int_{-\infty}^{\infty} dw \cdot w^{m_i+p_i} e^{-w^2/2} \left\{ \frac{\gamma_2 w^4}{a^4(s+L)} - \frac{(1+\gamma_2)w^2}{a^2(s+L)} + 1 \right\} \\ &= \gamma_2 \frac{a^{-m_i-p_i-5}}{(s+L)^2} \mathcal{N}(m_i+p_i+4) - a^{-m_i-p_i-1} \mathcal{N}(m_i+p_i) \\ &\quad - (1+\gamma_2) \frac{a^{m_i-p_i-3}}{s+L} \mathcal{N}(m_i+p_i+2) \end{aligned} \quad (2.35)$$

Returning to  $I_1$  in (2.16) we write

$$I_1 \sim \left( \frac{1}{s+L} \right)^{(n_{i1}+p_{i1}+3)/2} \left( \frac{1}{s/\gamma_2+L} \right)^{(n_{i2}+p_{i2}+3)/2} I_{11} \quad (2.36)$$

In terms of the singularity in  $l$  we can rewrite  $a^2$  in (2.18) as

$$a^2 = \left( \frac{c}{s+cL} \right)^{-1} \left( \frac{1}{s+L} \right) \left( \frac{\gamma_2}{s+\gamma_2 L} \right) \sim \frac{\gamma_2}{c} \frac{1}{s+L} \quad (2.37)$$

and similarly with  $(s/\gamma_i + L)^{-1}$  terms, etc., to obtain

$$\begin{aligned} I_{11} &\sim \left\{ \gamma_2 \left( \frac{\gamma_2}{c} \right)^{\frac{1}{2}(m_i+p_i+5)} \mathcal{N}(m_i+p_i+4) - (1+\gamma_2) \left( \frac{\gamma_2}{c} \right)^{-\frac{1}{2}(m_i+p_i+3)} \mathcal{N}(m_i+p_i+2) \right. \\ &\quad \left. + \left( \frac{\gamma_2}{c} \right)^{\frac{1}{2}} \mathcal{N}(m_i+p_i) \right\} \left( \frac{1}{s+L} \right)^{-\frac{1}{2}(m_i+p_i+1)} \end{aligned} \quad (2.38)$$

$$\begin{aligned}
I_1 &\sim \gamma_2^{\frac{1}{2}(n_{i2}+3p_{i2}+3)} \left\{ \gamma_2 \left( \frac{\gamma_2}{c} \right)^{-\frac{1}{2}(m_i+p_i+5)} \mathcal{N}(m_i+p_i+4) \right. \\
&- (1+\gamma_2) \left( \frac{\gamma_2}{c} \right)^{-\frac{1}{2}(m_i+p_i+3)} \mathcal{N}(m_i+p_i+2) + \left( \frac{\gamma_2}{c} \right)^{\frac{1}{2}} \mathcal{N}(m_i+p_i) \left. \right\} \\
&\cdot \left( \frac{1}{s+l^2} \right)^{-\frac{1}{2}(m_i+p_i+1)+\frac{1}{2}(n_i+3p_i)+3} \tag{2.39}
\end{aligned}$$

Using the consistency relations A and B, we simplify the exponent in (2.39) as

$$-\frac{1}{2}(m_i+p_i+1)+\frac{1}{2}(n_i+3p_i)+3 = \frac{n_i+p_i+2}{2} + \frac{p_i}{2} - \frac{m_i}{2} + \frac{3}{2} = \frac{1}{2} + \frac{p_i}{2} - \frac{m_i}{2} + \frac{3}{2} = 1 \tag{2.40}$$

since  $q_i = 2$ . Upon performing the  $s$ -integral we can evaluate  $u_i^{(1)}(x, t)$  in (2.17) as

$$\begin{aligned}
u_i^{(1)}(x, t) &\sim \sum_{\text{ind}} B_i(\text{ind})(-1)^{p_i} \frac{Q_0}{2\pi} \gamma_i^{-\frac{1}{2}} t^{-\frac{1}{2}} \exp\left(-\frac{\gamma_i x^2}{2t}\right) \gamma_2^{\frac{1}{2}(n_{i2}+3p_{i2}+3)} \\
&\left\{ \gamma_2 \left( \frac{\gamma_2}{c} \right)^{-\frac{1}{2}(m_i+p_i+5)} \mathcal{N}(m_i+p_i+4) - (1+\gamma_2) \left( \frac{\gamma_2}{c} \right)^{-\frac{1}{2}(m_i+p_i+3)} \right\} \mathcal{N}(m_i+p_i+2) \\
&+ \left( \frac{\gamma_2}{c} \right)^{\frac{1}{2}} \mathcal{N}(m_i+p_i) \ln\left(\frac{t/\gamma_i+l^2}{l^2}\right) \tag{2.41}
\end{aligned}$$

Using (2.10) with this expression and  $u_i \simeq u_i^{(0)} + \epsilon u_i^{(1)}$  one obtains an asymptotic expansion for leading order behavior in  $l^{-1}$ , in the original time unit,  $t'$ , as

$$u_i(x, t'; l, \epsilon) \sim \frac{T_0}{2\pi^{1/2}} \left( \frac{t'}{Q_1^2/D_i} \right)^{-1/2} \exp\left(\frac{-x^2}{4D_i t'}\right) \left\{ 1 + \epsilon A_i \log\left(\frac{2D_i t'}{l^2}\right) \right\} \tag{2.42}$$

$$\begin{aligned}
A_i &:= (2\pi)^{-1/2} \sum_{\text{ind}} B_i(\text{ind})(-1)^{p_i} \gamma_2^{\frac{1}{2}(n_{i2}+3p_{i2}+3)} \\
&\cdot \left\{ \gamma_2 \left( \frac{\gamma_2}{c} \right)^{-\frac{1}{2}(m_i+p_i+5)} \mathcal{N}(m_i+p_i+4) - (1+\gamma_2) \left( \frac{\gamma_2}{c} \right)^{-\frac{1}{2}(m_i+p_i+3)} \right\} \mathcal{N}(m_i+p_i+3) \\
&+ \left( \frac{\gamma_2}{c} \right)^{\frac{1}{2}} \mathcal{N}(m_i+p_i) \tag{2.43}
\end{aligned}$$

The analysis of this section involves approximations along the lines of (i) small  $\epsilon$ , (ii) small  $l$ , (iii) large  $t$  and  $x$ . The RG calculation of the next section will involve the additional approximation in the fixed point in that (iv) large  $k$  is used to approximate  $\infty$ . The interactions between these approximations presents a number of subtleties that have been discussed in detail in Caginalp [1996]. Briefly, the examination of large  $l$  is within a particular order in  $\epsilon$ . Stated differently, one can take  $l$  to zero more slowly than  $\epsilon$  in

order to avoid the possibility of cancellation of the singularity in  $l^{-1}$ . The approximations involving large  $t$  and  $x$  simplify the integrands and allow for evaluation near the lower limit of the integral.

**3. RENORMALIZATION AND SCALING ANALYSIS.** Given an asymptotic expression such as (2.42) one can use a precisely defined set of procedures in order to extract the leading order behavior and the decay exponent in  $t^{-1/2-\alpha}$ . The idea is based on rescaling the variables  $t$ ,  $x$ , and  $u$  for large  $t$  so that a limiting self-similarity is obtained in terms of a solution to a fixed point problems. The methodology is similar to Caginalp [1996] and is close in spirit to those of Goldenfeld, Martin, Oono and Liu [1990] and Creswick, Farach and Poole [1992].

The results can be expressed in a more general form than needed for our purposes, so any function in a particular asymptotic form has a characteristic exponent that can be determined through this sequence of transformations. Of course this means that a large class of parabolic systems share the same characteristic exponent that is determined by the value of the integral (2.43).

The objective of this section is to convert information involving the product of a power-law decay and an asymptotic expression into a modified power-law decay. The simplest perspective into the calculations is through the Taylor approximation

$$t^{-1/2}(1 + \epsilon A \log t) \cong t^{-1/2+\epsilon A}$$

which is ultimately responsible for the anomalous exponent. The transformations are dependent simply upon the form of the functions  $u_i$  and do not involve the differential equations directly at this point. This means that the methods can be employed in a broad class of problems for which such an asymptotic relation can be obtained.

**Proposition 3.1.** Suppose  $u_i$  ( $i = 1, \dots, M$ ) can be expressed in the form (in the original time variable  $t'$ )

$$u_i(x, t'; l, \epsilon) = \frac{T_0}{2\pi^{1/2}} \left( \frac{t'}{Q_1^2/D_i} \right)^{-1/2} \gamma_i^{1/2} \exp \left( \frac{-\gamma_i x^2}{4D_i t'} \right) \left\{ 1 + \epsilon A_i \ln \left( \frac{2D_i t'}{\gamma_i l^2} \right) \right\} \quad (3.1)$$

where  $A_i$  does not depend upon  $x, t', l$  or  $\epsilon$  and  $\epsilon A_i \log(2D_i t'/l^2) \ll 1$ . Then to leading order in  $\epsilon$ , one can write the dominant part of  $u_i$  as

$$u_i(x, t'; l, \epsilon) = \left( \frac{t'}{Q_1^2/d} \right)^{-1/2+\epsilon A_i} u_i^* \left( \frac{\gamma_i^{1/2} x}{(D_i t'/Q_1^2)^{1/2}}, \frac{Q_1^2}{D_i} \right) \quad (3.2)$$

so that the anomalous exponent is given by  $\alpha_i = -\epsilon A_i$ . The fixed point function  $u^*$  is given by

$$u^*(\zeta, \tau) = \frac{T_0 \gamma_i^{1/2}}{2\pi^{1/2}} \exp \left( \frac{-\zeta^2}{4D_i \tau} \right) \left\{ 1 + \epsilon A_i \log \left( \frac{2D_i}{l^2} \tau \right) \right\} \quad (3.3)$$

**Verification:** The derivation consists of several transformations that are valid through  $O(\epsilon)$ . Like the asymptotic analysis of the preceding section, this procedure is fairly general and may be applied to other systems of equations. Dependence of  $u_i$  on  $l$  and  $\epsilon$  is suppressed whenever possible.

**Stage 1: The RG equation.** The key step is to obtain an identity valid through  $O(\epsilon)$ , of the form

$$u_i(b^\phi x, bt') = Z_i(b)u_i(x, t') \quad (3.4)$$

that is valid for a particular choice of  $Z_i$  and  $\phi$  and for **all**  $b > 1$ . The form of (3.1) clearly forces  $\phi = \frac{1}{2}$ . To determine a function  $Z$  that will satisfy (3.4), one can write  $u(b^{1/2}x, bt')$  as

$$u_i(b^{1/2}x, bt') = \frac{T_0}{2\pi^{1/2}} b^{-1/2} \left( \frac{t'}{Q_1^2/D_i} \right)^{-1/2} \gamma_i^{1/2} \exp\left(\frac{-\gamma_i x^2}{4D_i t'}\right) \left\{ 1 + \epsilon A_i \log b \right\} \left\{ 1 + \epsilon A_i \log \left( \frac{2D_i t'}{\gamma_i t^2} \right) \right\} + O(\epsilon^2) \quad (3.5)$$

Then  $Z_i(b)$  can be defined (independently of  $l$ ) as

$$Z_i(b) := b^{-1/2}(1 + \epsilon A_i \log b) \quad (3.6)$$

so that the renormalization operator  $R_{b,\phi}$  (see Creswick et. al. [1992]) can be defined by

$$R_{b,\phi} u_i(x, t') = Z_i(b)u_i(b^{1/2}x, bt') \quad (3.7)$$

**Stage 2: The asymptotic fixed point.** Iterating the relation (3.4)  $k$  times one has

$$u_i(b^{k/2}x, b^k t') = Z(b)^k u_i(x, t'), \quad (3.8)$$

so that a fixed point of the iteration will exist only if the limit

$$u_i^*(x, t') := \lim_{k \rightarrow \infty} Z(b)^{-k} u_i(b^{k/2}x, b^k t') \quad (3.9)$$

is well defined. Assuming the existence of a fixed point, one can write (3.9) for large but finite  $k$  as

$$u_i(b^{k/2}x, b^k t') \simeq Z(b)^k u_i^*(x, t'). \quad (3.10)$$

The validity of equation (3.10) for large  $k$ ,  $x$  and  $t'$  implies the validity (set  $\bar{x} := b^{k/2}x$  and  $\bar{t} := b^k t'$ ) for large  $k$ ,  $\bar{x}$ ,  $\bar{t}$ , and any  $b > 1$ , of the equation

$$u_i(\bar{x}, \bar{t}) \simeq Z_i(b)^k u_i^*(b^{-k/2}\bar{x}, b^{-k}\bar{t}). \quad (3.11)$$

The arbitrariness of  $b$  means that  $u_i$  can be evaluated at any argument  $(\bar{x}, \bar{t})$  by setting  $b^k := \bar{t}/(Q_1^2/D_i)$ , so that the second argument remains unchanged as we examine different

values of  $(\bar{x}, \bar{t})$ . Hence, one can determine the entire domain in  $\mathbf{R}^2$  through a one-variable function  $u_i^*(\cdot, Q_1^2/D_i)$  as follows. With  $b$  set at this value, we write (3.11) as

$$u_i(\bar{x}, \bar{t}) \simeq \left[ Z_i \left\{ \left( \frac{\bar{t}}{Q_1^2/D_i} \right)^{1/k} \right\} \right]^k u_i^* \left( \frac{\bar{x}}{(D_i \bar{t}/Q_1^2)^{1/2}}, \frac{Q_1^2}{D_i} \right) \quad (3.12)$$

for large  $k$ ,  $\bar{x}$ , and  $\bar{t}$ . Note that  $Q_1^2/D_i$  serves to maintain the time unit and could be set to unity if dimensions are ignored. At this point could replace  $(\bar{x}, \bar{t})$  by  $(x, t')$  in (3.12) since both sets of variables are dimensional.

The significance of (3.12) is due to the expression of  $u_i(\bar{x}, \bar{t})$  as a product of a function of  $\bar{t}$  (which will determine the exponent) and a scaling function  $u_i^*$  of a single scaled variable. In the next two stages we determine these explicitly.

**Stage 3: The anomalous exponent.** The characteristic exponents with which the functions  $u_i$  decay in time is given by the fixed point attained in the limit of infinitely many iterations  $k$ :

$$\lim_{k \rightarrow \infty} \left[ Z \left\{ \left( \frac{\bar{t}}{Q_1^2/D_i} \right)^{1/k} \right\} \right]^k = (\text{const}) \bar{t}^{-1/2 - \alpha_i} \quad (3.13)$$

To calculate the limit and hence the exponent  $\alpha_i$ , we let  $t_1 := D_i \bar{t}/Q_1^2$  and utilize the asymptotic approximation (for large  $k$ )

$$\left( 1 + \frac{\epsilon A_i}{k} \log t_1 \right)^k \simeq t_1^{\epsilon A_i} \quad (3.14)$$

in the definition of  $Z_i$  in (3.6) to obtain

$$\left[ Z_i \left( t_1^{1/k} \right) \right]^k = t_1^{-1/2} \left( 1 + \frac{\epsilon A_i}{k} \log t_1 \right)^k \simeq t_1^{-1/2 + \epsilon A_i} \quad (3.15)$$

Consequently, the anomalous exponents are calculated as  $\alpha_i = -\epsilon A_i$  since one obtains

$$\lim_{k \rightarrow \infty} \left\{ Z_i \left( \frac{D_i \bar{t}}{Q_1^2} \right)^{1/k} \right\}^k = \left( \frac{D_i \bar{t}}{Q_1^2} \right)^{-1/2 + \epsilon A_i} \quad (3.16)$$

**Stage 4: Explicit calculation of the scaling functions  $u_i^*$ .** Using the calculation (3.16) in (3.12) one may write, upon replacing  $(\bar{x}, \bar{t})$  by  $(x, t')$ ,

$$u_i(x, t') \simeq \left( \frac{D_i t'}{Q_1^2} \right)^{-1/2 + \epsilon A_i} u_i^* \left( \frac{x}{(D_i t'/Q_1^2)^{1/2}}, \frac{Q_1^2}{D_i} \right) \quad (3.17)$$

Also, one can write (3.1) as

$$u_i(x, t'; l, \epsilon) = \frac{T_0}{2\pi^{1/2}} \left( \frac{D_i t'}{Q_1^2} \right)^{1/2} \gamma_i^{1/2} \exp \left( \frac{-\gamma_i x^2}{4D_i t'} \right) \left\{ 1 + \epsilon A_i \log \left( \frac{D_i t'}{Q_1^2} \right) \right\} \left\{ 1 + \epsilon A_i \log \left( \frac{2Q_1^2}{\gamma_i l^2} \right) \right\} \quad (3.18)$$

Under the assumption  $\epsilon A_i \log (D_i t' / Q_1^2) \ll 1$  one has the approximation

$$1 + \epsilon A_i \log \left( \frac{D_i t'}{Q_1^2} \right) \simeq \exp \left\{ \epsilon A_i \log (D_i t' / Q_1^2) \right\} = \left( \frac{D_i t'}{Q_1^2} \right)^{\epsilon A_i}$$

so that (3.18) can be written in the form

$$u_i(x, t'; l, \epsilon) \simeq \frac{T_0}{2\pi^{1/2}} \left( \frac{D_i t'}{Q_1^2} \right)^{-1/2 + \epsilon A_i} \exp \left( \frac{-\gamma_i x^2}{4D_i t'} \right) \cdot \left\{ 1 + \epsilon A_i \log \left( \frac{D_i t'}{Q_1^2} \right) \right\} \left\{ 1 + \epsilon A_i \log \left( \frac{2Q_1^2}{\gamma_i l^2} \right) \right\} \quad (3.19)$$

A comparison of (3.19) with (3.17) immediately yields the scaling function  $u^*$  which we write in the following form to group together the arguments:

$$u_i^* \left( \frac{x}{(D_i t' / Q_1^2)^{1/2}}, \frac{Q_1^2}{D_i} \right) = \frac{T_0 \gamma_i^{1/2}}{2\pi^{1/2}} \exp \left\{ \frac{\gamma_i \left( \frac{x}{(D_i t' / Q_1^2)^{1/2}} \right)^2}{4D_i (Q_1^2 / D_i)} \right\} \cdot \left\{ 1 + \epsilon A_i \log \left( \frac{2Q_1^2}{D_i} \cdot \frac{D_i}{l^2} \right) \right\} \quad (3.20)$$

Hence, the function can be expressed simply as

$$u_i^*(\zeta, \tau) = \frac{T_0}{2\pi^{1/2}} \gamma_i^{1/2} \exp \left( \frac{-\gamma_i \zeta^2}{4D_i \tau} \right) \left\{ 1 + \epsilon A_i \log \left( \frac{2D_i}{l^2} \tau \right) \right\} \quad (3.21)$$

thereby verifying Proposition 3.1. ///

Combining the calculations of Section 2 with this result, one obtains the leading order asymptotic decay for solutions to the parabolic system of equations under consideration.

**Proposition 3.2.** The dominant component of solutions to the system of differential equations (2.1) subject to Conditions A-D and initial conditions (2.7) satisfy the asymptotic relation (3.17) with  $u^*$  given by (3.21) and  $A_i$  given by (2.43). ///

While the derivation reduces the calculation of exponents and scaling function  $u^*$  to simple arithmetic, the actual existence of the fixed point  $u^*$  is subtle and its existence is contingent upon the boundedness of the nonlinear terms that are small in terms of the formal analysis.

**5. CONCLUSION.** A systematic renormalization and scaling technique has been utilized to calculate nonclassical decay exponents for a large class of parabolic systems. These techniques are quite general and can be applied, for example, to systems in higher spatial dimensions. Within this setting the renormalization and scaling methods can be regarded as an extension of standard asymptotic analysis.

While the calculations have assumed a small  $\epsilon$ , there is reason to expect that the anomalous exponent will persist as  $\epsilon$  is increased in a continuous manner (Goldenfeld et al [1990], Caginalp [1996]). This would provide a universality principle for nonlinear differential equations that is analogous to that of critical phenomena in statistical mechanics. In a previous paper, blow-up in a single parabolic equation was calculated using these methods. The calculations can be generalized to other systems of differential equations using the techniques of Section 2.

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