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Renormalization and scaling methods for quasi-static interface problems

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Abstract

We study the temporal evolution of an interface separating two phases for its large-time behavior by adapting renormalization group methods and scaling theory. We consider a full two-phase model in the quasi-static regime and implement a renormalization procedure in order to calculate the characteristic length of a self-similar system, $R(t)$, that is the time-dependent length scale characterizing the pattern growth. When the dynamical undercooling is non-zero ($\alpha \neq 0$), we find that $R(t)$ increases as $t^{-1/\lambda}$, where λ can take on values in the continuous spectrum, $[-3, -2]$. For $\alpha = 0$ the spectrum is $[-3, 0)$ so that the single value of $\lambda = -1$ is selected by the plane wave imposed by Jasnow and Vinals. It is also shown that in almost all of these cases, the capillarity length, d_0, (arising from the surface tension, σ_0) is not relevant for the large-time behavior even though it has a crucial role at the early stage evolution of an interface. The exception is $\lambda = -3$, i.e., $R(t) \sim t^{1/3}$, for which d_0 is invariant.

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1. Introduction

Spatial pattern formation arising from a variety of non-equilibrium growth phenomena has attracted much attention. A number of mathematical methods such as analytical methods, linear stability theory and large-scale computations have been used to study these

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problems. In many cases, the pattern arises through the motion of an interface separating two phases or liquids [5]. Early studies of pattern formation generally focused on the existence, the nature of steady states and their stability, etc., for example, the onset of stability for fluids, alloys, and Stefan-like supercooled solidification (see [18,14–16]). Analytical methods and linear stability have been a valuable tool in answering such key questions. In particular, the latter has been used to study the evolution of the interface for its small time behavior.

From a practical standpoint, the large-time evolution has been of great interest in a number of problems such as dendritic growth, directional solidification in binary alloys, and fluids. In many problems, this is a central issue in the interfacial phenomena and merits analysis. For example, in solidification of a binary alloy, a key issue involves the deposition of the impurities as the late-stage dendrites involve into fully solidified material. The pattern of impurities in the solid has a strong bearing on the mechanical properties such as brittleness.

A first aim along these lines has been the development of an analytical method, analogous to linear stability theory, that can be used to determine the characteristic length, $R(t)$, as a function of time if the pattern is self-similar. Progress toward this goal was made by Jasnow and Vinals [9,10], and Caginalp [1,2] who adapt renormalization group (RG) and scaling theory to study the large-time behavior of an interface.

The characteristic length, $R(t)$, is the time-dependent length scale governing the morphology of late-stage pattern growth. For example, it may be the radius of a circle which contains the pattern evolving self-similarly in time (see [11]).

We summarize now some results obtained for complex interface problems that arise frequently in applications. Using the terminology of thermal problems we write the sharp interface problem as follows. We consider a material occupying a spatial region, Ω, in d-dimensional space that can be in either of two phases, which assume to be call liquid and solid. The mathematical model consists of determining the temperature, $T(x,t)$, and the interface, $\Gamma(t)$, in the system of equations:

$$CT_t = K\Delta T \quad \text{in } \Omega, \tag{1}$$

$$lv_n = -K[\nabla T \cdot \widehat{n}]_-^+ \quad \text{on } \Gamma, \tag{2}$$

$$T = \frac{-\sigma_0}{[s]_{\mathrm{eq}}}(\kappa + \alpha v_n) \quad \text{on } \Gamma. \tag{3}$$

Here, C is the specific heat per unit volume, K is the thermal conductivity (and $D := K/C$), l is the latent heat per unit volume, σ_0 is the surface tension, $[s]_{\mathrm{eq}}$ is the entropy difference per unit volume between phases, α is the dynamical undercooling and $[\ldots]_-^+$ is the difference in the limiting values between the two sides of the interface. The variables v_n and κ denote the (normal) velocity and the sum of the principle curvatures at a point on the interface, respectively. In addition, $+$ denotes the phase with the higher internal energy, i.e., liquid and $-$ denotes the phase with the lower internal energy, i.e., solid.

Jasnow and Vinals utilized the following conditions: (i) the dynamical undercooling was set to zero ($\alpha = 0$); (ii) one of the two phases was suppressed so that the equations involved one of the phases; (iii) the quasi-static limit was considered by suppressing the time dependence in (1) (i.e., $CT_t = 0$); (iv) a plane wave solution was utilized (through flux conditions) and subtracted from the solutions. Under these conditions they found that the characteristic length, $R(t)$, of a system with single scale self-similarity must have the large-time behavior $R(t) \sim t$.

Subsequently, Caginalp examined the problem above under the conditions (i) $\alpha \neq 0$; (ii) two-phases are present; (iii) the fully-dynamic case is considered ($CT_t \neq 0$); (iv) both with a particular plane wave and without. Under these conditions RG led to the conclusion that $R(t) \sim t^{1/2}$ (assuming R is non-singular as d_0 approaches zero).

This suggests the following questions. (a) What feature of the equations is responsible for the difference in the scaling exponents? (b) How is the transition made between the different regimes? (c) What insights can we obtain for other problems involving nonlinear dynamics?

There is also another issue that is illuminated by the RG process. As we will see in the next section, the RG methodology also distinguishes between those physical parameters that are "relevant" and those that are "irrelevant". An irrelevant variable (in terms of large-time behavior) is one whose value can be set to zero and eliminated without influencing the scaling exponent (i.e., 1 or 1/2 in the two cases above). Both of the analyses above show that the capillarity length, which is the length scale associated with the surface tension, σ_0, is irrelevant for large-time in most cases. This is in sharp contrast with the crucial role it has for stability in short time where it essentially determines the nature of the interface evolution. A large capillarity length means that the interface seeks to minimize the curvature, making the interface more rounded. Consequently, the irrelevance of the capillarity length is one of the key surprises presented by the RG analysis.

In this paper, we consider the full two-phase interface problem in the quasi-static regime in a d-dimensional space where $d > 2$. The growth process for sufficiently long time is analyzed in the context of a general geometry and more general conditions on the degree of undercooling. The quasi-static regime is important since it is a good approximation for many materials with common boundary conditions, as the temperature quickly approaches a solution to Laplace's equation. Thus, a key difference in long-term behavior between quasi-static and fully dynamic would be significant in theoretical and practical terms. The main result of our work is that without reference to a plane wave the characteristic length, $R(t)$, varies as $t^{-1/\lambda}$ where $\lambda \in [-3, -2]$, under the single scale self-similarity assumption when the dynamical undercooling is non-zero ($\alpha \neq 0$). For $\alpha = 0$ the spectrum is $[-3, 0)$ so that the single value of $\lambda = -1$ is selected by the plane wave imposed through the boundary conditions by Jasnow and Vinals [10]. This work indicates that the pattern evolves with different forms as λ varies in a continuous spectrum and extends the earlier results and those of Jasnow and Vinals who obtained a single growth form for the interface problem in the same regime (see Fig. 1).

Furthermore, the results confirm, as in [10,2], that for almost all values of λ the capillarity length, d_0, which is the length scale associated with the surface tension, is not relevant to the scaling of the large-scale behavior of an interface. This is an intriguing consequence since we know that the surface tension plays the stabilizing role in the early stage growth so that it has a critical influence for short time. Indeed, while large capillarity length tends to suppress instabilities small capillarity length permits it. The only exception arises at the value $\lambda = -3$ (so that $R(t) \sim t^{1/3}$) for which the capillarity length, d_0, is invariant.

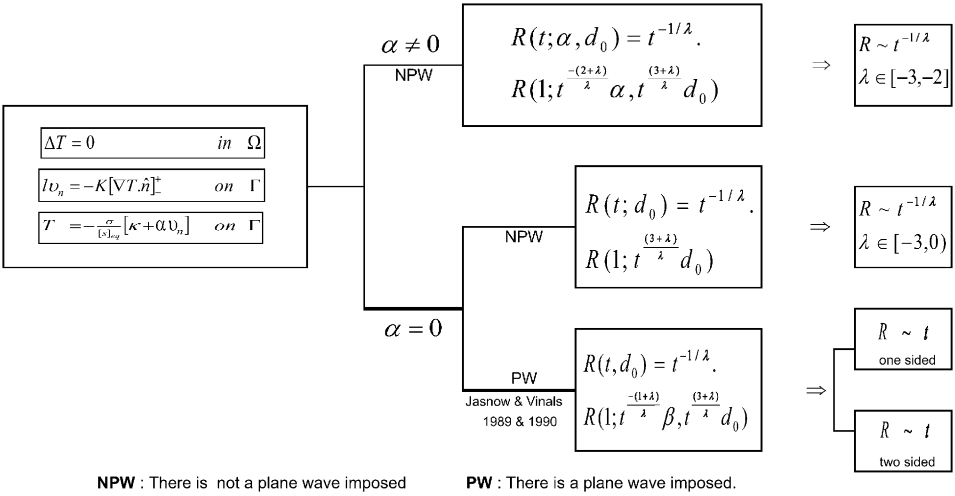

Fig. 1. The quasi-static model.

The methodology we use is similar to that in Caginalp [2], and basically utilizes RG transformations that are identities involving Green's function representation for Poisson's equation. The outline of the paper is as follows: In Section 2, we first rewrite, for the case $\alpha \neq 0$, the basic equations in terms of a Green's function representation by introducing a phase parameter (in Section 2.1). In Section 2.2, the RG analysis is implemented in several steps so that the equations are first transformed and then converted back into the original form by renormalizing physical parameters. In Section 3, the case $\alpha = 0$ is studied.

We have also studied the important connection between dynamic and static renormalization methods through a Green's function approach. In particular, the transition between exponents in different regimes can be understood in terms of the elliptic limit of the parabolic fundamental solution (see [3]).

The applications of RG approach to a variety of problems in applied mathematics show that it has the potential to be a powerful tool to study other physical problems (see, for example, the texts by Goldenfeld [6], and Creswick et al. [4], and the papers by Goldenfeld et al. [7], Zhang and Graham [19], Moise and Temam [13], Golstein et al. [8], Merdan and Caginalp [12], and other references therein).

2. The RG analysis in the quasi-static regime

We address the question of isolating the factors behind the different scaling regimes. In particular we begin by using (1)–(3) with $CT_t = 0$, i.e., the quasi-static regime.

The basic steps in the RG process can be presented as follows. In this section, the dynamical undercooling is non-zero, i.e. $\alpha \neq 0$.

2.1. The model and Green's function representation

We begin the calculations by writing Eqs. (1) and (2) as a single equation. In order to do this, we utilize the heat equation

$$CT_t = K\Delta T, \tag{4}$$

where C is the specific heat per unit volume and K is the thermal conductivity and $D := K/C$. Following Caginalp [2], Eqs. (4) and (2) can be reformulated and written as a single equation by defining locally a signed distance, r (defined a sufficiently small distance from the interface), which is positive on the liquid phase, and introducing a phase variable $\varphi(r, t)$ that is a step function having the value -1 in the solid phase and $+1$ in the liquid phase. One then has

$$CT_t - K\Delta T = -\frac{l}{2}\,\varphi_t. \tag{5}$$

This formulation is known as Oleinik formulation that is related to a continuously varying function φ in the phase field equation (see Caginalp [1,2] and Oleinik [17] for more details). Multiplying both sides of Eq. (5) by $1/K$ and setting $CT_t = 0$ we obtain

$$\Delta T = \frac{l}{2K}\,\varphi_t. \tag{6}$$

Treating the phase change as a source term with support along the interface, $\Gamma(t)$, and using the Green's formulation, one can express the solution of (6) as

$$\begin{aligned} T(x) = {} & \int_\Omega \mathrm{d}^d y\, G(\vec{x}-\vec{y})\left(\frac{l}{2K}\,\varphi_t(\vec{y}, t)\right) \\ & + \int_{\partial\Omega}\left(T(\vec{y})\frac{\partial G}{\partial \nu}(\vec{x}-\vec{y}) + G(\vec{x}-\vec{y})\frac{\partial T}{\partial \nu}(\vec{y})\right)\mathrm{d}^{d-1}\sigma_y, \end{aligned} \tag{7}$$

where the Green's function that we use is

$$G(\vec{x}-\vec{y}) = \begin{cases} \dfrac{1}{2(2-d)\omega_d}|\vec{x}-\vec{y}|^{2-d} & \text{if } d > 2, \\ \dfrac{1}{2\pi}\log|\vec{x}-\vec{y}| & \text{if } d = 2. \end{cases} \tag{8}$$

Here, the simplest Green's function for infinite domains is implemented. Since we are interested in very large domains, this is a good approximation. In this paper we consider the case $d \geqslant 3$. The results for $d = 2$ are similar though more technical.

We now examine the region near the interface to evaluate the first integral (i.e $\int_\Omega \ldots$) in (7). The function $\varphi_t(\vec{x}, t)$ will vanish outside of the interfacial region. Across the interface, it will behave like a *delta function*. In order to perform the integral, we use local coordinates $(\vec{r}, \vec{\sigma})$ which are the signed normal to the interface (positive toward to the liquid phase) and the tangential vector, respectively. We consider a suitable local neighborhood of the interface in order to eliminate problems such as the uniqueness of the signed normal. Note that for a sufficiently thin region (of width δ) containing the interface, the local coordinate system $(\vec{r}, \vec{\sigma})$ can be defined unambiguously. With this notation, we can write the normal velocity, v_n, at each point on the interface as

$$v_n = -r_t(\vec{x}, t) \tag{9}$$

Let $\widetilde{\varphi}(x,t)$ be a smoothing of the step function $\varphi(x,t)$ where the transition from -1 to $+1$ appears on a small distance scale. To leading order, the smoothing function $\widetilde{\varphi}(x,t)$ and its derivatives are functions of $r(x,t)$. By defining a variable ϕ as a function $r(x,t)$, to leading order we have

$$\varphi(x,t) = \widetilde{\varphi}(x,t) = \phi(r(x,t)). \tag{10}$$

If the interface is sufficiently smooth and the thickness of the interface is sufficiently small, then the transition region will be in this local region and this approximation can be used in order to compute the integration across the interface in (7). In particular, one has

$$\varphi_t(x,t) = r_t \phi_r(r(x,t)) = -v_n \phi_r(r(x,t)) \tag{11}$$

and also, for enough small δ,

$$\int_{-\delta}^{\delta} \phi_r r(x,t)\, \mathrm{d}r = 2. \tag{12}$$

Using then these new definitions (7) is rewritten as

$$T(x) = \int_{\Omega} \mathrm{d}^d y\, G(\vec{x} - \vec{y}) \left(\frac{l}{2K}\right) (-v_n \phi_{r_{\vec{y}}}(r(\vec{x}, t)) + \mathrm{BI}, \tag{13}$$

where BI denotes the integral that is taken over the boundary of the domain in (7). Since the derivatives of ϕ vanish just outside of the interfacial region, we can perform the integral in the normal direction thereby reducing the integral over Ω to one over Γ, with the result,

$$T(x) = \int_{\Gamma(t)} \mathrm{d}^{d-1}\sigma_y G(\vec{x} - \vec{y}) \left(\frac{l}{2K}\right) (-2v_n) + \mathrm{BI}. \tag{14}$$

For the points on the interface, one can combine (14) and (3). Recalling $D = K/C$ and neglecting BI term since it is far away from the interface one then has

$$-\frac{\sigma_0}{[s]_{\mathrm{eq}}}(\kappa + \alpha v_n(\vec{x}, t)) = -\frac{l}{C}\frac{1}{D}\int_{\Gamma(t)} \mathrm{d}^{d-1}\sigma_y G(\vec{x} - \vec{y}) v_n(\vec{y}, t). \tag{15}$$

Following [2] we now define the standard capillarity length to be

$$d_0 := \frac{\sigma_0/[s]_{\mathrm{eq}}}{l/C} \tag{16}$$

and rewrite (15) as

$$d_0(\kappa + \alpha v_n(\vec{x}, t)) = \frac{1}{D}\int_{\Gamma(t)} \mathrm{d}^{d-1}\sigma_y G(\vec{x} - \vec{y}) v_n(\vec{y}, t). \tag{17}$$

Dividing the variables in the equation above by appropriate reference length, L_0, and time, T_0, scales etc., we convert all constants and variables in (17) to their dimensionless counterparts (see [2] for further details), and write the equation entirely in dimensionless variables in order to compare pure numbers after a RG procedure. Using the dimensionless units and replacing $\vec{\eta}$ in the place of $\vec{x}$ for the points on the interface one writes Eq. (17) as

$$d_0\{\kappa(\vec{\eta},t)+\alpha v_n(\vec{\eta},t)\}=\frac{1}{D}\int_{\Gamma(t)} \mathrm{d}^{d-1}\sigma_y G(\vec{\eta}-\vec{y})v_n(\vec{y},t). \tag{18}$$

2.2. Renormalization group analysis of the interface equation

We now implement a renormalization procedure as follows:

Step 1: The first step is to make the algebraic substitutions

$$b\vec{\eta} \text{ for } \vec{\eta} \quad \text{and} \quad b^{-\lambda}t \text{ for } t \tag{19}$$

into (18) for any $b>0$ and $\lambda \in R$, which will be determined later, so that one has

$$d_0\left\{\kappa(b\vec{\eta},b^{-\lambda}t)+\alpha v_n(b\vec{\eta},b^{-\lambda}t)\right\}=\frac{1}{D}\int_{\Gamma(b^{-\lambda}t)} \mathrm{d}^{d-1}\sigma_y G(b\vec{\eta}-\vec{y})v_n(\vec{y},b^{-\lambda}t). \tag{20}$$

Next we define new variables

$$\vec{y'}=y/b \quad \text{and} \quad \sigma_{y'}=\sigma_y/b \tag{21}$$

in order to rescale space. These two substitutions into (20) yield

$$\begin{aligned} &d_0\{\kappa(b\vec{\eta},b^{-\lambda}t)+\alpha v_n(b\vec{\eta},b^{-\lambda}t)\} \\ &\quad =\frac{1}{D}\int_{by'\in\Gamma(b^{-\lambda}t)} b^{d-1}\mathrm{d}^{d-1}\sigma_{y'}G(b\vec{\eta}-b\vec{y'})v_n(b\vec{y'},b^{-\lambda}t). \end{aligned} \tag{22}$$

Note that the surface integral in (22) is over those points for which $y\in\Gamma(b^{-\lambda}t)$ which is identical (algebraically) to $by'\in\Gamma(b^{-\lambda}t)$. The latter will be equivalent to $y'\in\Gamma(t)$ upon assuming single scale self-similarity in (24) below.

Step 2: The second step involves the examination of the scaling of individual terms. A purely algebraic transformation for the Green's function, for $d\geqslant 3$, leads to the result

$$G(b\vec{\eta}-b\vec{y'})=b^{2-d}G(\vec{\eta}-\vec{y'}). \tag{23}$$

We assume the *single scale self-similarity* for the scaling of the physical quantities; i.e., all physical lengths and all physical time measurements in the problem scale as

$$\xi(b\vec{\eta},b^{-\lambda}t)=b\xi(\vec{\eta},t), \tag{24}$$

$$\mathrm{T}(b\vec{\eta},b^{-\lambda}t)=b^{-\lambda}\mathrm{T}(\vec{\eta},t), \tag{25}$$

respectively, (see [2,10]). Note that (24) implies $by'\in\Gamma(b^{-\lambda}t)\Leftrightarrow y'\in\Gamma(t)$. One interpretation of the first relation, for example, is that if one rescales the position on the interface,

$\Gamma(t)$, by b, and the time by $b^{-\lambda}$, then the position in the z-direction, which is $\xi := h/L_0$ in the calculation above, will change by a factor of b.

As a result of these assumptions, i.e. (24) and (25), one can obtain the scaling relations for the (normal) velocity, υ_n, and the curvature, κ, which has units of $1/length$, as

$$\upsilon_n(b\vec{\eta}, b^{-\lambda}t) = b^{1+\lambda}\upsilon_n(\vec{\eta}, t), \tag{26}$$

$$b\kappa(\vec{\eta}, t) = \kappa(b\vec{\eta}, b^{-\lambda}t). \tag{27}$$

Substituting the relations above into (22) and simplifying the terms lead to the new interface equation below that can be compared with the original equation (17), after the physical parameters are renormalized

$$\frac{d_0}{b^{3+\lambda}}\left\{\kappa(\vec{\eta}, t) + \frac{\alpha}{b^{-2-\lambda}}\upsilon_n(\vec{\eta}, t)\right\} = \frac{1}{D}\int_{y'\in\Gamma(t)} \mathrm{d}^{d-1}\sigma_{y'} G(\vec{\eta} - \vec{y}')\upsilon_n(\vec{y}', t). \tag{28}$$

Step 3: At this stage in the renormalization process, we rescale the physical parameters in order that the new equation has the same form as the original equation. The key observation here is that the new equation (28) is identical to the original, (17) upon replacing

$$d_0 \to \frac{d_0}{b^{3+\lambda}} \quad \text{and} \quad \alpha \to \frac{\alpha}{b^{-2-\lambda}}. \tag{29}$$

In summary, the process of algebraic substitutions (i.e. $b\vec{\eta} \to \vec{\eta}$ and $b^{-\lambda}t \to t$), as done in Step 1 and the (single scale) self-similarity assumption together with the scaling of the physical parameters (29) allow us transform the new interface equation back into its original form.

Since it is assumed that the system evolves in a self-similar manner with a single length scale, all physical quantities having units of length must grow at a rate proportional to a characteristic length, $R(t)$, that depends on $(t; \alpha, d_0)$. Hence, R must satisfy the same relationship as the length ξ does (see (24)) so that one has the following self-similarity relation:

$$R(b^{-\lambda}t; \alpha, d_0) = bR(t; \alpha/b^{-2-\lambda}, d_0/b^{3+\lambda}). \tag{30}$$

Equality (30) expresses the relation between the characteristic lengths of two systems with different parameters and at different times and also describes the necessary changes in the physical parameters, (α, d_0). The algebraic substitution $t = b^{\lambda}t_1$ into scaling equation (30) yields

$$R(t_1; \alpha, d_0) = bR(b^{\lambda}t_1; \alpha/b^{-2-\lambda}, d_0/b^{3+\lambda}). \tag{31}$$

Step 4: Recalling that, from Step 1, the calculations are valid for any $b > 0$ and any real-valued parameter λ that was to be determined. One then chooses $b = t_1^{-1/\lambda}$, (so that $b^{\lambda}t_1 = 1$), and rewrites identity (31), omitting the subscript 1 on t_1, as

$$R(t; \alpha, d_0) = t^{-1/\lambda}R(1; \alpha/t^{(2+\lambda)/\lambda}, d_0/t^{-(3+\lambda)/\lambda}). \tag{32}$$

We now examine this new identity in terms of its implications for the parameter λ.

Analysis of the parameter λ: The value of λ clearly determines the long-time asymptotics of the characteristic length, $R(t)$, provided that R is not singular in the second and third variables at 0. Both the cases $\lambda < -3$ and $\lambda > 0$ lead to the result that d_0 approaches ∞ as $t \to \infty$. These, however, yield fixed points that are physically not meaningful. Similarly, if $\lambda \in (-2, 0)$, then α approaches ∞ for large t, while d_0 approaches 0 which yield also the nonphysical fixed points. Hence, any possible value for λ which yields the nontrivial fixed point lies in the interval $[-3, -2]$. This indicates that the characteristic length, $R(t)$, increases as $t^{-1/\lambda}$ as λ varies in the continuous spectrum $[-3, -2]$.

The result also confirms, once again, for $\lambda \in (-3, -2]$ that the capillarity length, d_0, is essentially irrelevant for large time, which is sharp contrast with its stabilizing role for short times (see Refs. [14–16]). The only exception is the value $\lambda = -3$ (i.e. $R(t) \sim t^{1/3}$) for which the capillarity length, d_0, is invariant for large time. In this case, the scaling does not depend on the non-singularity of R as a function of d_0.

3. The case $\alpha = 0$

In this section, we set $\alpha = 0$ in Eq. (3) so that it has the form

$$T = \frac{-\sigma_0}{[s]_{\text{eq}}}\,\kappa \tag{33}$$

and examine the large-time characteristic of the characteristic length, $R(t)$, corresponding to the system of Eqs. (1), (2) and (33). Following Section 2.1 one rewrites the equations of the form

$$d_0\kappa = \frac{1}{D}\int_{\Gamma(t)} \mathrm{d}^{d-1}\sigma_y G(\vec{x} - \vec{y}) \upsilon_n(\vec{y}, t). \tag{34}$$

The RG analysis, following Section 2.2, yields the identity

$$R(t; \alpha, d_0) = t^{-1/\lambda} R(1; d_0/t^{-(3+\lambda)/\lambda}). \tag{35}$$

Analysis of the parameter λ, once again, determines the large-time characteristic of the characteristic length, $R(t)$. Since we expect a positive growth, λ must be negative, i.e. $\lambda < 0$. On the other hand, if $\lambda < -3$, then d_0 approaches ∞ as $t \to \infty$ that leads to the physically irrelevant fixed points. The values of λ which are physically relevant belong to the continuous spectrum, $[-3, 0)$. The single value of $\lambda = -1$ is selected from this spectrum by the plane wave imposed by Jasnow and Vinals [10].

Similarly, the result also shows, for $\lambda \in (-3, 0)$, that the capillarity length, d_0, is essentially irrelevant for large time. Once again, the exceptional case arises for the value of $\lambda = -3$ at which the capillarity length, d_0, is unchanged during the scaling of the large scale behavior. This yields $R(t) \sim t^{1/3}$ with no assumption of non-singularity of R. Moreover, the only scaling which d_0 is invariant has $t^{1/3}$ behavior.

4. Conclusions

We have performed a RG analysis for the large-scale dynamical behavior of the full two-phase interface problem, defined by the system of Eqs. (1)–(3) in the quasi-static regime. The calculations involve the implementation of renormalization group methods once a Green's function representation is introduced for the equations. Two cases were considered for the coefficient of the dynamical undercooling: $\alpha = 0$ and $\alpha \neq 0$. The latter condition includes the effect of a lower temperature on the interface that is associated with motion. We assume that the system evolves self-similarly with a single length scale and find that the characteristic length, $R(t)$, evolves as $t^{-1/\lambda}$ without reference a plane wave. For the case $\alpha \neq 0$ we find that a continuous spectrum of λ is possible, namely, $\lambda \in [-3, -2]$. The case $\alpha = 0$ corresponds to the completely quasi-static case as the dynamical undercooling effect is suppressed. Here the single length scale self-similarity implies the continuous spectrum $\lambda \in [-3, 0)$. When a particular plane wave is imposed [10] a single value, namely, $\lambda = -1$, is selected from this spectrum.

The difference in the exponents of the characteristic length suggests that there is an important difference between the fully-dynamic and static regime.

Another important conclusion resulting from this and prior works is that in almost all of these cases, the capillarity length, d_0, associated with the surface tension is irrelevant for the large-time behavior. This is a very important consequence since we know that the capillarity length is a crucial factor for the initial velocity and the linear stability of an interface. An interesting question is how the role of the capillarity length changes from the early stage growth to the late-stage growth.

The study of the late-stage interface behavior using RG provides a complement to the systematic approach provided by linear stability theory. As methodology is developed for these two regimes, the most challenging problem may be the understanding of the transition between the short-term and the long-term asymptotics and crossover behaviors.

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