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A RENORMALIZATION GROUP APPROACH TO BLOW-UP AND EXTINCTION EXPONENTS

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Abstract: Renormalization and scaling methods (RG) are applied systematically using asymptotic analysis to nonlinear differential equations involving blow-up and extinction. This complements earlier results involving decay and establishes a unified RG approach that facilitates calculation of the key exponents involved in these phenomena.

The methodology also demonstrates the importance of dimensional analysis in the development of the singularity. The analysis leads to an understanding of the influence of spatial dimensions as well as dimensional compatibility in nonlinear equations.

In particular, it shows that the addition of (possibly nonlinear) terms that are dimensionally compatible with the leading diffusion term in a parabolic equation often do not influence the exponent or coefficient of the singularity in the basic blow-up problems. Similarly, the renormalization and scaling methods demonstrate that the singularity arises in a way that is independent of other nonlinearities in the equations. Similar conclusions are obtained for the large time and space decay of solutions to systems of nonlinear parabolic differential equations. The analysis also facilitates an understanding of "universality" or the dependence of the key exponents only on particular aspects of the differential equations.

Key words: Renormalization group, scaling, differential equations, blow-up, extinction.

1. INTRODUCTION. Renormalization group methods (RG) have provided a powerful tool for calculation of key exponents that are otherwise extremely difficult to evaluate (see text by Goldenfeld [1]). The methods that were first used for quantum field theory and statistical mechanics, particularly critical phenomena, have evolved into a broad philosophy rather

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than a single technique, as each new application often involves different methods. The application of RG to a spectrum of problems such as fractals, self-avoiding random walk, percolation, etc., illustrates the central themes that provide insight into new problems (see Creswick, Farach and Poole [2]).

One aspect of this recent development involves understanding blow-up, extinction and decay in nonlinear differential equations. Decay problems using renormalization group techniques were studied by Goldenfeld, Martin, Oono and Liu [3], Bricmont, Kupiainen and Lin [4], and Caginalp [5] (see other references therein).

In this paper we demonstrate that the basic methodology can be used to understand some important features of blow-up and extinction in finite time for parabolic equations. This renormalization group methodology leads to a convenient tool for calculating blow-up exponents.

An extensive collection of exact and approximate results for blow-up in nonlinear problems has been obtained during the past 20 years and discussed thoroughly in the text of Samarskii, Galaktionov, Kurdyumov and Mikhailov [6].

These problems provide a useful testing ground for the application of RG methods to singularities in nonlinear differential equations.

2. BLOW-UP AND EXTINCTION WITH $\mathbf{u}^r$ TERMS. We consider blow-up and extinction problems for differential equations of the form

$$u_t = C\nabla\cdot\left(u^s\nabla u\right)+\sum_{r,\rho}A(x,t;r,\rho)u^r u_x^\rho+\sum_{\alpha,\beta,\gamma}B(x,t;\alpha,\beta,\gamma)x^\alpha u^\beta u_x^\gamma \quad (2.1)$$

where x is the radial coordinate in $d-$dimensional space, $t\in(0,\infty)$, A and B are continuous functions of their arguments x and t, and the parameters ρ, α, β, γ satisfy the conditions of dimensional consistency

$$\beta+\gamma=s+1, \qquad \gamma-\alpha=2. \qquad (2.2a,b)$$

We consider radially symmetric domains and solutions to (2.1), so that the first term in (2.1) can be written as

$$\nabla\cdot\left(u^s\nabla u\right)=(u^s u_x)_x+\frac{d-1}{x}u^s u_x \qquad (2.3)$$

and the second term in (2.3) above can be incorporated as a term in the second sum with $B(-1,s,1):=d-1$. With the constant C incorporated into t through the time scale, one has the equation

$$u_t=\left(u^s u_x\right)_x+\sum_{r,\rho}A(x,t;r,\rho)u^r u_x^\rho+\sum_{\alpha,\beta,\gamma}B(x,t;\alpha,\beta,\gamma)x^\alpha u^\beta u_x^\gamma. \quad (2.4)$$

We perform the calculation constant functions A and B, as it will be clear that the situation is similar for continuous functions. We consider the domain $x \in \Omega := (-1, 1)$ and initial and boundary conditions

$$u(x, 0) =: u_0(x) \geq 0, \qquad u(\pm 1, t) = u_0(\pm 1) \geq 0. \tag{2.5}$$

Alternatively, we can consider $\Omega := \mathbf{R}$, subject to the restriction that u_0 has compact support.

The calculation can be divided into several steps.

Stage 1. Obtaining a scaling relation. We examine the possible values of p and q such that any solution $u(x, t)$ of (2.4) implies that for any $b > 0$ the function

$$u_b(x, t) = b^p u(b^q x, bt) \tag{2.6}$$

is also a solution. This can be done systematically as follows. Let D_1 and D_2 denote differentiation with respect to the first and second variables (i.e., space and time), respectively.

Computing the derivatives, such as

$$\frac{\partial}{\partial t} u_b(x, t) = b^{p+1} D_2 u(b^q x, bt) \tag{2.7}$$

one obtains the following equation upon inserting u_b into equation (2.4):

$$\begin{aligned} b^{p+1} D_2 u = {} & b^{ps+p+2q} D_1(u^s D_1 u) + A b^{pr+(p+q)\rho} u^r (D_1 u)^\rho \\ & + B p^{-\alpha q + p\beta + (p+q)\gamma} (b^q x)^\alpha u^\beta (D_1 u)^\gamma \end{aligned} \tag{2.8}$$

where u is evaluated at $(b^q x, bt)$. Hence, $u_b(x, t)$ satisfies (2.4) if

$$p + 1 = p + ps + 2q, \qquad p + 1 = pr + (p + q)\rho,$$

$$p + 1 = -\alpha q + p\beta + (p + q)\gamma. \tag{2.9a, b, c}$$

Identity (2.9c) is already implied by the dimensional constraints imposed upon the parameters, and so does not contain any further constraints. Also, since (2.9a,b) are independent of α, β and γ, this means that the scaling relation to be defined through p and q will be independent of the $Bx^\alpha u^\beta u_x^\gamma$ term.

Identities (2.9a) and (2.9b) imply

$$p = \frac{\rho - 2}{2(1 - r) + \rho(s + 1)}, \qquad q = \frac{r - 1 + \rho - s}{2(r - 1) + \rho(2 - s)}. \tag{2.10}$$

provided the denominators are nonzero.

Note that the calculations are similar if one replaces x with $x - x_0$ and/or t with $t - t_0$ throughout. We continue to denote the shifted function by $u(\cdot,\cdot)$ when no confusion is likely to arise.

Stage 2. The RG transformation and fixed point. Suppose a blow-up occurs at (x_0, t_0). For some fixed $k \in \mathbf{Z}^+$, let

$$\overline{x} := b^{qk}(x - x_0), \qquad \overline{t} := b^k(t - t_0) \tag{2.11}$$

and repeatedly apply the transformation

$$R_{b,q}u(x-x_0, t-t_0) := u_b(x-x_0, t-t_0) = b^p u\big(b^q(x-x_0), b(t-t_0)\big). \tag{2.12}$$

The fixed point u^*, if it exists, will satisfy

$$u^*(x - x_0, t - t_0) = \lim_{k\to\infty} b^{pk} u\big(b^{qk}(x - x_0), b^k(t - t_0)\big). \tag{2.13}$$

Now consider large but finite k, and write

$$u(\overline{x}, \overline{t}) \simeq b^{-pk} u^*(b^{-qk}\overline{x}, b^{-k}\overline{t}). \tag{2.14}$$

For any small $\overline{t} \in \mathbf{R}^+$ we can evaluate $u(\overline{x}, \overline{t})$ by setting $b^k = -\overline{t}$ which yields

$$u(\overline{x}, \overline{t}) \simeq (-\overline{t})^{-p} u^*\big(\overline{x}(-\overline{t})^{-q}, -1\big). \tag{2.15}$$

Letting $\zeta := \overline{x}(-\overline{t})^{-q} = (x - x_0)(t_0 - t)^{-q}$ and $f(\zeta) := u^*(\zeta, -1)$, we can write this as

$$u(\overline{x}, \overline{t}) \simeq (-\overline{t})^{-p} f(\zeta) \tag{2.16}$$

or, using the original variables, one has

$$u(x - x_0, t - t_0) \simeq (t_0 - t)^{-p} f\big((x - x_0)(t_0 - t)^{-q}\big). \tag{2.16$'$}$$

Substituting equation (2.16′) into the differential equation (2.4) and utilizing the relation

$$\zeta_t = q(t_0 - t)^{-1}\zeta \tag{2.17}$$

yields the equation

$$(t_0-t)^{-p-1}[pf + q\zeta f'] = (t_0-t)^{-ps-p-2q}(f^s f')' + A(t_0-t)^{-pr-\rho(p+q)} f^r (f')^\rho$$
$$B(t_0 - t)^{q\alpha-(p+q)\gamma-\beta p}\zeta^\alpha f^\beta (f')^\gamma \tag{2.18}$$

where f denotes $f\big(\zeta(x,t)\big)$.

We know from the dimensional relations (2.10) that the exponents are all equal so that f satisfies

$$(f^s f')' - pf - q\zeta f' + Af^r(f')^\rho + B\zeta^\alpha f^\beta (f')^\gamma = 0 \tag{2.19}$$

Stage 3. Asymptotics for fixed times near t_0. The nature of the blow-up or extinction depends on the values of the parameter p in (2.16) and the solutions of the ordinary differential equation for f in (2.19).

We can consider the problem as an asymptotic perturbation problem in fixed time t close to t_0 by defining a small parameter as $\epsilon := t_0 - t$ and a stretched variable $\xi := \epsilon^{-q}(x - x_l)$ for some x_l to be determined as the location of the transition layer (not necessarily x_0), and writing

$$u(x,t) = \epsilon^{-p} f\big(\epsilon^{-q}(x - x_l)\big). \tag{2.20}$$

(See O'Malley [7] for transition layer methods.) The properties of the blow-up in the most general case thus depend on the nature of the solutions, f, to the ODE (2.19). Of course, it is evident from (2.16') that the $(t - t_0)^{-p}$ will dominate in some scaling regime with respect to t and x.

In the subsequent analysis we illustrate the asymptotic methodology for $s \leq 0$ and $\rho = 0$. For $s > 0$, nonconstant profiles for f are possible. For example, in the case $\rho = B = 0$, see pp. 187-197 of Samarskii et al [6].

The Case $\mathbf{s} \leq 0$, $\rho = 0$, and $\mathbf{B} = 0$ or $\mathbf{B} \neq 0$ with $\gamma > 0$. Note that any stationary points of (2.19) satisfy

$$f = \left(\frac{p}{A}\right)^{1/(r-1)} = \left(\frac{1}{A(r-1)}\right)^{1/(r-1)} \tag{2.21}$$

so that for $A > 0$ and $r > 1$ there is a unique solution to (2.21) and an interior transition layer for (2.19) is not possible. For $A < 0$ and $r < 1$ there is again a unique solution. We consider these two cases in which any transition layer (for small ϵ) must be a boundary layer.

With t fixed and varying x we can regard (2.19) as an ODE in ξ. Upon defining the shifted variable $y = x - x_l$, and focusing on the left boundary $x_l = 1$, we define

$$f(\xi) =: F(y) \tag{2.22}$$

so that (F, y) represent the 'outer' expansion while (f, q) represent the 'inner' expansion, and the derivatives are related through

$$\frac{\partial f(\xi)}{\partial \xi} = \epsilon^{q} \frac{\partial F(y)}{\partial y}. \tag{2.23}$$

In terms of the outer variable the basic equation (2.19) can be written as

$$\epsilon^{2q}(F^s F')' - pF - qyF' + AF^r(F')^\rho + B\epsilon^{2q} y^\alpha F^\beta (F')^\gamma = 0 \tag{2.24}$$

with primes now denoting differentiation with respect to y.

Using the procedure of matched asymptotic analysis, we approximate $F(y)$ and $f(\xi)$ by

$$F(y) \simeq F_0(y) + \epsilon F_1(y) + \cdots \epsilon^{N'} F_{N'}(y) \tag{2.25}$$

$$f(y) \simeq f_0(y) + \epsilon f_1(y) + \cdots \epsilon^{N'} f_{N'}(y) \tag{2.26}$$

where the F_j and f_j are independent of ϵ. The ϵ^{2q} term can be expected to force a transition layer at some point that we have defined as $y = 0$ (i.e. $x = x_l$).

The outer solution is expected to be valid far from the transition region and does not need to satisfy the boundary condition at x_l, while the inner solution is expected to satisfy that boundary condition and describe the details of the transition region.

The two expansions are connected by means of a set of conditions that constitute boundary conditions. The first of these (which is all that is necessary for our purposes) is the condition that the outer limit of the inner expansion should equal the inner limit of the outer expansion, i.e.,

$$lim_{y\to 0} F_0(y) = lim_{\xi \to 0} f_0(\xi) \tag{2.27}$$

which will provide the second boundary condition for f_0.

The next step is to obtain a series of equations by considering different orders in ϵ for the inner and outer equations. In particular we formally separate the different orders in ϵ in the two equations and seek solutions at each level.

Using this procedure first for the outer equation (2.24) one has:

$O(1)$ **Outer Expansion:** The first order equation for F_0,

$$F_o'(y) + \frac{p}{qy} F_0 = \frac{A}{ay} F_0^r, \tag{2.28}$$

is Bernoulli's equation and has solution

$$F_0(y) = \left(\frac{A}{p} + C y^{(r-1)p/q} \right)^{1/(1-r)} \tag{2.29}$$

A solution that is bounded and nonzero in the limit $y \to 0$ implies the choice $C = 0$ so that the leading term is the constant,

$$F_0(y) = \left(\frac{1}{A(r-1)} \right)^{1/(r-1)} \tag{2.30}$$

$O(1)$ **Inner Expansion:** The full inner equation, given by (2.19), is the $O(1)$ equation which is subject to the condition (2.27) implying the boundary condition

$$lim_{y\to\infty} f_0(y) = \left(\frac{1}{A(r-1)} \right)^{1/(r-1)} \tag{2.31}$$

The other boundary condition is given by (2.5) and the definition of ξ as

$$f(0) = \epsilon^p u_0(-1), \tag{2.32}$$

so that formally setting ϵ to zero leads to the boundary condition

$$f_0(0) = 0 \tag{2.33}$$

for the leading order if $p > 0$ (i.e. blow-up). For $p < 0$, using zero boundary conditions leads to (2.33).

The chief result necessary for calculating the blow-up profile is (2.30) which, together with (2.16′) leads to the asymptotic form

$$u(x - x_0, t - t_0) \sim \left(\frac{1}{A(r-1)}\right)^{1/(r-1)} (t_0 - t)^{1/(1-r)} \tag{2.34}$$

for large ζ, i.e. small $t - t_0$.

Hence, for $A > 0$ and $r > 1$, one has blow-up in u, while $A < 0$ and $r < 1$ implies extinction in finite time (under the constraints listed above).

Remarks:

1. The calculations demonsrate that the addition of a linear or nonlinear term which is dimensionally consistent with the basic diffusion term cannot change the nature (i.e. exponent and coefficient) of the blow-up or extinction.

2. The spatial dimension also does not play a role in the exponent or leading coefficient since the Laplacians for different spatial dimensionalities differ by terms that are of course dimensionally identical.

3. The factor u^s plays a limited role in the nature of blow-up since the exponent p does not involve s when $\rho = 0$ (and nonzero ρ does not permit blow-up). The influence of s arises solely from its role in determining the nature of solutions f to (2.19). Under the conditions stated after (2.20) one has, to leading order, constant f solutions, so that the critical exponents of (2.34) are completely independent of s. In the general case, the value of s can influence the critical exponents through (2.16'). Nevertheless, the argument of f in (2.16') is $(x - x_0)(t_0 - t)^{-q}$ so that there will always be some scaling regime in which the $(t_0 - t)^{-p}$ dominates, independent of f. Of course, changing s will generally change the blow-up or extinction time t_0, as do the other terms discussed above.

4. A generalization of the terms in the second sum of (2.4) to second order derivatives can be treated similarly at a formal level, although there are difficulties with existence and uniqueness in some cases. The terms are then of the form $B(x, t; \alpha, \beta, \gamma) x^\alpha u^\beta u_x^\gamma u_{xx}^\delta$ while the dimensional constraints are now

$$\beta + \gamma + \delta = s + 1, \qquad 2\delta + \gamma - \alpha = 2. \tag{2.35}$$

The formal calculations would suggest that the addition of such terms to (2.4) would not alter the blow-up coefficient or the exponent. We test this numerically in some cases as discussed below. Some of the early work on numerical computation of differential equations with blow-up was done by Berger and Kohn [8].

Numerical Calculations. We perform computations on partial and ordinary differential equations to check the basic conclusions obtained above. All of the numerical computations were performed using a Mathematica 3.0 differential equations package (NDSolve) that implements an implicit finite difference scheme. The default setting for the maximum number of steps permitted, $MaxSteps = 200$ has been modified to $MaxSteps = 3000$. The starting step size is 0.01 (also smaller than the default setting) unless indicated otherwise.

The *Interpolation Precision* specifies the number of digits of precision to use inside the interpolation function object generated by NDSolve. We use the default setting that results in the same setting as the *Working Precision* described below.

NDSolve computations terminate when either the *Accuracy Goal* or the *Precision Goal* has been met. The *Accuracy Goal* effectively specifies the absolute error allowed in solutions, while *Precision Goal* specifies the relative error. We use the default setting of Automatic for *Accuracy Goal* and *Precision Goal* which yield goals equal to the *Working Precision* (the number of digits used in internal precision, i.e., 16 in this case) minus 10. In particular, one obtains 6 digit accuracy and precision, in the sense described below. For *Accuracy Goal*$\leq a$ and *Precision Goal*$\leq p$, a quantity x will have an error bound given by

$$\text{Error in } |x| \leq 10^{-a} + 10^{-p}|x| = 10^{-6} + 10^{-6}|x|.$$

In order to identify the blow-up or decay profile, we define

$$s(t) := \frac{u(0,t)^{1-r}}{A(r-1)}, \tag{2.36}$$

so that $s(t) \simeq t_0 - t$ is expected if (2.34) is valid. Once a solution is computed, one can plot $s(t)$ and determine if the slope is -1 in order to verify both the exponent and the slope of the blow-up or extinction.

Thus in Figures 1-5 below, the first, (a), displays the profile $u(0,t)$ between 0 and t_0^* (an approximation to t_0). Figures (b) display the profile $u(0,t_0^*)$, while Figures (c) display $s(t)$ and Figures (d) display an approximation to ds/dt, and the printout is the approximation to $s'(t_0^*)$.

Computation 1. Letting $s = 0$, $r = 3$, $\rho = 0$ and $B = 0$ in (2.4) one has the equation

$$u_t = u_{xx} + u^3 \tag{2.37}$$

subject to the initial condition $u_0 := 2(1 - x^2)$ with $u(\pm 1, t) = 0$, which is a standard blow-up equation [see e.g. Samarskii et. al. (6)]. With the starting step size as 0.005 one has the emerging blow-up profile, with $t_0^* = 0.251$, shown in Figure 1(a) with $s(t)$ asymptotically approaching linearity characterized by slope -1.

Computation 2. Changing s to -1, and adding a term $u^{-2}u_x^2$ in the equation above, we perform the same computations on the equation

$$u_t = (u^{-1}u_x)_x + u^3 + u^{-2}u_x^2. \tag{2.38}$$

One obtains a similar blow-up (Figure 2), with $t_0^*= 0.057$. A profile for $s(t)$ that is similar to the previous confirms that the coefficient of the blow-up as well as the exponent are unchanged.

Computation 3. Adding a nonlinear term involving u_{xx} to the right hand side of the equation (2.37) we have the equation

$$u_t = u_{xx} + u^3 + 0.2xu^{-1}u_x u_{xx} \tag{2.39}$$

which we study subject to $u_0(x) := 2(1 - x^2) + 1$ with boundary conditions $u(\pm, t) = 1$ in order to avoid artificial singularities due to the u^{-1} term. Once again the blow-up is of the same form as Figures 3(c) and 3(d) confirm that $s(t) \simeq t_0 - t$ so that the exponent and coefficient are preserved.

Computations with other nonlinear terms subject to the restrictions (2.2) yield similar results.

Analogous results for extinction are shown below with $r := 1/2$ and $A := -1$ so that $s(t) := 2u(0, t)^{1/2}$, and the expected decay exponent of $(t_0 - t)^2$ and coefficient of 1/4 in (2.34) are exhibited if $s(t)$ has the form $(t_0 - t)$.

Computation 4. We consider the basic equation

$$u_t = u_{xx} - u^{1/2} \tag{2.40}$$

subject to the initial condition $u_0 := 2(1 - x^2)$ with $u(\pm 1, t) = 0$. With the starting step size as 0.2 one has the extinction profile, with $t_0^* = 1.327$, shown in Figure 4 with a $s(t)$ asymptotically approaching linearity of slope -1 with the last computed slope as -1.107.

Computation 5. Adding the additional nonlinear term, u_x^2/u, to the previous equation we study

$$u_t = u_{xx} - u^{1/2} + u^{-1}u_x^2. \tag{2.41}$$

and obtain the results of Figure 5. Now $s(t)$ has last computed slope of -1.02 so that u has the same extinction behavior (exponent and coefficient) as in equation (2.40) as indicated by the asymptotic form (2.34).

Finally, the ordinary differential equation (2.24) can be studied numerically for a range of values of ϵ to confirm the asymptotical analysis above. For example, we compute (2.24) with the parameters $s = 0$, $r = 3$, $\rho = 0$ so that $p = q = 0.5$ and subject to the boundary conditions $F'(0) = 0$ and $F(1) = 0$. Figures 6(a)-(d) show the computations for $\epsilon = 0.03$, $\epsilon = 0.02$, $\epsilon = 0.017$, $\epsilon = 0.013$. The two numbers below the graphs indicate the values $F'(1)$ and $F(1)$ respectively. The derivative $u'(1,0)$ has the values -8.96, -15.36, -18.70 and -25.61 respectively for the cases $\epsilon = 0.03$, 0.02, 0.017 and 0.013 respectively.

The derivative times -1 versus $1/\epsilon$ displays an essentially perfect linear relationship, thereby confirming the nature of the transition layer constructed in the asymptotic analysis.

3. OTHER EQUATIONS EXHIBITING BLOW-UP OR EXTINCTION. We discuss three other equations for which blow-up or extinction occurs under appropriate initial and boundary conditions. Consider first radially symmetric solutions to

$$u_t = C\nabla \cdot \left(e^{su}\nabla u\right) + \sum_{\lambda,\rho} A(x,t;\lambda,\rho)e^{\lambda u}u_x^{\rho} + \sum_{\alpha,\gamma} B(x,t;\alpha,\gamma)x^{\alpha}e^{su}u_x^{\gamma} \quad (3.1)$$

where x is again the radial coordinate in d-dimensions, and A and B are continuous and (α, γ) satisfy the spatial dimensional consistency condition $\gamma - \alpha = 2$.

Since we can write the diffusion term as

$$\nabla \cdot \left(e^{su}\nabla u\right) = \left(e^{su}u_x\right)_x + \frac{d-1}{r}e^{su}u_x \quad (3.2)$$

in d-dimensional radial symmetry, the second term in (3.2) is of the form of the last term in (3.1). Without loss of generality, then, one can consider the $(e^{su}u_x)_x$ term alone in (3.1). As in Section 2, we consider first a single constant coefficients, i.e.,

$$u_t = \left(e^{su}u_x\right)_x + Ae^{\lambda u}u_x^{\rho} + Bx^{\alpha}e^{su}u_x^{\gamma} \quad (3.3)$$

for $t \in \mathbf{R}^+$ on a bounded interval $\Omega \subset \mathbf{R}$ which we take as $(-1,1)$ subject to initial and boundary conditions

$$u(x,0) =: u_0(x) > 1, \qquad u(\pm 1, t) = u_0(\pm 1) > 1 \quad (3.4)$$

where $u_0 : \Omega \to \mathbf{R}$ is smooth. Alternatively, we can consider $\Omega := \mathbf{R}$ with u_0 having compact support.

The results are generalized in a straightforward way to the general case in which A and B are functions of space and time.

With the transformation $u =: log(w)$ equation (3.3) can be written as

$$w_t = \left(w^s w_x\right)_x - w^{s-1}w_x^2 + Aw^{\lambda-\rho+1}w_x^{\rho} + Bx^{\alpha}w^{s-\gamma+1}w_x^{\gamma}. \quad (3.5)$$

This equation is now in the form of (2.1) with $r := \lambda - \rho + 1$ and the second and fourth terms of (3.5) provided that $\gamma - \alpha = 2$, which is assumed in the sequel. Consequently, these will not influence the nature of the blow-up, by the results of Section 2, which implies that the solution scales as

$$w(x - x_0, t - t_0) \simeq (t - t_0)^{-p} f\big((x - x_0)(t_0 - t)^{-q}\big) \tag{3.6}$$

with the parameters p and q given by

$$p = \frac{\rho - 2}{2(\rho - \lambda) + \rho(s+1)}, \qquad q = \frac{\lambda - s}{2\lambda - \rho s} \tag{3.7}$$

and f is determined, as in Section 2, by an ordinary differential equation obtained by substitution of w into (3.5), namely the analog of (2.19), which is

$$(f^s f')' - pf - q\zeta f' - f^{s-1}(f')^2 + Af^{\lambda-\rho+1}(f')^\rho + \mathrm{B}\rho^\alpha f^{s-\gamma+1}(f')^\gamma. \tag{3.8}$$

Consequently, blow-up occurs under the conditions described in Section 2. In particular, if $s \leq 0$ and $\rho := 0$ then $p = \lambda^{-1}$ by (3.7) and

$$w(x, t - t_0) \sim \left(\frac{1}{A\lambda}\right)^{1/\lambda} (t_0 - t)^{-1/\lambda} \tag{3.9}$$

the same result as (2.34) with $r := \lambda + 1$, the exponent of w in (3.5).

In terms of the original variable, $u = log(w)$ one has

$$u(x - x_0) \sim -\frac{1}{\lambda} log(t_0 - t). \tag{3.10}$$

This agrees with the result of Dold [9] who studied the particular case

$$u_t = \Delta u + A(x, t)e^u \tag{3.11}$$

asymptotically and numerically, concluding that a singularity of the form of (3.10) occurs with $\lambda = 1$, and "is remarkably independent of both conditions leading to thermal runaway and the topology of the thermal-runaway region" (p. 386).

Other papers in which related ideas have been discussed include Wayne [10], Bacry et. al. [11] and Budd and Galaktionov [12].

Remarks.

1. A very large set of equations described by (3.1) have solutions that exhibit blow-up of the form of (3.10) with identical exponents and coefficients that are independent of the spatial dimension and the number

and type of nonlinearities of the form expressed in (3.1) provided they satisfy the dimensional constraint $\gamma - \alpha = 2$.

2. The terms $\sum_{\alpha,\gamma} B(x,t;\alpha,\gamma)x^\alpha e^{su}u_x^\gamma$ do not influence the singularity provided the dimensionality condition is satisfied. The criterion of dimensionality rather than nonlinearity is a surprising feature of these phenomena.

Extinction involving Δu^m. Next, we consider a problem in which the techniques are used in the absence of dimensional inconsistency. In particular, we set $A := 0$ in (2.1) and look for radial solutions of the equation

$$\frac{\partial u}{\partial t} = \Delta(u^m) + \sum_{\alpha,\beta,\gamma} B(x,t;\alpha,\beta,\gamma)x^\alpha u^\beta u_x^\gamma \tag{3.12}$$

in d-dimensional domain Ω, subject to initial conditions $u(x,0) = u_0(x)$ and the dimensional consistency conditions

$$m = \beta + \gamma, \qquad \gamma - \alpha = 2. \tag{3.13}$$

Note that the Laplacian in radial coordinates is

$$\Delta(u^m) = (u^m)_{xx} + \frac{d-1}{x}(u^m)_x$$

and it suffices, as in Section 2, to consider only the first in terms of the scaling relationships.

We apply the same methodology by assuming that $u(x,t)$ is a solution and determine p and q such that

$$u_b(x,t) = b^p u(b^q x, bt). \tag{3.14}$$

Substitution of (3.14) into (3.12) shows that u_b is a solution provided the following identities are satisfied:

$$p+1 = p + p(m-1) + 2q, \qquad p+1 = -\alpha q + \beta p + (p+q)\alpha. \tag{3.15a, b}$$

The first equation arises from the balance of the time and Laplacian terms, while the second arises from the time and the additional nonlinear terms, analogous to (2.9a) and (2.9c). Equation (3.15a) establishes a relation between p and q, namely,

$$p = \frac{1-2q}{1-m}, \qquad or \qquad q = \frac{1-p(m-1)}{2}. \tag{3.16}$$

Upon utilizing the dimensional consistency conditions, (3.13), it is clear that (3.15b) is identical to (3.16) so that no further constraint is imposed. This means that the nature of extinction (in particular the exponent in the asymptotically self-similar profile) cannot be changed by the presence of these terms.

In this case, there is no uniquely determined pair (p, q) but rather a relation between the two. However, we can proceed as in Section 2 and write

$$u_b(x,t) = b^{(1-2q)/(1-m)} u(b^q x, bt). \tag{3.17}$$

Upon defining $\zeta := x(t_0 - t)^{-q}$ and using the procedure of Stage 2 leading to (2.16') and (2.19) one obtains

$$u(x, t-t_0) \simeq (t_0 - t)^{-(1-2q)/(1-m)} f(\zeta) \tag{3.18}$$

where $f(\zeta)$ is the solution to the elliptic equation

$$(f^m)'' + \frac{d-1}{\zeta}(f^m)' - pf - q\zeta f' + \sum_{\alpha,\beta,\gamma} B\zeta^\alpha f^\beta (f')^\gamma = 0. \tag{3.19}$$

The existence of asymptotically self-similar solutions in the form of (3.18) is then contingent upon the issue of existence of nontrivial solutions to (3.19) which in turn depends upon the dimension of the space and the boundary conditions. We summarize a known result as follows (see, for example, King [13] and references contained therein). For $B := 0$ in (3.12), $d \geq 3$ and m satisfying

$$0 < m < 1 - 2/d$$

and $u_0 \in L^1 \cap L^P$ with $P > (1-m)/(2d)$ and

$$u_0(x) \sim x^{-(d-2)/m} \qquad \text{as} \qquad x \to \infty$$

in order to preserve a finite mass condition, one has extinction in the form of (3.18).

We consider a class of third order initial value problems that are related to the generalized Korteweg-de Vries (KdV) equation

$$u_t + u^r u_x + \epsilon u_{xxx} = 0. \tag{3.20}$$

The blow-up properties of KdV equations have been studied extensively (see Bona et al [14], [15] and [16], and Laedke et. al. [17] and references therein). See also Snoussi et. al. [18] for related work.

We consider, for smooth functions A and B, the following generalization of this equation:

$$u_t + u_{xxx} + A(x,t)u^r u_x + \sum_{\alpha,\beta,\gamma} B(x,t;\alpha,\beta,\gamma) u^\alpha u_x^\beta u_{xx}^\gamma = 0 \tag{3.21}$$

where α, β, γ satisfy the dimensional restrictions

$$\alpha + \beta + \gamma = 1 \qquad \beta + 2\gamma = 3 \qquad (3.22, \quad 3.23)$$

subject to initial condition $u(x, 0) = u_0(x)$ and suitable boundary conditions, e.g. the value of u at the two endpoints and the derivative at one endpoint. The dimensional conditions (3.23) assure compatibility of the last term with the leading diffusion term, u_{xxx}. Examples include terms such as $u^{-2}u_x^3$ and $u^{-1}u_x u_{xx}$.

Using the scaling relation of (2.6) and substituting it into (3.21) one has from the first three terms the identity

$$p = 2/(3r). \qquad (3.24)$$

The last term leads to an identity, analogous to (2.9c), that reproduces the dimensional conditions (3.23). Hence, one concludes that if $u(x, t)$ is a solution then so is

$$u_b(x, t) = b^{2/(3r)} u\left(b^{1/3}x, bt\right) \qquad (3.25)$$

for any positive value of b, and similarly with $x - x_0$ and $t - t_0$ replacing x and t, respectively.

The scaling analysis of (2.11)-(2.16) then proceeds identically with the result [using the scaled variables in (2.16)],

$$u(\bar{x}, \bar{t}) \simeq (-\bar{t})^{-2/(3r)} f(\zeta), \qquad (3.26)$$

or, using the original variables, one has

$$u(x - x_0, t - t_0) \simeq (t_0 - t)^{-2/(3r)} f\left((x - x_0)(t_0 - t)^{-1/3}\right) \qquad (3.27)$$

with f satisfying the ordinary differential equation

$$\frac{2}{3r} f + \frac{\zeta}{3} f' + A f^r f' + f''' + B f^\alpha (f')^\beta (f'')^\gamma = 0 \qquad (3.28)$$

in terms of the scaled variable $\zeta := (x - x_0)(t_0 - t)^{-1/3}$. Solutions to (3.21) that exhibit blow-up for fixed x near x_0 with t approaching t_0, i.e., large ζ, require small f' for large ζ. Hence, the term involving B in (3.28) will not change the boundedness of f and the leading order singularity will remain as $(t_0 - t)^{-2/(3r)}$. For $B := 0$ this agrees with the known result (p. 343 of Bona et. al. [14]).

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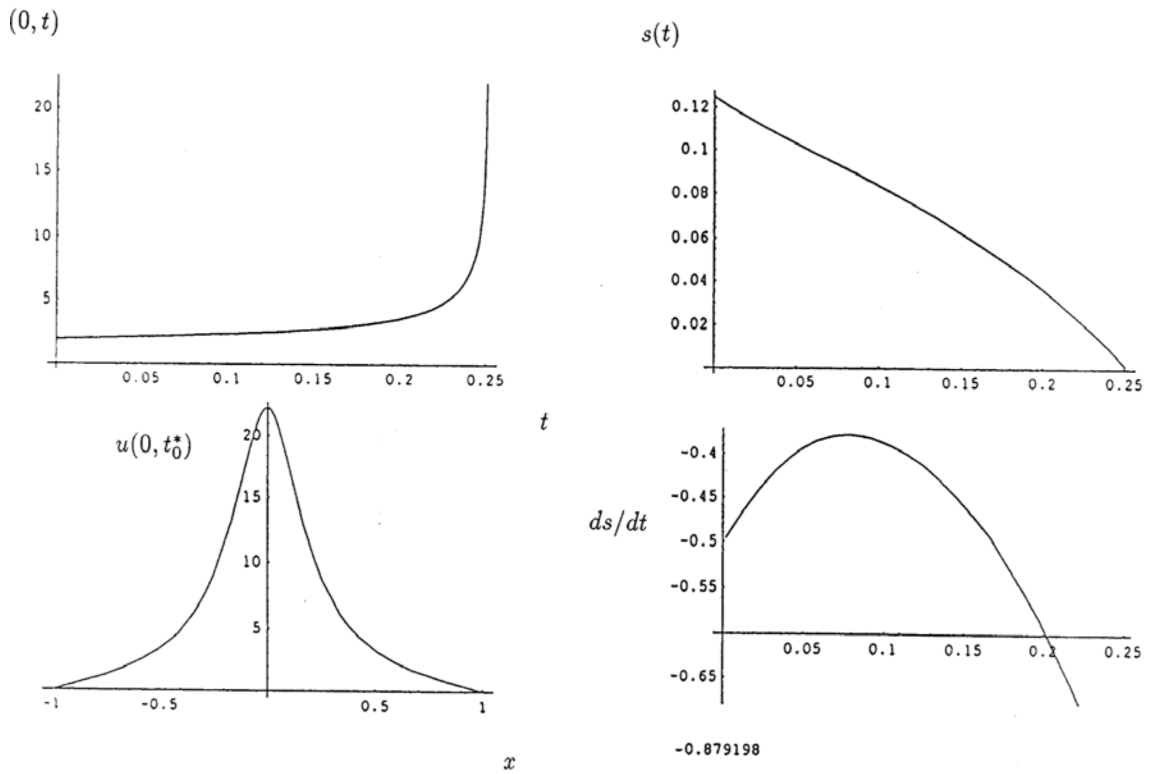


Figure 1

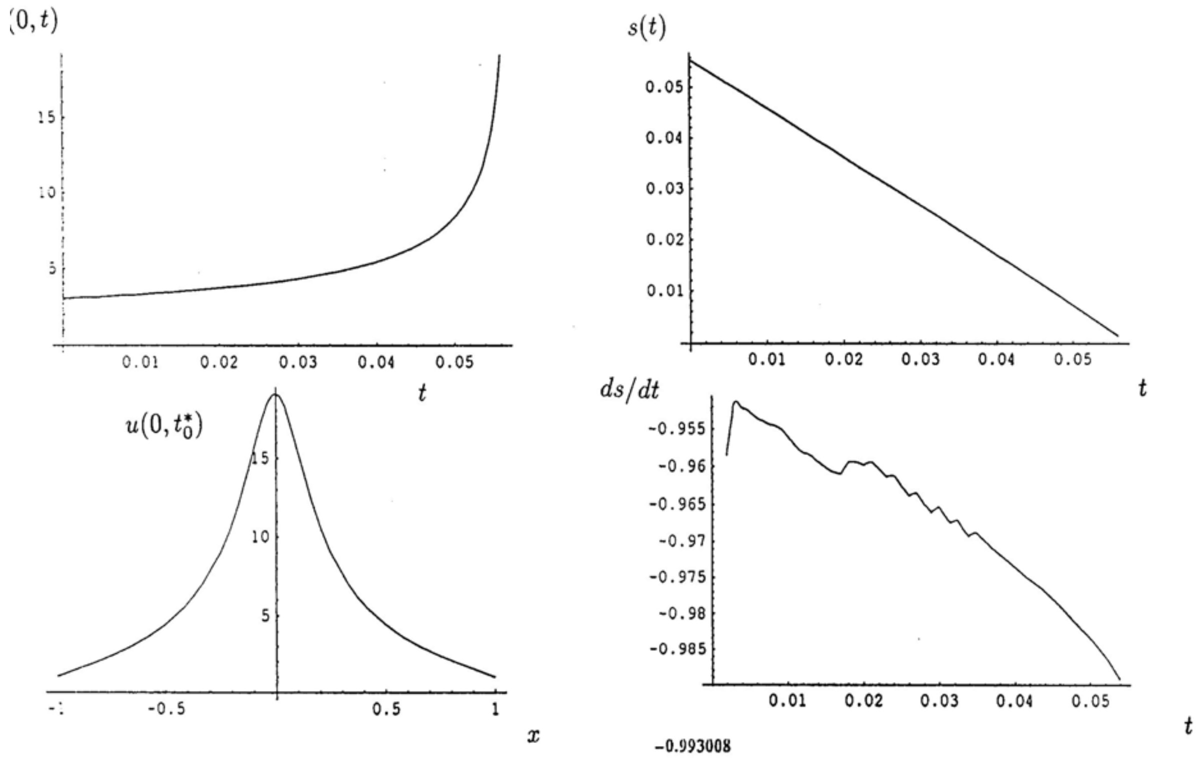


Figure 2

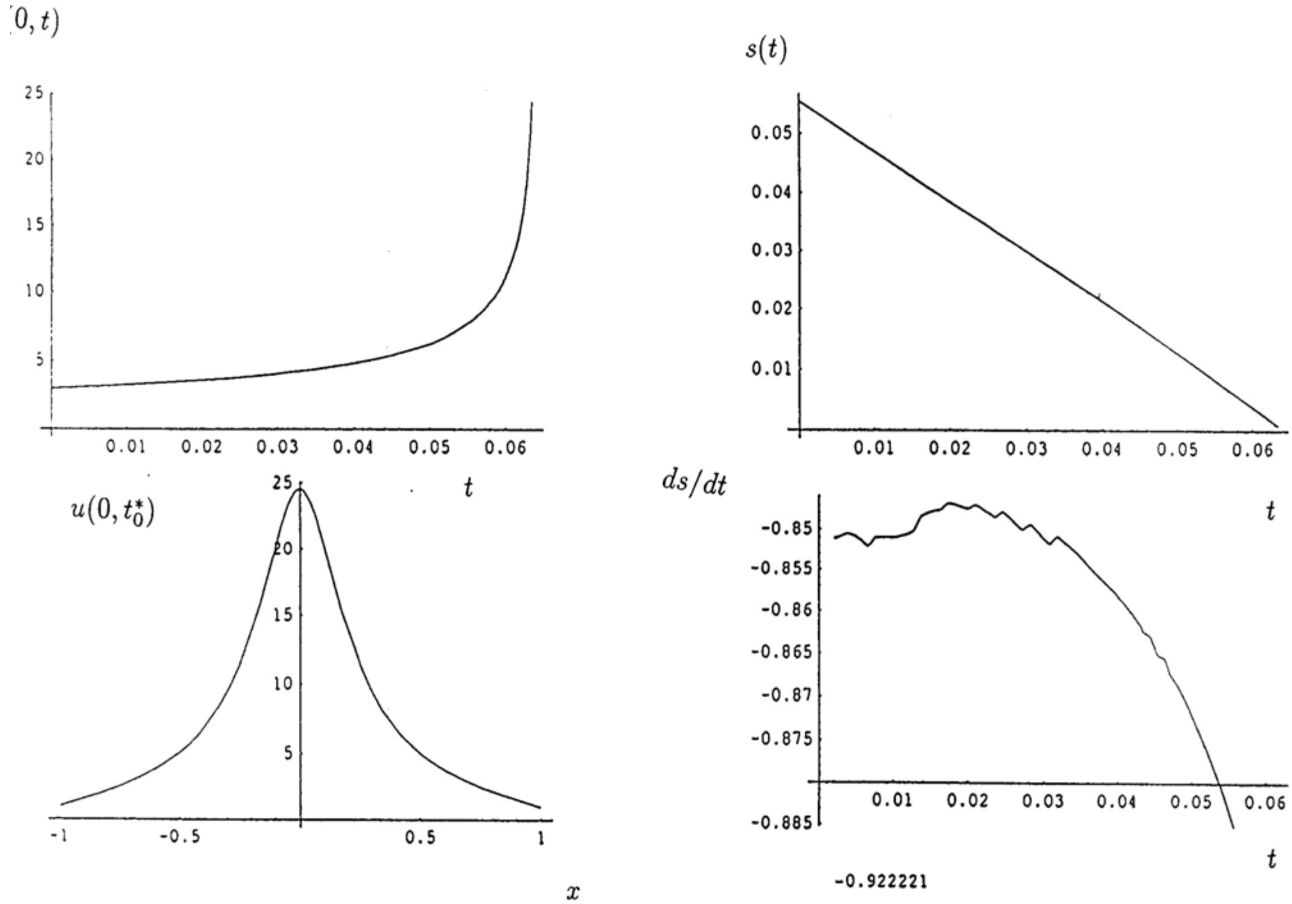
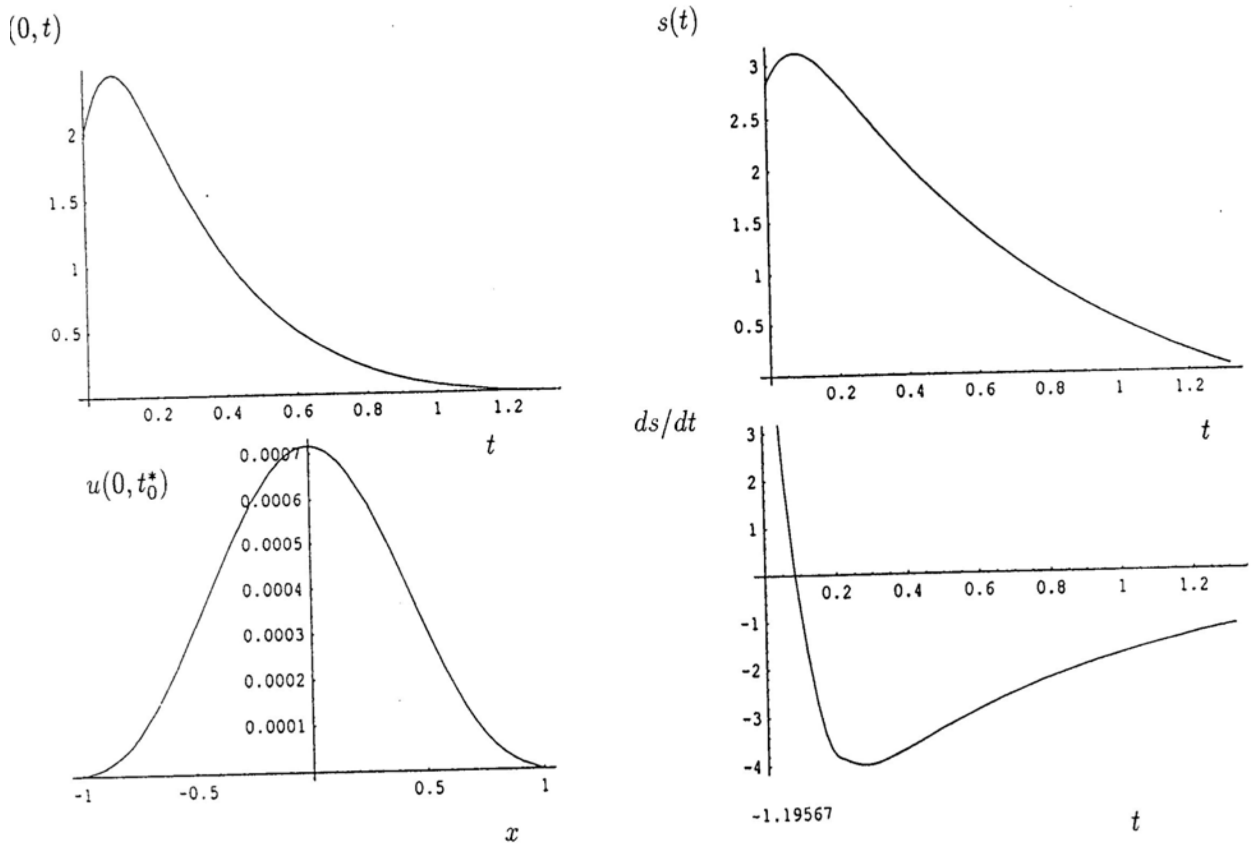
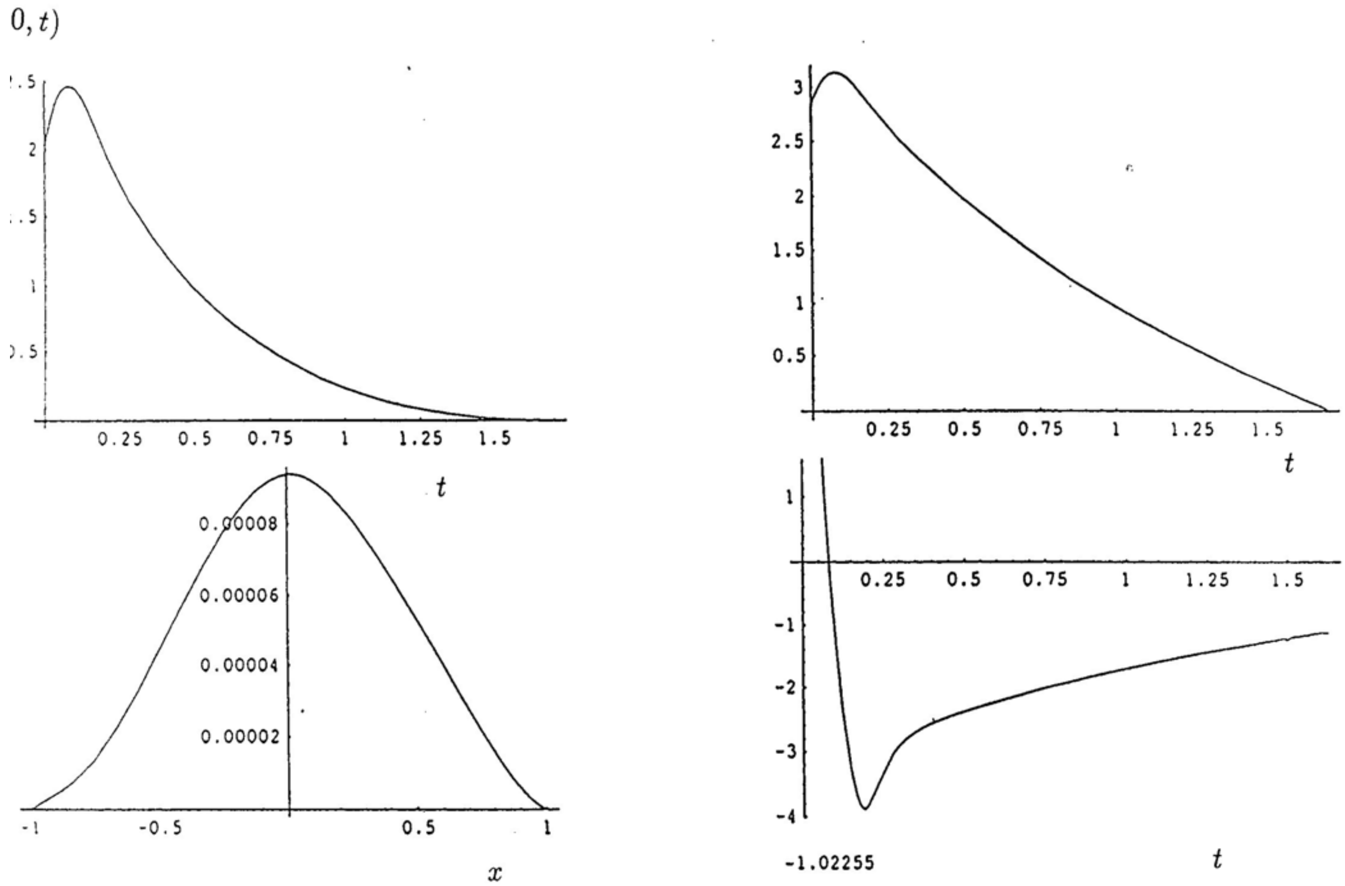

(0, t)
25
20
15
10
5
0.01
0.02
0.03
0.04
0.05
0.06
t
s(t)
0.05
0.04
0.03
0.02
0.01
0.01
0.02
0.03
0.04
0.05
0.06
t
u(0, t*0)
25
20
15
10
5
-1
-0.5
0.5
1
x
ds/dt
-0.85
-0.855
-0.86
-0.865
-0.87
-0.875
-0.885
0.01
0.02
0.03
0.04
0.05
0.06
t
-0.922221

Figure 3



(0, t)
2
1.5
1
0.5
0.2
0.4
0.6
0.8
1
1.2
t
s(t)
3
2.5
2
1.5
1
0.5
0.2
0.4
0.6
0.8
1
1.2
t
u(0, t*0)
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
-1
-0.5
0.5
1
x
ds/dt
3
2
1
0.2
0.4
0.6
0.8
1
1.2
-1
-2
-3
-4
-1.19567
t

Figure 4



0, t)
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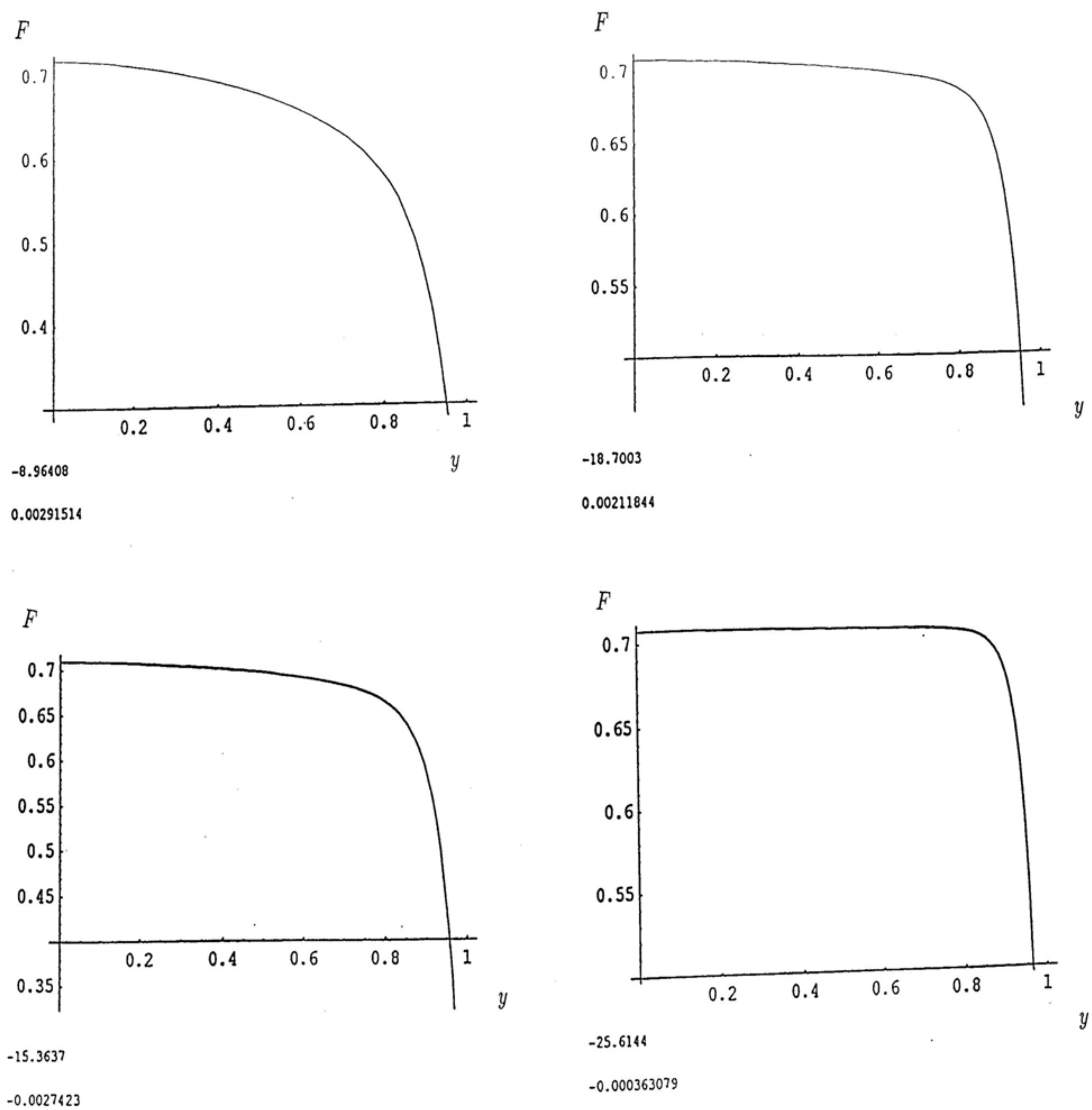
0.5
0.75
1
1.25
1.5
t
0.00008
0.00006
0.00004
0.00002
-1
-0.5
0.5
1
x
3
2.5
2
1.5
1
0.5
0.25
0.5
0.75
1
1.25
1.5
t
1
-1
-2
-3
-4
-1.02255
t

Figure 5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0.6
0.5
0.4
0.2
0.4
0.6
0.8
1
y
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0.00291514
F
0.7
0.65
0.6
0.55
0.2
0.4
0.6
0.8
1
y
-18.7003
0.00211844
F
0.7
0.65
0.6
0.55
0.5
0.45
0.35
0.2
0.4
0.6
0.8
1
y
-15.3637
-0.0027423
F
0.7
0.65
0.6
0.55
0.2
0.4
0.6
0.8
1
y
-25.6144
-0.000363079

Figure 6