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DECAY OF SOLUTIONS TO NONLINEAR PARABOLIC EQUATIONS: RENORMALIZATION AND RIGOROUS RESULTS

H. Merdan

Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260

G. Caginalp

Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260

Abstract. Scaling and renormalization group (RG) methods are used to study parabolic equations with a small nonlinear term and find the decay exponents. The determination of decay exponents is viewed as an asymptotically self similar process that facilitates an RG approach. These RG methods are extended to higher order in the small coefficient of the nonlinearity. The RG results are verified in some cases by rigorous proofs and other calculational methods.

1. **Introduction.** Renormalization group (RG) methods that were originally developed for quantum field theory and statistical mechanics (see text by Goldenfeld [10]) have been broadened in recent years to include a spectrum of problems such as fractals, self-avoiding random walk, difference equation (see text by Creswick, Farach and Poole [8] for an excellent survey) and have evolved into a broad philosophy rather than a single technique, as each new application often involves different methods. These methods have provided a powerful tool for calculation of key exponents that are otherwise extremely difficult to evaluate (see [10] and [8]).

Creswick et. al. demonstrate how some of the classical problems such as self-avoiding random walk can be understood clearly from an RG perspective. The diversity of the problems that have been understood through RG suggests that it has the potential to become a systematic tool of applied mathematics. Since differential equations are central to much of applied mathematics, it is important to examine RG in this context, particularly within classes of equations for which we can verify some of the results independently.

For differential equations which exhibit self-similarity at an asymptotic fixed point there are several aspects including (1) decay of solutions for large time and space, (2) finite time blow-up of solutions; and (3) finite time extinction of solutions. In particular the calculation of the exponent that characterizes decay, blow-up or extinction is a key question. Another aspect for systems of equations describing

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interface problems is (4) the large time evolution of the interface, and, for example, the exponent that characterizes the growth of the interface as a function of time.

Decay problems adapting RG methods and scaling theory has been studied by Goldenfeld, Martin, Oono and Liu [9], Bricmont, Kupiainen and Lin [2], and Caginalp [6, 7] (see other references therein). In particular, the decay exponent for the porous medium equation with a small nonlinear term was calculated using RG methods in [9].

We can write an important set of goals as follows: (a) render these methods more systematic within the context of applied mathematical methods, (b) define large classes of differential equations for which these methods lead to simple rules for asymptotic decay of solutions, (c) understand these classes of equations in terms of "universality classes" whereby different equations have similar behavior, (d) determine whether the methods can be implemented for higher order in ε, (e) verify the exponent results of RG [6] through different types of calculations, (f) prove the RG results rigorously.

For parts (a)-(c) a first step was undertaken in [6] where the equation

$$u_t = \frac{1}{2} u_{xx} + \varepsilon F(x, u, u_x, u_{xx}) \tag{1.1}$$

was considered in an infinite domain, where $F(x, u, u_x, u_{xx})$ is of the form $x^m u^n u_x^p u_{xx}^q$ and ε is a small parameter. The parameters (m, n, p, q) are constrained by dimensionality so that the nonlinear term has the same physical dimensions as u_{xx}, i.e.,

$$n + p + q = 1 \text{ and } p + 2q - m = 2. \tag{1.2}$$

The large time decay was found to be of the following form:

$$u(x, t) \sim t^{-\frac{1}{2}-\alpha} u^*(xt^{-1/2}, 1)$$

where α is a simple function of the powers of x, u, u_x, and u_{xx} in F. These results were also generalized to systems of parabolic equations [7].

In this work we focus on some of the key issues outlined above (particularly (d)-(f)) with two general goals for the case $q = 0$. First, we want to extend the RG analysis, particularly to examine terms of $O(\varepsilon^2)$ and higher. We also show that the RG process can be used to establish upper bounds for decay exponents. In particular, we determine the transform operators and perform an RG calculation that yields higher order terms beyond $O(\varepsilon)$. In fact this is an infinite series that can be summed to yield exact exponents in some cases. In other cases, if the $O(\varepsilon^2)$ term is negative, one can bound the exponent from above.

Second, we want to resolve rigorously and exactly the exponents for some nonlinearities described above. As part of this process we prove in some cases that the exponents obtained in [6] above are, in fact, the first terms of a convergent expansion in ε. In addition to the renormalization and rigorous calculations, we produce an iterative expansion. In particular, we transform the equations so that the exponent can be calculated exactly by solving iteratively a set of ordinary differential equations. The solution involves closed form integrals that can be evaluated in terms of error functions.

A subset of the exponents obtained in [6] using RG methods are proved rigorously using shooting methods. The rigorous and exact calculations confirming the results further bolster the observation that dimensionality expressed in (1.2) above is a key feature that governs the decay of solutions. The dimensionality criteria establishes large classes of equations with similar decay properties.

This paper is organized as follows. In section 2 we rewrite the equation (1.1) in terms of the fundamental solution, treating the nonlinear term as a source term. We apply asymptotic analysis in order to write the solution in terms of an integral that is in the appropriate form for the RG treatment. In section 3 we write the RG transformation for arbitrary order in ε. In section 4 we present alternative methods for calculating exponents that demonstrate agreement with the RG methods. The results are summarized in the conclusion (Section 5). A proof of a theorem that confirms earlier RG results is presented in the Appendix.

Decay of exponents to solutions of nonlinear equations have also been studied by related self-similarity methods in [3]- [5], [12], [16]. Renormalization group techniques have also been utilized in other dynamical differential equation problems (see, for example, [11], [13], [15], [17] and references therein).

2. **Renormalization group calculations.** Let ε be a small, positive, dimensionless number and consider the diffusion equation with the nonlinearity of the form

$$C_p u_{t'} = K\{u_{xx} + 2\varepsilon F[x, u, u_x, u_{xx}]\} \tag{2.1}$$

where C_p and K are constants (with $D := K/C_p$) and the nonlinear term, F, is a linear sum of the terms of the form $x^m u^n u_x^p u_{xx}^q$ where the integers m, n, p, q satisfy (1.2). Defining $t := 2Dt'$ (*which has units of* $(length)^2 = area$) we simplify notation and use (2.1) of the form

$$u_t = \frac{1}{2}u_{xx} + \varepsilon F[x, u, u_x, u_{xx}] \tag{2.2a}$$

for the remainder of the paper. We consider

$$u(x, 0; l) := g(x, l) = \frac{Q_0}{(2\pi l^2)^{1/2}} \exp\left(\frac{-x^2}{2l^2}\right) \tag{2.2b}$$

as the initial condition in which l is a small parameter in order to study the decay from a sharply peaked Gaussian and $Q_0 := T_0 Q_1$ with T_0 having temperature units and Q_1 length units. Our procedure is to extract, for each order in ε, the leading order behavior in l^{-1}, so that only positive contributions to the decay are significant in the $O(\varepsilon^2)$ and higher. A key step in this process is to obtain a transformation that rescales variables. While RG methods usually involve an identity in this transformation, we utilize the basic ideas by using an identity up to a particular order in ε.

Asymptotics of the heat equation with small nonlinearity. We obtain a basic solution for the equation (2.2a) with the initial condition (2.2b) below. Using the Green's Function

$$G(x, t) := \frac{1}{(2\pi t)^{1/2}} \exp\left(\frac{-x^2}{2t}\right) \tag{2.3}$$

and taking the nonlinearity F as a source term one can express the solution of (2.2a, b) as

$$u(x, t) = \int_{-\infty}^{\infty} G(x-y, t)g(y)dy + \varepsilon \int_0^t \int_{-\infty}^{\infty} G(x-y, t-s)F[y, u(y, s), ...]dyds. \tag{2.4}$$

We solve (2.4) using an asymptotic expansion for small ε and write the formal sum as

$$u(x, t; \varepsilon, l) = u_0(x, t; l) + \varepsilon u_1(x, t; l) + \varepsilon^2 u_2(x, t; l) + \varepsilon^3 u_3(x, t; l) + \cdots \tag{2.5}$$

so that l is not yet treated as a small number in comparison with ε here. Following [6] we write

$$u_0(x,t;l) = \frac{Q_0}{[2\pi(t+l^2)]^{1/2}} \exp\left(\frac{-x^2}{2(t+l^2)}\right), \tag{2.6a}$$

$$\frac{\partial u_0}{\partial x} = \left(\frac{-x}{t+l^2}\right) u_0, \tag{2.6b}$$

$$u_1(x,t;l) = \frac{Q_0}{(2\pi)^{1/2}} t^{-1/2} e^{-x^2/(2t)} \{(-1)^p (1 \cdot 3 \cdots |2p-3|)\} \log\left(\frac{t+l^2}{l^2}\right) \tag{2.7}$$

for $p \geq 1$ and $q := 0$, and

$$u_1(x,t;l) = \frac{Q_0}{(2\pi)^{1/2}} t^{-1/2} e^{-x^2/(2t)}$$

$$\cdot \left\{ \sum_{j=0}^{q} (-1)^{j+p} (1 \cdot 3 \cdots |2p+4q-2j-3|) \right\} \log\left(\frac{t+l^2}{l^2}\right) \tag{2.8}$$

for $q \neq 0$. The derivative of u_1 is given by $\frac{\partial u_1}{\partial x} = \left(\frac{-x}{t}\right) u_1$ and we rewrite it as

$$\frac{\partial u_1}{\partial x} = \left(\frac{-x}{t+l^2}\right) u_1 + \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right) u_1 \tag{2.9}$$

using the equality

$$\left(\frac{x}{t+l^2}\right) - \left(\frac{x}{t}\right) = \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right). \tag{2.10}$$

We proceed by using u_0 and u_1 to generate the next term of (2.5), namely u_2, and by using u_0, u_1 and u_2 to generate the u_3 term, and so on. The nonlinear term is taken as $x^m u^n u_x^p u_{xx}^q$ subject to (1.2) for the simplicity of the calculations as in [6]. In this work, we consider the case $q = 0$ only, so that the nonlinearity will be completely specified by $p \geq 1$, as $n = 1-p$ and $m = p-2$ (see (1.2)), and the nonlinear term is given by $F[x,u,u_x,u_{xx}] := x^{p-2}u^{1-p}u_x^p$. We should also mention here that p will be taken as $p \geq 1$ and $p \in Z^+$ throughout the paper. In addition, in the subsequent analysis the calculations are formally valid for

$$\varepsilon A \log(t/l^2) << 1.$$

In other words, for a given small ε the expansion is valid in some intermediate region of large t. Continuity arguments may be used to conjecture that the decay exponents obtained remain valid for arbitrarily large time. This has been confirmed in terms of the examples for which we have rigorous theorems or exact calculations. For the problem under consideration the result can be stated as follows:

Proposition 2.1. *Consider the equation* (2.2a) *with the initial condition* (2.2b). *One has to leading order in l within $O(\varepsilon^r)$ the solution*

$$u(x,t;\varepsilon,l) = \frac{Q_0}{\sqrt{2\pi}} t^{-1/2} e^{-x^2/(2t)} \sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log(t/l^2)\right]^j \tag{2.11}$$

for $p \geq 1$ and $q := 0$, where $A := A(p) := (-1)^p(1 \cdot 3 \cdots |2p-3|)$, where only non-negative terms contribute for $O(\varepsilon^2)$ and beyond.

VERIFICATION: We calculate the terms u_2 and u_3 below and the calculation of the rest of the terms can be done following a similar procedure in the calculation of the u_3 term. In the calculation of each term, we first state the estimates, then calculate the term, and finally prove the lemmas.

2.1. THE CALCULATION OF THE u_2 TERM.

I. The estimates. We have the following:

LEMMA 2.1. *We state the result (see* [1]*, p. 302, 7.4.4) as follow. Let*

$$L := \int_{-\infty}^{\infty} y^{2(p-1)} e^{-y^2/(2t)} dy. \tag{2.1.1}$$

It follows that $L = \sqrt{2\pi}\{1 \cdot 3 \cdots |2p-3|\} t^{p-\frac{1}{2}}$.

LEMMA 2.2. *Let*

$$L_{2,1} := \int_0^t (s+l^2)^{-p} s^{p-1} \log\left(\frac{s+l^2}{l^2}\right) ds. \tag{2.1.2}$$

We then have the following bounds

$$C_{2,1}^{(1)} \left(\frac{t}{l^2}\right)^{p+1} \le L_{2,1} < C_{2,1}^{(2)} \left(\frac{t}{l^2}\right)^{p+1} \qquad \text{for } t \le l^2 \tag{2.1.2a}$$

$$\frac{1}{2!}\left[\log\left(\frac{t}{l^2}\right)\right]^2 + C_{2,1}^{(3)} < L_{2,1}$$

$$< \frac{1}{2!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^2 + C_{2,1}^{(4)} \qquad \text{for } t > l^2 \tag{2.1.2b}$$

where $C_{2,1}^{(i)}$ *is a constant depending on* p *for* $i = 1, 2, 3, 4$.

LEMMA 2.3. *Let*

$$L_{2,2} := \int_0^t pl^2 (s+l^2)^{-p} s^{p-2} \log\left(\frac{s+l^2}{l^2}\right) ds. \tag{2.1.3}$$

We then have the following bounds

$$0 \le L_{2,2} < C_{2,2}^{(1)} \left(\frac{t}{l^2}\right)^{p} \qquad \text{for } t \le l^2 \tag{2.1.3a}$$

$$0 < L_{2,2} < C_{2,2}^{(2)} \qquad \text{for } t > l^2 \tag{2.1.3b}$$

where $C_{2,2}^{(i)}$ *is a constant depending on* p *for* $i = 1, 2$.

We prove Lemma 2.2 and Lemma 2.3 after the calculation of u_2 (see III. Proofs of lemmas) .

II. The calculation of the u_2 term. Using u_0 and u_1 we calculate the u_2 term in (2.5). In order to do this, we need to substitute $u := u_0 + \varepsilon u_1$ into (2.4) so that we first need to find $(u_0+\varepsilon u_1)^{(1-p)}(u_0+\varepsilon u_1)_x^p$. Using (2.6$b$) and (2.9) one has

$$(u_0+\varepsilon u_1)_x = \left(\frac{-x}{t+l^2}\right)(u_0+\varepsilon u_1) + \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right)(\varepsilon u_1). \tag{2.1.4}$$

Applying now the Binomial Formula to this we obtain

$$
\begin{aligned}
(u_0+\varepsilon u_1)^{(1-p)}(u_0+\varepsilon u_1)_x^p &= \left(\frac{-x}{t+l^2}\right)^p (u_0+\varepsilon u_1) + \lambda_1 \left(\frac{-x}{t+l^2}\right)^p \left(\frac{l^2}{t}\right)(\varepsilon u_1) \\
&\quad + \sum_{n=2}^{p} \lambda_n \left(\frac{-x}{t+l^2}\right)^p \left(\frac{l^2}{t}\right)^n \frac{(\varepsilon u_1)^n}{(u_0+\varepsilon u_1)^{n-1}} \\
&\cong \left(\frac{-x}{t+l^2}\right)^p (u_0+\varepsilon u_1) + \lambda_1 \left(\frac{-x}{t+l^2}\right)^p \left(\frac{l^2}{t}\right)(\varepsilon u_1) \\
&\quad + \sum_{n=2}^{p} \lambda_n \left(\frac{-x}{t+l^2}\right)^n \left(\frac{l^2}{t}\right)^n \frac{\varepsilon^n {u_1}^n}{{u_0}^{n-1}} \qquad (2.1.5)
\end{aligned}
$$

where $\lambda_n := \binom{p}{n} = \frac{p!}{(p-n)!n!}$. Substituting (2.1.5) into (2.5) and retaining up to $O(\varepsilon^2)$ terms lead to the expression

$$u_2(x,t;l) := I + J \qquad (2.1.6)$$

where

$$I := \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) y^{2(p-1)} (-1)^p (s+l^2)^{-p} u_1(y,s)dyds \qquad (2.1.6a)$$

$$J := \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) y^{2(p-1)} (-1)^p (s+l^2)^{-p} \lambda_1 l^2 s^{-1} u_1(y,s)dyds. \qquad (2.1.6b)$$

i. Evaluation of I integral. Using (2.3) and (2.7) we write (2.1.6a) as

$$
\begin{aligned}
I &:= \int_0^t \int_{-\infty}^{\infty} \frac{(t-s)^{-1/2}}{(2\pi)^{1/2}} \exp\left(\frac{-(x-y)^2}{2(t-s)}\right) y^{2(p-1)} (-1)^p (s+l^2)^{-p} \\
&\quad \cdot \frac{Q_0}{(2\pi)^{1/2}} s^{-1/2} e^{-y^2/(2s)} \{(-1)^p (1\cdot 3 \cdots |2p-3|)\} \log\left(\frac{s+l^2}{l^2}\right) dyds \\
&\cong \frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/(2t)} \{(-1)^{2p} (1\cdot 3 \cdots |2p-3|)\} \\
&\quad \cdot \int_0^t ds (s+l^2)^{-p} \log\left(\frac{s+l^2}{l^2}\right) s^{-1/2} \int_{-\infty}^{\infty} y^{2(p-1)} e^{-y^2/(2s)} dy. \qquad (2.1.7)
\end{aligned}
$$

The approximations involve replacing $t-s$ by t, and $x-y$ by x to obtain the $t^{-1/2}e^{-x^2/(2t)}$ term above. The justification (see [6], p. 9-12) is based on the Laplace's method for integrals, since the main contribution to the integral must arise from the regions near $y=0$ and $s=0$ for small l. Letting now

$$\gamma := \gamma(x,t) := \frac{Q_0}{\sqrt{2\pi}} t^{-1/2} e^{-x^2/(2t)} \qquad (2.1.8a)$$

$$A := A(p) := (-1)^p (1\cdot 3 \cdots |2p-3|) \qquad (2.1.8b)$$

and applying Lemma 2.1 to (2.1.7) one has

$$I \cong \gamma A^2 \int_0^t (s+l^2)^{-p} s^{p-1} \log\left(\frac{s+l^2}{l^2}\right) ds =: \gamma A^2 L_{2,1}(s;l,p). \qquad (2.1.9)$$

Now using Lemma 2.2 we have the following bounds for the first integral:

$$\gamma A^2 \left\{\frac{1}{2!}\left[\log\left(\frac{t}{l^2}\right)\right]^2 + C_{2,1}^{(3)}\right\} < I < \gamma A^2 \left\{\frac{1}{2!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^2 + C_{2,1}^{(4)}\right\}. \qquad (2.1.10)$$

ii. Evaluation of J integral. Similarly, following the procedure in the previous calculation one writes (2.1.6b) as

$$J \cong \gamma A^2 \int_0^t \lambda_1 l^2 (s+l^2)^{-p} s^{p-2} \log\left(\frac{s+l^2}{l^2}\right) ds =: \gamma A^2 L_{2,2}(s;l,p). \tag{2.1.11}$$

Using now Lemma 2.3 one has the following bounds for the integral J

$$0 < J < \gamma A^2 C(p), \tag{2.1.12}$$

where $C(p)$ is a constant depending on p. Combining (2.1.6), (2.1.10) and (2.1.12) one has

$$\gamma A^2 \left\{ \frac{1}{2!}\left[\log\left(\frac{t}{l^2}\right)\right]^2 + C_1 \right\} < u_2(x,t;l) < \gamma A^2 \left\{ \frac{1}{2!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^2 + C_2 \right\} \tag{2.1.13}$$

where C_1 and C_2 are constants. Furthermore, combining (2.1.6), (2.1.9) and (2.1.11) one can express $u_2(x,t;l)$ as

$$u_2(x,t;l) \cong \gamma A^2 [L_{2,1}(s;l,p) + L_{2,2}(s;l,p)]. \tag{2.1.14a}$$

so that using (2.10) one obtains the derivative of u_2 of the form:

$$\frac{\partial u_2}{\partial x} = \left(\frac{-x}{t+l^2}\right) u_2 + \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right) u_2. \tag{2.1.14b}$$

III. Proofs of lemmas.

Proof of Lemma 2.2. We begin by labeling some basic inequalities:

$$0 < log(1+z) < z \qquad for\ z > 0 \tag{2.1.15a}$$

$$\frac{z}{2c} \leq \log(1+z) \qquad for\ 0 \leq z < 1 \leq c \tag{2.1.15b}$$

$$log(z) < log(1+z) \qquad for\ z > 0. \tag{2.1.15c}$$

Letting $z := s/l^2$ we write (2.1.2) as

$$\begin{aligned} L_{2,1} &:= \int_0^t l^{-2p}(1+\frac{s}{l^2})^{-p} s^{p-1} \log\left(\frac{s+l^2}{l^2}\right) ds \\ &= \int_0^{t/l^2} (1+z)^{-p} z^{p-1} \log(1+z) dz. \end{aligned} \tag{2.1.16}$$

Upper and lower bounds for $t \leq l^2$: Using the inequality $(1+z)^{-p} \leq 1$ and (2.1.15a) one has

$$L_{2,1} < \int_0^{t/l^2} z^{p-1} z dz = \frac{1}{p+1}\left(\frac{t}{l^2}\right)^{p+1}. \tag{2.1.17}$$

Note that $(1+z)^{-p} \geq 2^{-p}$ for $z \leq 1$ (since $t \leq l^2$). Using then this inequality and (2.1.15b) one has

$$L_{2,1} \geq \int_0^{t/l^2} 2^{-p} z^{p-1} \frac{z}{2} dz = \frac{1}{2^{p+1}(p+1)}\left(\frac{t}{l^2}\right)^{p+1}. \tag{2.1.18}$$

Upper and lower bounds for $t > l^2$: In this case, we split the integral (2.1.16) into two parts as follows:

$$\begin{aligned} L_{2,1} &= \int_0^1 (1+z)^{-p} z^{p-1} \log(1+z) dz + \int_1^{t/l^2} (1+z)^{-p} z^{p-1} \log(1+z) dz \\ &=: L_{2,1}^{(1)} + L_{2,1}^{(2)} \end{aligned} \tag{2.1.19}$$

so that using (2.1.17) and (2.1.18) we have

$$0 < \frac{1}{2^{p+1}(p+1)} \le L_{2,1}^{(1)} < \frac{1}{p+1}. \tag{2.1.20}$$

Next we obtain an upper bound and a lower bound for $\mathrm{L}_{2,1}^{(2)}$. Using the inequality $z^{p-1} \le (1+z)^{p-1}$ for $z \ge 0$ we have

$$\begin{aligned} L_{2,1}^{(2)} &\le \int_1^{t/l^2} (1+z)^{-1} \log(1+z) dz \\ &= \frac{1}{2!} \left[\log\left(\frac{t+l^2}{l^2} \right) \right]^2 + const. \end{aligned} \tag{2.1.21}$$

Using the infinite series

$$\frac{1}{1+z} = \frac{1}{z} \frac{1}{1+\frac{1}{z}} = \frac{1}{z}(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots) \qquad for\ z > 1 \tag{2.1.22a}$$

one obtains

$$(1+z)^{-p} = z^{-p}(1 + \frac{\nu_1(p)}{z} + \frac{\nu_2(p)}{z^2} + \cdots). \tag{2.1.22b}$$

Using now (2.1.22b) and (2.1.15c) one has

$$\begin{aligned} L_{2,1}^{(2)} &> \int_1^{t/l^2} z^{-1}(1 + \frac{\nu_1(p)}{z} + \frac{\nu_2(p)}{z^2} + \cdots) \log(z) dz \\ &= \int_1^{t/l^2} z^{-1} \log(z) dz + \sum_{j=1}^{\infty} \int_1^{t/l^2} \nu_j(p) z^{-1-j} \log(z) dz \\ &> \frac{1}{2!} \left[\log\left(\frac{t}{l^2} \right) \right]^2 + C(p), \end{aligned} \tag{2.1.23}$$

where $C(p)$ is a constant. □

Proof of Lemma 2.3. Following the proof of Lemma 2.2 one writes (2.1.3) as

$$L_{2,2} = \int_0^{t/l^2} p(1+z)^{-p} z^{p-2} \log(1+z) dz \tag{2.1.24}$$

so that for $t \le l^2$ one has

$$0 \le L_{2,2} < \int_0^{t/l^2} z^{p-2} z dz = \left(\frac{t}{l^2} \right)^p. \tag{2.1.25}$$

For the large t/l^2, one similarly splits the integral into two parts as follows:

$$\begin{aligned} L_{2,2} &= \int_0^1 (1+z)^{-p} z^{p-2} \log(1+z) dz + \int_1^{t/l^2} (1+z)^{-p} z^{p-2} \log(1+z) dz \\ &=: L_{2,2}^{(1)} + L_{2,2}^{(2)} \end{aligned}$$

so that utilizing (2.1.25) one has

$$0 \le L_{2,2}^{(1)} < 1. \tag{2.1.26}$$

For the second part of the integral, namely $L_{2,2}^{(2)}$, using the inequality $(1+z)^{-p} < z^{-p}$ for $z > 1$ (to obtain the following upper bound) one has

$$\begin{aligned} 0 < L_{2,2}^{(2)} &= \int_1^{t/l^2} z^{-2}\log(1+z)dz \\ &< \int_1^{\infty} z^{-2}\log(1+z)dz < const. \end{aligned} \tag{2.1.27}$$

□

2.2. The calculation of the u_3 term

I. The estimates. We have the following:

Lemma 2.4. *Let*

$$L_{3,1} := \int_0^t (s+l^2)^{-p} s^{p-1}\{L_{2,1} + L_{2,2}\}ds \tag{2.2.1}$$

where $L_{2,1}$ and $L_{2,2}$ are as defined in Lemma 2.2 and in Lemma 2.3, respectively. We then have the following bounds

$$C_{3,1}^{(1)}\left(\frac{t}{l^2}\right)^{2p+1} \le L_{3,1} < C_{3,1}^{(2)}\left(\frac{t}{l^2}\right)^{2p+1} + C_{3,1}^{(3)}\left(\frac{t}{l^2}\right)^{2p} \qquad for\ t \le l^2 \tag{2.2.1a}$$

$$\begin{aligned} \frac{1}{3!}\left[\log\left(\frac{t}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t}{l^2}\right)\right) &< L_{2,1} \\ < \frac{1}{3!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t+l^2}{l^2}\right)\right) & \qquad for\ t > l^2 \end{aligned} \tag{2.2.1b}$$

where $C_{3,1}^{(j)}$ is a constant depending on p for $j = 1, 2, 3$.

Lemma 2.5. *Let*

$$L_{3,2} := \int_0^t \lambda_1 l^2 (s+l^2)^{-p} s^{p-1}\{L_{2,1} + L_{2,2}\}ds \tag{2.2.2}$$

where $L_{2,1}$ and $L_{2,2}$ are as in Lemma 2.4. We then have the following bounds

$$0 \le L_{3,2} < C_{3,2}^{(1)}\left(\frac{t}{l^2}\right)^{2p} + C_{3,2}^{(2)}\left(\frac{t}{l^2}\right)^{2p-1} \qquad for\ t \le l^2 \tag{2.2.2a}$$

$$0 \le L_{3,2} < C_{3,2}^{(3)}(p) \qquad for\ t > l^2 \tag{2.2.2b}$$

where $C_{3,2}^{(i)}$ is a constant depending on p for $i = 1, 2, 3$.

Lemma 2.6. *Let*

$$L_k := \int_0^t \lambda_n l^{2n}(s+l^2)^{-p+(\frac{n-1}{2})} s^{p-(\frac{3n+1}{2})}\left[\log\left(\frac{s+l^2}{l^2}\right)\right]^n ds \tag{2.2.3}$$

for $p \ge 2$, $n \ge 2$, $p \ge n$ and $p,\ n \in Z^+$. We then have the following bounds

$$0 \le L_k < C_k^{(1)}(p,n)\left(\frac{t}{l^2}\right)^{p-\frac{n}{2}+\frac{1}{2}} \qquad for\ t \le l^2 \tag{2.2.3a}$$

$$0 \le L_k < C_k^{(2)}(p,n) \qquad for\ t > l^2 \tag{2.2.3b}$$

where $C_k^{(1)}$ and $C_k^{(2)}$ are constants depending on p and n.

The lemmas above are proved in the Appendix (see Part B).

II. The calculation of the u_3 term. Following a similar procedure in the calculation of the u_2 term we calculate u_3 so that we first need to find $(u_0+\varepsilon u_1+\varepsilon^2 u_2)^{(1-p)}(u_0+\varepsilon u_1+\varepsilon^2 u_2)_x^p$. Using (2.1.4) with together (2.1.14b) we write $(u_0+\varepsilon u_1+\varepsilon^2 u_2)_x$ as

$$(u_0+\varepsilon u_1+\varepsilon^2 u_2)_x = \left(\frac{-x}{t+l^2}\right)(u_0+\varepsilon u_1+\varepsilon^2 u_2) + \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right)(\varepsilon u_1+\varepsilon^2 u_2). \tag{2.2.4}$$

Similarly, applying the Binomial Formula we obtain

$$\begin{aligned}(u_0+\varepsilon u_1+\varepsilon^2 u_2)^{(1-p)}(u_0+\varepsilon u_1+\varepsilon^2 u_2)_x^p &\cong \left(\frac{-x}{t+l^2}\right)^p (u_0+\varepsilon u_1+\varepsilon^2 u_2)\\ &+ \lambda_1\left(\frac{-x}{t+l^2}\right)^p\left(\frac{l^2}{t}\right)(\varepsilon u_1+\varepsilon^2 u_2)\\ &+ \sum_{n=2}^{p}\lambda_n\left(\frac{-x}{t+l^2}\right)^p\left(\frac{l^2}{t}\right)^n\frac{\varepsilon^n {u_1}^n}{{u_0}^{n-1}}.\end{aligned} \tag{2.2.5}$$

Substituting (2.2.5) into (2.5) and retaining up to $O(\varepsilon^3)$ terms yield the expression

$$u_3(x,t;l) := M + N + Q \tag{2.2.6}$$

where

$$M := \int_0^t\int_{-\infty}^{\infty} G(x-y,t-s)y^{2(p-1)}(-1)^p(s+l^2)^{-p}u_2(y,s)dyds \tag{2.2.6a}$$

$$N := \int_0^t\int_{-\infty}^{\infty} G(x-y,t-s)y^{2(p-1)}(-1)^p(s+l^2)^{-p}\lambda_1 l^2 s^{-1}u_2(y,s)dyds \tag{2.2.6b}$$

$$Q := \int_0^t\int_{-\infty}^{\infty} G(x-y,t-s)y^{2(p-1)}(-1)^p(s+l^2)^{-p}\lambda_2 l^4 s^{-2}\frac{(u_1(y,s))^2}{u_0(y,s)}dyds. \tag{2.2.6c}$$

i. Evaluation of M integral. Using (2.3) and (2.1.14a) we write (2.2.6a) as

$$\begin{aligned}M &:= \int_0^t\int_{-\infty}^{\infty}\frac{(t-s)^{-1/2}}{(2\pi)^{1/2}}\exp\left(\frac{-(x-y)^2}{2(t-s)}\right)y^{2(p-1)}(-1)^p(s+l^2)^{-p}\\ &\cdot\frac{Q_0}{(2\pi)^{1/2}}s^{-1/2}e^{-y^2/(2s)}\{(-1)^p(1\cdot 3\cdots|2p-3|)\}^2[L_{2,1}(s;l,p)+L_{2,2}(s;l,p)]dyds\\ &\cong \frac{Q_0}{2\pi}t^{-1/2}e^{-x^2/(2t)}\{(-1)^{3p}(1\cdot 3\cdots|2p-3|)^2\}\\ &\cdot\int_0^t ds(s+l^2)^{-p}[L_{2,1}(s;l,p)+L_{2,2}(s;l,p)]s^{-1/2}\int_{-\infty}^{\infty}y^{2(p-1)}e^{-y^2/(2s)}dy.\end{aligned} \tag{2.2.7}$$

As done in the previous calculation, we approximate $t^{-1/2}e^{-x^2/(2t)}$ by replacing $t-s$ by t, and $x-y$ by x (see 2.1.7). Applying now Lemma 2.1 to (2.2.7) and using (2.1.8a,b) we have

$$M \cong \gamma A^3\int_0^t (s+l^2)^{-p}s^{p-1}[L_{2,1}(s;l,p)+L_{2,2}(s;l,p)]ds =: \gamma A^3 L_{3,1}(s;l,p). \tag{2.2.8}$$

Using now Lemma 2.4 we have the following bounds for the first integral:

$$\gamma A^3\left\{\frac{1}{3!}\left[\log\left(\frac{t}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t}{l^2}\right)\right)\right\} < M$$

$$< \gamma A^3 \left\{ \frac{1}{3!} \left[\log\left(\frac{t+l^2}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t+l^2}{l^2}\right)\right)\right\}. \tag{2.2.9}$$

ii. Evaluation of N integral. Following a similar procedure in (i) one writes (2.2.6b) as

$$N \cong \gamma A^3 \int_0^t \lambda_1 l^2 (s+l^2)^{-p} s^{p-2} [L_{2,1}(s;l,p) + L_{2,2}(s;l,p)] ds =: \gamma A^3 L_{3,2}(s;l,p) \tag{2.2.10}$$

so that applying Lemma 2.5 yields the following bounds for the second integral

$$0 \leq N < \gamma A^3 C(p), \tag{2.2.11}$$

where $C(p)$ is a constant depending on p.

iii. Evaluation of Q integral. Using (2.3), (2.6a) and (2.7) we first write (2.2.6c) as

$$\begin{aligned} Q := &\int_0^t \int_{-\infty}^{\infty} \frac{(t-s)^{-1/2}}{(2\pi)^{1/2}} \exp\left(\frac{-(x-y)^2}{2(t-s)}\right) y^{2(p-1)} (-1)^p (s+l^2)^{-p} \lambda_2 l^4 s^{-2} \\ &\cdot \frac{Q_0}{(2\pi)^{1/2}} \{(-1)^p (1\cdot 3 \cdots |2p-3|)\}^2 s^{-1} (s+l^2)^{1/2} \\ &\cdot \left[\log\left(\frac{s+l^2}{l^2}\right)\right]^2 e^{-2y^2/(2s)} e^{y^2/(2(s+l^2))} dy ds \end{aligned}$$

and applying then Laplace's method for integrals (see 2.1.7) we obtain

$$\begin{aligned} Q \cong &\frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/(2t)} \{(-1)^{3p} (1\cdot 3\cdots |2p-3|)^2\} \\ &\cdot \int_0^t ds \lambda_2 l^4 (s+l^2)^{-p+\frac{1}{2}} \left[\log\left(\frac{s+l^2}{l^2}\right)\right]^2 s^{-3} \\ &\cdot \int_{-\infty}^{\infty} y^{2(p-1)} e^{-2y^2/(2s)} e^{y^2/(2(s+l^2))} dy. \end{aligned} \tag{2.2.12a}$$

Letting now, for $n \geq 2$,

$$\begin{aligned} \widehat{L}_n := &\int_0^t ds \lambda_n l^{2n} (s+l^2)^{-p+(\frac{n-1}{2})} \left[\log\left(\frac{s+l^2}{l^2}\right)\right]^2 s^{-3n/2} \\ &\cdot \int_{-\infty}^{\infty} y^{2(p-1)} \exp(-ny^2/(2s)) \exp((n-1)y^2/(2(s+l^2))) dy \end{aligned} \tag{2.2.13}$$

and using (2.1.8a, b) we rewrite (2.2.12a) as

$$Q \cong: \gamma A^2 (-1)^p (2\pi)^{-1/2} \widehat{L}_3(s,l;p). \tag{2.2.12b}$$

Now consider, for $n \geq 2$,

$$\begin{aligned} L_k^{(n)} &:= \int_{-\infty}^{\infty} y^{2(p-1)} \exp\left(\frac{-ny^2}{2s}\right) \exp\left(\frac{(n-1)y^2}{2(s+l^2)}\right) dy \\ &= \int_{-\infty}^{\infty} y^{2(p-1)} \exp\left(\frac{-y^2}{2s}\right) \exp\left(\frac{-(n-1)y^2}{2s}\right) \exp\left(\frac{(n-1)y^2}{2(s+l^2)}\right) dy \end{aligned} \tag{2.2.14a}$$

so that

$$L_k^{(n)} \leq \int_{-\infty}^{\infty} y^{2(p-1)} \exp\left(\frac{-y^2}{2s}\right) dy$$

since $\exp\left(\frac{-(n-1)y^2}{2s}\right)\exp\left(\frac{(n-1)y^2}{2(s+l^2)}\right) \le 1$. Thus, applying Lemma 2.1 one has

$$L_k^{(n)} \le (2\pi)^{1/2}\{1\cdot 3\cdots|2p-3|\}s^{p-\frac{1}{2}}. \tag{2.2.14b}$$

Using (2.2.14a, b) (with n=2) and (2.1.8a,b) we rewrite (2.2.12a) as

$$Q \le \gamma A^3 \int_0^t \lambda_2 l^4 (s+l^2)^{-p+\frac{1}{2}} s^{p-\frac{7}{2}} \left[\log\left(\frac{s+l^2}{l^2}\right)\right]^2 ds =: \gamma A^3 L_3(s;l,p) \tag{2.2.15}$$

so that applying Lemma 2.6 (2.2.3b with $n=2$) we obtain the following bounds for the third integral

$$0 \le Q < \gamma A^3 C(p,n), \tag{2.2.16}$$

where C is a constant depending on p and n. Once again, combining (2.2.6), (2.2.9), (2.2.11), and (2.2.16) yields the following bounds for u_3:

$$\gamma A^3\{\left\{\frac{1}{3!}\left[\log\left(\frac{t}{l^2}\right)\right]^2 + O\left(\log\left(\frac{t}{l^2}\right)\right)\right\} < u_3(x,t;l)$$

$$< \gamma A^3\left\{\frac{1}{3!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t+l^2}{l^2}\right)\right)\right\}. \tag{2.2.17}$$

As done before (see (2.1.14)), one can also express u_3 as

$$u_3(x,t;l) := \gamma A^3 L_{3,1}(s;l,p) + \gamma A^3 L_{3,2}(s;l,p) + \gamma A^2(-1)^p(2\pi)^{-1/2}\widehat{L}_3(s,l;p) \tag{2.2.18}$$

that can be used to obtain the derivative of u_3 in order to calculate the next term in (2.5).

Following a similar procedure above, the u_k term can be calculated so that combining this with (2.6a), (2.7), (2.1.13), (2.1.17) in (2.5) leads to leading order in ε and to leading order in l within $O\left(\varepsilon^k\right)$ the solution (2.11).

3. **The renormalization group transformations.**

If one has an asymptotic relation such as (2.11), one can then calculate the anomalous exponent explicitly and obtain the similarity solution for large time and space. The arguments below are within the context of formal applied analysis without reference to numerical procedures or physical analogies. We follow the methodology in [6] and [14]. For the problem under consideration, we state the result as follows using true dimensions:

PROPOSITION 3.1. *Suppose u can be expressed as*

$$u(x,t';\varepsilon,l) = \frac{T_0}{2\pi^{1/2}}\left(\frac{t'}{Q_1^2/D}\right)^{-1/2} e^{-x^2/(4Dt')}\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log(2Dt'/l^2)\right]^j \tag{3.1}$$

where A is independent of x, t', ε and l. Then, to leading order in ε^r, u can be expressed as

$$u(x,t';\varepsilon,l) = \left(\frac{t'}{Q_1^2/D}\right)^{-\frac{1}{2}+\varepsilon A} u_r^*\left(\frac{x}{(Dt'/Q_1^2)^{1/2}}, \frac{Q_1^2}{D}\right) \tag{3.2}$$

so that the anomalous exponent is given by only εA. The fixed point function u_r^* has the following form

$$u_r^*(\xi,\tau) = \frac{T_0}{2\pi^{1/2}}\exp\left(-\frac{\xi^2}{4D\tau}\right)\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log\left(\frac{2D}{l^2}\tau\right)\right]^j. \tag{3.3}$$

VERIFICATION. We verify the claim for arbitrary $r \in Z^+$ below by dividing the derivation into five stages. Particular, the verification was done for $r = 1$ and $r = 2$ in [6] and [14], respectively.

Stage 1. We first need to find an identity (*up to* $O(\varepsilon^r)$) of the form

$$u(b^{\phi}x, bt') = Z_r(b)u(x, t') \tag{3.4}$$

which is valid for a particular choice of Z_r and ϕ and all $b > 1$. One can easily see that the exponential term in (3.1) yields $\phi = \frac{1}{2}$. We next rewrite (3.1) up to $O(\varepsilon^r)$ as

$$u(b^{1/2}x, bt') = \frac{T_0}{2\pi^{1/2}} \left(\frac{t'}{Q_1^2/D}\right)^{-1/2} e^{-\frac{x^2}{4Dt'}} \left\{\sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log(2Dt'/l^2)\right]^j\right\} \tag{3.5}$$
$$\cdot\, b^{-1/2} \left\{\sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log(b)\right]^j\right\}$$

that leads to the expression

$$Z_r(b) := b^{-1/2} \left\{\sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log(b)\right]^j\right\}. \tag{3.6}$$

Note that Z_r does not depend upon l.

Following [14] and [6] we now define the operator as

$$R_{b,\phi}u(x, t') := \frac{1}{Z_r(b)} u(b^{1/2}x, bt'). \tag{3.7}$$

Stage 2. By iteration we have (suppressing ε and l and ignoring $O(\varepsilon^{r+1})$ terms)

$$u(b^{k/2}x, b^k t') = Z_r(b)^k u(x, t'). \tag{3.8}$$

A fixed point of this iteration will exist only if

$$u_r^*(x, t') := \lim_{k\to\infty} Z_r(b)^{-k} u(b^{k/2}x, b^k t') \tag{3.9}$$

is well defined. Under the assumption of the existence of a fixed point in this formal derivation, we rewrite this for large but finite k as

$$u(b^{k/2}x, b^k t') \cong Z_r(b)^k u_r^*(x, t'). \tag{3.10}$$

Letting now $\bar{x} := b^{k/2}x$ and $\bar{t} := b^k t'$ one rewrites the last equation so that one has (for large k)

$$u(\bar{x}, \bar{t}) \cong \left[Z_r(b)\right]^k u_r^* \left(\bar{x}b^{-k/2}, \bar{t}b^{-k}\right). \tag{3.11}$$

This means that for any large $\bar{t}$ we can determine the u profile by setting $b^k := \bar{t}/(Q_1^2/D)$ (so that the second argument does not change as $\bar{t}$ varies), and write (3.11) as

$$u(\bar{x}, \bar{t}) \cong \left[Z_r\left({t_1}^{1/k}\right)\right]^k u_r^* \left(\bar{x}t_1^{-1/2}, Q_1^2/D\right) \tag{3.12}$$

by letting $t_1 := D\bar{t}/Q_1^2$.

Stage 3. The limit below (if it exists)

$$\lim_{k\to\infty} \left[Z_r\left({t_1}^{1/k}\right)\right]^k = \lim_{k\to\infty} t_1^{-1/2} \left\{\sum_{j=0}^{r} \frac{1}{j!} \left[\frac{\varepsilon A \log(t_1)}{k}\right]^j\right\}^k \tag{3.13}$$

will determine the scaling exponent. Letting $y := \frac{\varepsilon A \log(t_1)}{k}$ one can show that

$$\sum_{j=0}^{r} \frac{1}{j!} \left[\frac{\varepsilon A \log(t_1)}{k} \right]^j = \frac{1}{r!} \sum_{j=0}^{r} \frac{r!}{(r-j)!} y^{r-j}$$
$$= \frac{1}{r!} \prod_{j=1}^{r} (y + \beta_j) = \frac{\beta_1 \beta_2 \cdots \beta_r}{r!} \prod_{j=1}^{r} \left(1 + \frac{y}{\beta_j} \right) \tag{3.14}$$

where $\beta_1, \beta_2, \cdots, \beta_r$ are roots of the polynomial $\prod_{j=1}^{r} (y + \beta_j)$ such that $\frac{\beta_1 \beta_2 \cdots \beta_r}{r!} = 1$, and $\sum_{j=1}^{r} (1/\beta_j) = 1$ (see [14] for $r = 2$). Utilizing now the asymptotic expansion $e^x \cong 1 + x$ for small x we have

$$\left\{ \sum_{j=0}^{r} \frac{1}{j!} \left[\frac{\varepsilon A \log(t_1)}{k} \right]^j \right\}^k \cong \prod_{j=1}^{r} \exp \left[\frac{\varepsilon A}{\beta_j} \log(t_1) \right] = t_1^{\varepsilon A}. \tag{3.15}$$

that yields the result

$$\lim_{k \to \infty} \left[Z_r \left({t_1}^{1/k} \right) \right]^k = t_1^{-\frac{1}{2} + \varepsilon A}. \tag{3.16}$$

Stage 4. We first substitute (3.16) into (3.12), and then drop the superbar since (3.12) is valid for arbitrary large $\bar{t}$. This yields the identity

$$u(x, t') = \left(Dt'/Q_1^2 \right)^{-\frac{1}{2} + \varepsilon A} u_r^* \left(x \left(Dt'/Q_1^2 \right)^{-1/2}, Q_1^2/D \right) \tag{3.17}$$

so that the anomalous exponent or "dimension" is $\alpha = -\varepsilon A$.

Stage 5. To obtain u_r^* we first rewrite (3.1) as

$$u(x, t'; \varepsilon, l) = \frac{T_0}{2\pi^{1/2}} \left(\frac{t'}{Q_1^2/D} \right)^{-1/2} e^{-\frac{x^2}{4Dt'}} \left\{ \sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log \left(\frac{Dt'}{Q_1^2} \right) \right]^j \right\}$$
$$\cdot \left\{ \sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log \left(\frac{2Q_1^2}{l^2} \right) \right]^j \right\}. \tag{3.18}$$

Following a procedure similar to (3.14)-(3.15) one can show

$$\sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log \left(\frac{Dt'}{Q_1^2} \right) \right]^j \cong \prod_{j=1}^{r} \exp \left[\frac{\varepsilon A}{\beta_j} \log \left(\frac{Dt'}{Q_1^2} \right) \right] = \left(\frac{Dt'}{Q_1^2} \right)^{\varepsilon A} \tag{3.19}$$

so that one has

$$u(x, t'; \varepsilon, l) = \frac{T_0}{2\pi^{1/2}} \left(\frac{Dt'}{Q_1^2} \right)^{-1/2 + \varepsilon A} \exp \left(-\frac{x^2}{4Dt'} \right) \left\{ \sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log \left(\frac{2Q_1^2}{l^2} \right) \right]^j \right\}. \tag{3.20}$$

Comparison of (3.20) with (3.17) yields u_r^* as

$$u_r^*\left(\frac{x}{(Dt'/Q_1^2)^{1/2}},\frac{Q_1^2}{D}\right) = \frac{T_0}{2\pi^{1/2}}\exp\left(-\frac{\left(\frac{x}{(Dt'/Q_1^2)^{1/2}}\right)^2}{4D\left(Q_1^2/D\right)}\right) \cdot\left\{\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)\right]^j\right\}$$

which is (3.3).

4. **Exact results.** In this section we consider exact solutions to some special cases of (2.2a) subject to constraints (1.2) with the aim of checking the RG calculations. In each of the calculations below we transform (2.2a) into

$$\varphi_\tau = \frac{1}{2}\left[\varphi_{\xi\xi}+\xi\varphi_\xi+\varphi_\xi^2\right]+\varepsilon F\left[\xi,1,\varphi_\xi,\varphi_{\xi\xi}+\varphi_\xi^2\right] \tag{4.1}$$

using the change of variables

$$u(x,t) := e^{\varphi(\xi,\tau)}, \quad \tau := \log(t+t_0) \quad \text{and} \quad \xi := x(t+t_0)^{-1/2} \tag{4.2}$$

where F is taken as in section 2. This transformation was utilized in [14] to determine exact solutions for the following two cases:

Example 1. The equation

$$u_t = \frac{1}{2}u_{xx}+\varepsilon x^{-1}u_x \tag{4.3}$$

has the exact (non-negative) solution

$$u\left(x,t-t_0\right) = t^{-\frac{1}{2}-\varepsilon}e^{\frac{-x^2}{2t}}. \tag{4.4}$$

Examples 2. The non-linear equation

$$u_t = \frac{1}{2}u_{xx}+\varepsilon u^{-1}u_x^2 \tag{4.5}$$

has the exact (non-negative) solution

$$u\left(x,t-t_0\right) = t^{\frac{-1}{2(1-2\varepsilon)}}\exp\left(\frac{-x^2}{(1-2\varepsilon)2t}\right). \tag{4.6}$$

These examples confirm the RG results ([6] and Section 3).

A related asymptotic method for equations of the form (4.1) is described in [14]. Briefly, this procedure involves using the formal expansion for the special variables,

$$\varphi\left(\xi,\tau;\varepsilon\right) = \phi_0\left(\xi,\tau\right)+\varepsilon\phi_1\left(\xi,\tau\right)+\varepsilon^2\phi_2\left(\xi,\tau\right)+\cdots \tag{4.7}$$

in equation (4.1). Substituting this expansion into equation (4.1) we obtain a hierarchy of equations in terms of order in ε. The zeroth order equation is the nonlinear equation

$$\phi_{0\tau} = \frac{1}{2}\left[\phi_{0\xi\xi}+\xi\phi_{0\xi}+\phi_{0\xi}^2\right] \tag{4.8}$$

while the remaining equations involve the linear operator

$$\mathbf{L}\phi = \frac{1}{2}\left[\phi_{\xi\xi}-\xi\phi_\xi\right]. \tag{4.9}$$

In particular, the first order equation is

$$\phi_{1\tau}-\mathbf{L}\phi_1 = \Omega_0\left(\xi\right) \tag{4.10}$$

where

$$\Omega_0(\xi) := F\left[\xi, 1, \phi_{0\xi}, \phi_{0\xi\xi} + \phi_{0\xi}^2\right]. \tag{4.11}$$

One can verify that the zeroth order nonlinear equation has the solution

$$\phi_o(\xi,\tau) = -\frac{1}{2}\xi^2 - \frac{1}{2}\tau. \tag{4.12}$$

The coefficient of τ yields the leading (classical) exponent $\alpha_0 = 1/2$, i.e., $t^{-1/2}$. The linear equations can be solved with the constraints imposed by the boundary conditions. The analysis yields the next term in the exponent

$$\alpha_1 = -\frac{\int_0^\infty \Omega_0(\eta)\, e^{-\eta^2/2} d\eta}{\int_0^\infty e^{-\eta^2/2} d\eta} \tag{4.13}$$

that provides the correction to the classical decay, i.e. $t^{-1/2}$, and agrees with the RG calculations.

Similarly, higher order corrections can be generated by analyzing successive linear equations.

Self-similar solutions. Another method for calculating these exponents involves self-similarity methods. We consider the case $q = 0$ so that we write (4.1) as

$$\varphi_\tau = \frac{1}{2}\left[\varphi_{\xi\xi} + \varphi_\xi^2 + \xi\varphi_\xi\right] + \varepsilon\xi^{p-2}\varphi_\xi^p. \tag{4.14}$$

One is looking for an exact solution to this of the form

$$\varphi(\xi,\tau;\varepsilon) = \phi(\xi;\varepsilon) - \alpha(\varepsilon)\tau, \tag{4.15}$$

where $(\alpha,\phi) \in R^1 \mathrm{x} C^2(R)$ is the unknown, so that

$$u(x, t - t_0) = t^{-\alpha} e^{\phi\left(x/\sqrt{t}\right)}. \tag{4.16}$$

It is then equivalent to solve

$$\begin{gathered} \ddot{\phi} + \dot{\phi}^2 + \xi\dot{\phi} + \varepsilon 2\xi^{p-2}\dot{\phi}^p + 2\alpha = 0 \quad \text{on } (-\Xi, \Xi) \\ \lim_{\xi\to\pm\Xi} e^{\phi(\xi;\varepsilon)} = 0, \qquad \lim_{\xi\to\pm\Xi} \dot{\phi} e^{\phi} = 0 \end{gathered} \tag{4.17}$$

where Ξ is either finite or infinite and $\cdot := \frac{d}{d\xi}$. Note that if Ξ is finite, then $u = u_x \equiv 0$ for $|x| \geq \Xi\sqrt{t}$ and we have a compactly supported self similar solution to (2.2a).

Suppose we are looking for an even solution, i.e., $\phi(-\xi) = \phi(\xi)$. Setting $w(\xi;\varepsilon) = \dot{\phi}(\xi;\varepsilon)$ one now has

$$\begin{gathered} \dot{w} + \xi w + w^2 + \varepsilon 2\xi^{p-2} w^p + 2\alpha = 0 \quad \text{on } (0, \Xi) \\ w(0) = 0, \qquad \lim_{\xi\to\Xi} w = -\infty, \qquad \lim_{\xi\to\Xi} w e^{\int_0^\xi w(\eta)d\eta} = 0. \end{gathered} \tag{4.18}$$

Using shooting methods one can show the existence of a unique solution (α,ϕ) (see the Appendix, Part A). Introducing the expansion

$$\alpha(\varepsilon) = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots \tag{4.19}$$

$$w(\xi;\varepsilon) = w_0(\xi) + \varepsilon w_1(\xi) + \varepsilon^2 w_2(\xi) + \cdots \tag{4.20}$$

one has a sequence of IVP's:

$$\dot{w}_0 = -2\alpha_0 - \xi w_0 - w_0^2, \qquad w_0(0) = 0 \tag{4.21}$$

$$\dot{w}_1 = -2\alpha_1 - \xi w_1 - 2w_0 w_1 - 2\xi^{p-2} w_0^p, \qquad w_1(0) = 0 \tag{4.22}$$

$$\dot{w}_2 = -2\alpha_2 - \xi w_2 - (w_1^2 + 2w_0 w_2) - 2p\xi^{p-2} w_0^{p-1} w_1, \qquad w_2(0) = 0 \tag{4.23}$$

$$\dot{w_k} = -2\alpha_k + \xi w_k + \theta\left(\xi, w_0, \cdots, w_{k-1}; p\right), \qquad w_k(0) = 0 \quad for\ k \geq 3 \tag{4.24}$$

where θ is a known function of these variables.

Note that the initial conditions in (4.21)-(4.24) ensure (through the expansion (4.20) for w) the first of the three conditions in (4.18). The second condition can be guaranteed by imposing it on w_0 alone, since the remaining terms are of the lower order in ε. The third condition in (4.18) can be written as

$$(w_0(\xi) + \varepsilon w_1(\xi) + \cdots)e^{\int_0^\xi w_0(\eta)d\eta} e^{\varepsilon \int_0^\xi w_1(\eta)d\eta + \varepsilon^2 \int_0^\xi w_2(\eta)d\eta + \cdots} \to 0 \tag{4.25}$$

and can be ensured by imposing the condition

$$\lim_{\xi \to \Xi} w_i e^{\int_0^\xi w(\eta)d\eta} = 0, \qquad \text{for } i = 0, 1, 2, \ldots. \tag{4.26}$$

With this additional condition we proceed to solve (4.21)-(4.24) for w_i and α_i. Note that the first of these is nonlinear while all of the others are linear in terms of the differentiated function. Accordingly the treatment differs in the two cases. One can easily verify that equation (4.21), subject to the limiting condition above, has a solution

$$\alpha_0 = \frac{1}{2} \text{ and } w_0(\xi) = -\xi \quad \Rightarrow \quad \phi_0(\xi) = -\frac{1}{2}\xi^2. \tag{4.27}$$

Further discussion of nonlinear equations of this type can be found in the Appendix.

The remaining equations (4.22)-(4.24) can be solved successively by multiplying by the integrating factor. In particular, upon multiplication by $e^{-\xi^2/2}$, (4.22) has the form

$$\frac{d}{d\xi}\left[e^{-\xi^2/2} w_1(\xi)\right] = e^{-\xi^2/2}\left[-2\alpha_1 - 2(-1)^p \xi^{p-2}\right] \tag{4.28}$$

so that integration on $[0, \infty)$ and utilizing the condition (4.26)($\lim_{\xi \to \Xi} w_1 e^{-\xi^2/2} = 0$) yields the value

$$\alpha_1 = \frac{(-1)^{p+1}\int_0^\infty \eta^{2p-2} e^{-\eta^2/2} d\eta}{\int_0^\infty e^{-\eta^2/2} d\eta} == (-1)^{p+1}\left(1 \cdot 3 \cdot \cdots \cdot |2p-3|\right). \tag{4.29}$$

Thus, this result yields the same exponent as the formal RG calculation (Caginalp [6]).

One needs to find a solution of (4.28) to obtain the next term, α_2, which is important in determining the most singular term in the anomalous exponent for nonlinear diffusion, and also to compare the results obtained by the RG methods. Substituting the value above for α_1 (as well as $w_0(\xi) = -\xi$) into (4.22) we obtain a first order linear IVP with variable coefficients that can be solved by standard methods. Thus , a solution to (4.35) is given by

$$w_1(\xi) = 0 \qquad for\ p = 1 \tag{4.30}$$

$$w_1(\xi) = \sum_{k=1}^{p-1} \frac{(-1)^p 2\left(1 \cdot 3 \cdot \cdots \cdot |2p-3|\right)}{\left(1 \cdot 3 \cdot \cdots \cdot |2k-1|\right)} \xi^{2k-1} \qquad for\ p \geq 2. \tag{4.31}$$

Next, we substitute these expressions for w_0 and w_1 into (4.23), and utilize the same methods to obtain the value of α_2 as

$$\alpha_2 = \frac{(-1)^p\, p \int_0^\infty \eta^{2p-3} w_1(\eta) e^{-\eta^2/2} d\eta - \frac{1}{2}\int_0^\infty w_1^2(\eta) e^{-\eta^2/2} d\eta}{\int_0^\infty e^{-\eta^2/2} d\eta}. \tag{4.32}$$

Evaluating these integrals yields the following values:

$$\alpha_2 = 0 \qquad for\ p = 1 \tag{4.33}$$

$$\alpha_2 = 2p\,(1\cdot 3\cdot\cdots\cdot|2p-3|)\sum_{k=1}^{p-1}\frac{(1\cdot 3\cdot\cdots\cdot|2(k+p)-5|)}{(1\cdot 3\cdot\cdots\cdot|2k-1|)}$$

$$-2\,(1\cdot 3\cdot\cdots\cdot|2p-3|)^2\sum_{r=1}^{p-1}\sum_{j=1}^{p-1}\frac{(1\cdot 3\cdot\cdots\cdot|2(r+j)-3|)}{(1\cdot 3\cdot\cdots\cdot|2r-1|)\,(1\cdot 3\cdot\cdots\cdot|2j-1|)}\quad for\ p\geq 2. \tag{4.34}$$

Note that the sign of α_2 would depend on that of the nonlinear term F. In addition, following the similar procedure above one can calculate α_k for $k \geq 3$.

REMARK. Note that if $p = 1$ in (4.14) one has the equation corresponding to (4.3) which has a solution

$$u(x,t) = t^{-\frac{1}{2}-\varepsilon}e^{-x^2/2t}$$

that is identical to (4.4).

5. **Conclusion.** We have developed the renormalization group ideas to higher order in ε by deriving the operator Z in (3.4) that allows us to write expressions such as (3.2). Our procedure is to extract, for each order in ε, the leading order behavior in l^{-1}, in a large but finite interval, so that only positive contributions to the decay are significant in $O(\varepsilon^2)$ and higher. A key step in this process is to obtain a transformation that rescales variables. While RG methods usually involve an identity in this transformation, we utilize the basic ideas by using an identity up to a particular order in ε.

For example, the equation

$$u_t = \frac{1}{2}u_{xx} + \varepsilon x^{-1}u_x$$

is characterized by the large time behavior

$$u(x,t) \sim t^{-1/2-\varepsilon/2}$$

since we can characterize all of the higher order terms as an exponential that is the sum of a convergent infinite series. The exponential with logarithmic terms as arguments can then be written as $t^{-\varepsilon/2}$.

The methodology presented in this paper can be expected to be useful in other problems in which there is an asymptotically self-similar structure. Examples of other such situations are finite-time blow up and extinction of solutions to nonlinear differential equations.

The rigorous results presented in this paper confirm that some of the earlier RG calculations are in fact valid for arbitrarily large time and space.

6. **Appendix.** PART A.

THEOREM 6.1. *Suppose that $F(x,u,p,q)$ is independent of q, that F/p^2 is smooth such that it and its first derivative are uniformly bounded. Then there exists a unique positive number, $\alpha(\varepsilon)$, such that the equation (2.2a) has a solution of the form*

$$u(x,t,\varepsilon) = t^{-\alpha(\varepsilon)}e^{\phi(xt^{-1/2},\varepsilon)}, \tag{A1}$$

where α and ϕ have the limiting properties

$$\lim_{\varepsilon\to 0}\alpha(\varepsilon) = \frac{1}{2}, \qquad \lim_{\varepsilon\to 0}\phi(\xi,\varepsilon) = -\frac{1}{4}\xi^2. \tag{A2}$$

Proof. We let Ψ be defined by

$$F(1,1,p,q)=\Psi(p)p^2 \tag{A3}$$

and assume that the Ψ and Ψ' are bounded by 1 in absolute value, since the larger constants can be absorbed into ε in (2.2). We use a shooting argument to prove the existence of a solution of self-similar type, i.e., ($A1$ above). We look for even solutions ϕ, i.e., $\phi(-\xi)=\phi(\xi)$, and set $w(\xi)=\phi'(\xi)$. Hence it is equivalent to the equation,

$$w'+w^2+\frac{1}{2}\xi w+\varepsilon F(\xi,1,w,w'+w^2)+\alpha=0 \tag{A4}$$

for all $\xi\in(0,\Xi)$, where $(0,\Xi)$ is the maximal existence interval, subject to [see (4.18)]

$$w(0)=0,\qquad \lim_{\xi\to\Xi} w=-\infty,\qquad \lim_{\xi\to\Xi} we^{\int_0^\xi w(\eta)d\eta}=0. \tag{A5}$$

Proposition 6.1. *For all $\alpha\in\mathbf{R}$ and $\xi\in[0,\Xi(\alpha))$ one has the inequality*

$$\frac{\partial w}{\partial\alpha}<0. \tag{A6}$$

Proof. The ODE for w is an initial value problem with smooth coefficients and so there is a unique solution for each α. Suppose that $\alpha_1<\alpha_2$. We show that the corresponding solutions, w_1 and w_2, cannot intersect. Initially, the right hand side of (A4) implies $w_1>w_2$ since $w_1(0)=w_2(0)=0$. Suppose for the purpose of contradiction that for some ξ_0 one has $w_1(\xi_0)=w_2(\xi_0)$. Then w_1 and w_2 satisfy the respective equations

$$w_1'=-\alpha_1-w_1^2\left(1+\varepsilon\Psi(\xi_0w_1)-\frac{1}{2}\xi w_1,\right. \tag{A7}$$

$$w_2'=-\alpha_2-w_2^2\left(1+\varepsilon\Psi(\xi_0w_2)-\frac{1}{2}\xi w_2.\right. \tag{A8}$$

Hence, $w_1'(\xi_0)-w_2'(\xi_0)=-\alpha_1+\alpha_2>0$, so that w_1' dominates w_2' at this point so that w_1 cannot cross below w_2. We can write this inequality for any α and α^* as

$$\frac{w(\xi;\alpha)-w(\xi;\alpha^*)}{\alpha-\alpha^*}\le 0, \tag{A9}$$

so that taking the limit as $\alpha\to\alpha^*$ we obtain the bound for the derivative

$$\frac{\partial w}{\partial\alpha}=\lim_{\alpha\to\alpha^*}\frac{w(\xi;\alpha)-w(\xi;\alpha^*)}{\alpha-\alpha^*}\le 0. \tag{A10}$$

□

Proposition 6.2. *If $\alpha>\frac{1}{2}+C^2\varepsilon$ for sufficiently large $C\in\mathbf{R}^+$ then $\Xi(\alpha)<\infty$. Furthermore, $lim_{\xi\to\Xi(\alpha)}we^{\int_0^\xi w(\eta)d\eta}=-1$.*

Proof. Suppose $\alpha\ge\frac{1}{2}+C^2\varepsilon$ for some large $C\in\mathbf{R}^+$. Then for any ξ we can write

$$\left\{z-\left(\frac{1}{2}+\frac{C^2}{2}\varepsilon\right)\xi\right\}'\ge\frac{C^2\varepsilon}{2}+(1-\varepsilon)z\left\{z-\frac{1}{1-\varepsilon}\frac{\xi}{2}\right\}. \tag{A11}$$

Using $1+C^2\varepsilon>(1-\varepsilon)^{-1}$ in the left hand side one can rewrite this as

$$\left\{z-\frac{1}{1-\varepsilon}\frac{\xi}{2}\right\}'\ge\frac{C^2}{2}\varepsilon+(1-\varepsilon)z\left\{z-\frac{1}{1-\varepsilon}\frac{\xi}{2}\right\}. \tag{A12}$$

Let $Z(\xi) := z - \frac{1}{1-\varepsilon}\frac{\xi}{2}$. Since the initial condition implies $Z(0) = 0$, then clearly Z is initially positive in the equation above. For comparison we consider the equation

$$Y' = \frac{C^2}{2}\varepsilon + (1-\varepsilon)Y^2 \tag{A13}$$

which has solutions

$$Y(\xi) = C\left(\frac{\varepsilon}{1-\varepsilon}\right)^{1/2} \tan\left\{C\sqrt{\varepsilon(1-\varepsilon)}(\xi + C_1)\right\}. \tag{A14}$$

Since tan diverges for finite values of its argument, $Y(\xi)$ diverges for finite ξ. Comparing $Y(\xi)$ with $Z(\xi)$ for the same initial conditions we see that Z (and hence z) also diverge for finite ξ. □

PROPOSITION 6.3. *If $\alpha < \frac{1}{2} - C^2\varepsilon$ for sufficiently large $C \in \mathbf{R}$ then $\Xi(\alpha) = \infty$ and $lim_{\xi\to\infty} w = 0$. Furthermore, one has $lim_{\xi\to\infty} w e^{\int_0^\xi w(\eta)d\eta} = 0$.*

Proof. Again using $z = -w$ one has

$$\left(z - \frac{1}{2}\xi\right)' = -C^2\varepsilon + z(z - \frac{1}{2}\xi) + \varepsilon z^2\Psi(-\xi z). \tag{A15}$$

As a consequence of the initial condition $z(0) = 0$ and the inequality $z'(0) = \frac{1}{2} - C^2\varepsilon > 0$ one has $z(\xi) > 0$ at least for some interval $(0, \xi_0)$. If at some point ξ_1, one has $z(\xi_1) = 0$ then the middle terms vanish and one obtains

$$z'(\xi_1) = \frac{1}{2} - C^2\varepsilon > 0. \tag{A16}$$

Consequently, one has the result that $z(\xi) > 0$ for all ξ. Next, we prove that $z(\xi) < \xi/2$ for all ξ. We first prove that this is the case at least when $\xi < 2C$. Initially, $(z - \frac{1}{2}\xi)\mid_{\xi=0} = 0$ and $(z - \frac{1}{2}\xi)' < 0$ so that $z - \frac{1}{2}\xi < 0$ at least for some maximal interval $(0, \xi_1)$. Suppose that $z(\xi_1) = \frac{1}{2}\xi_1$. Then we have

$$(z - \frac{1}{2}\xi)'\mid_{\xi=\xi_1} = -C^2\varepsilon + \varepsilon(\frac{1}{2}\xi_1)^2\Psi < 0 \tag{A17}$$

if $\xi_1 < 2C$. Consequently, $z - \frac{1}{2}\xi < 0$ on this interval. □

PROPOSITION 6.4. *Suppose ξ_0 is such that $z(\xi_0) < (2\varepsilon)^{-\frac{1}{2}}$. Then one cannot have a neighborhood of ξ_0 such that in this neighborhood, $\xi < \xi_0$ implies $z'(\xi_0) < 0$ while $\xi > \xi_0$ implies $z'(\xi_0) > 0$.*

Proof. From (A15) we have for all ξ,

$$\begin{aligned} z'' &= z'(z - \frac{1}{2}\xi) + z(z' - \frac{1}{2}) \\ &\quad + 2\varepsilon z z'\Psi + \varepsilon z^2\Psi'(-z - \xi z'). \end{aligned} \tag{A18}$$

If $z' = 0$ then this becomes

$$z'' = -\frac{1}{2}z - \varepsilon z^3\Psi'. \tag{A19}$$

Since $\mid \Psi' \mid \leq 1$ one has that $z'' < 0$ so long as $\varepsilon z^3 < z/2$, i.e.,

$$z < (2\varepsilon)^{-1/2}. \tag{A20}$$

Hence, z' cannot change sign from negative to positive so long as z satisfies this inequality. □

PROPOSITION 6.5. *At $\xi = C/2$ one has*

$$z(C/2) - \frac{1}{2}(C/2) \leq -(\frac{1}{2} - \frac{1}{96})C^3\varepsilon. \tag{A21}$$

Proof. Consider the interval $0 \leq \xi \leq C/2$ in which $z < \xi/2$. Then the original ODE for z implies

$$\begin{aligned}(z - \frac{1}{2}\xi)' &\leq -C^2\varepsilon + z(z - \frac{1}{2}\xi) + \varepsilon z^2 \\ &\leq -C^2\varepsilon + \varepsilon z^2\end{aligned} \tag{A22}$$

so that integrating this expression results in the inequality

$$\begin{aligned}\int_0^{C/2} (z - \frac{1}{2}\xi)' d\xi &\leq \int_0^{C/2} \left\{-C^2\varepsilon + \varepsilon z^2\right\} \\ &\leq -(\frac{1}{2} - \frac{1}{96})C^3\varepsilon\end{aligned} \tag{A23}$$

from which the conclusion follows. □

In view of the estimates obtained above, we can write, for some small $|\gamma|$, the inequality

$$\begin{aligned}(z - \frac{1}{2}\xi)' &= -C^2\varepsilon + z\left\{(1 + \varepsilon\Psi)z - \frac{1}{2}\xi\right\} \\ &\leq -C^2\varepsilon - \frac{\xi z}{2 + \gamma}.\end{aligned} \tag{A24}$$

We compare z satisfying this inequality with solutions of the equation for y below:

$$y' = C_0 - \frac{\xi y}{2 + \gamma} \tag{A25}$$

subject to the initial condition $y(C/2) := z(C/2)$ with $C_0 := \frac{1}{2} - C^2\varepsilon$. Solutions to this equation have the form

$$y(\xi) = C_0 e^{-\xi^2/(2+\gamma)^2} \int e^{s^2/(2+\gamma)^2} ds + C_1 e^{-\xi^2/(2+\gamma)^2}. \tag{A26}$$

Note that both terms on the right hand side approach zero with the first term dominating as it diminishes as $1/\xi$. Hence one has the bound

$$0 < z(\xi) < y(\xi) \tag{A27}$$

so that $z(\xi) \to 0$ as $\xi \to \infty$. By standard degree theory arguments we obtain the conclusion that for some $\alpha = \alpha(\varepsilon)$ satisfies

$$\frac{1}{2} - C^2\varepsilon < \alpha(\varepsilon) < \frac{1}{2} + C^2\varepsilon \tag{A28}$$

and the boundary conditions. The conclusion of Theorem 6.1 follows. □

Theorem 6.1 thus proves rigorously (for the subset of nonlinearities defined by the hypothesis) the RG calculations of the decay exponents. The results also confirm that the decay exponents are valid for arbitrarily large time and space.

PART B. *Proof of Lemma 2.4.* We first find an upper and a lower bound for the case $t \leq l^2$. We first set $z := s/l^2$. Using (2.1.2a) and (2.1.3a) we write (2.2.1) as

$$L_{3,1} < \int_0^{t/l^2} (1+z)^{-p} z^{p-1} \{C_{2,1}^{(2)} z^{p+1} + C_{2,2}^{(1)} z^p\} dz$$
$$= \frac{C_{2,1}^{(2)}}{2p+1} \left(\frac{t}{l^2}\right)^{2p+1} + \frac{C_{2,2}^{(1)}}{2p} \left(\frac{t}{l^2}\right)^{2p} \quad \text{(B1)}$$

since $(1+z)^{-p} \leq 1$. Similarly, using (2.1.2b), (2.1.3b), and also the inequality $(1+z)^{-p} \geq 2^{-p}$ for $z \leq 1$ we obtain the following lower bound

$$L_{3,1} \geq \int_0^{t/l^2} (1+z)^{-p} z^{p-1} \{C_{2,1}^{(1)} z^{p+1}\} dz$$
$$= \frac{C_{2,1}^{(1)}}{2^p(2p+1)} \left(\frac{t}{l^2}\right)^{2p+1}. \quad \text{(B2)}$$

For the case $t > l^2$, we first split the integral (2.2.1) into two parts as follows:

$$L_{3,1} := \int_0^{l^2} (s+l^2)^{-p} s^{p-1} \{ L_{2,1} + L_{2,2}\} ds + \int_{l^2}^{t} (s+l^2)^{-p} s^{p-1} \{L_{2,1} + L_{2,2}\} ds$$
$$:= L_{3,1}^{(1)} + L_{3,1}^{(2)} \quad \text{(B3)}$$

so that one has, from (B1) and (B2),

$$\frac{C_{2,1}^{(1)}}{2^p(2p+1)} \leq L_{3,1}^{(1)} < \frac{C_{2,1}^{(2)}}{2p+1} + \frac{C_{2,2}^{(1)}}{2p}. \quad \text{(B4)}$$

To find an upper bound and a lower bound for $L_{3,1}^{(2)}$ we apply Lemma 2.2 and Lemma 2.3 so that $L_{3,1}^{(2)}$ is bounded above by

$$L_{3,1}^{(2)} < \int_1^{t/l^2} (1+z)^{-p} z^{p-1} \{\frac{1}{2!}[\log(1+z)]^2\} dz + (C_{2,1}^{(4)} + C_{2,2}^{(2)}) \int_1^{t/l^2} (1+z)^{-p} z^{p-1} dz$$
$$= \frac{1}{3!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^3 + (C_{2,1}^{(4)} + C_{2,2}^{(2)}) \left[\log\left(\frac{t+l^2}{l^2}\right)\right] + C(p) \quad \text{(B5)}$$

by using the inequality $z^{p-1} \leq (1+z)^{p-1}$ for $z > 1$. Similarly, utilizing (2.1.22a, b) $L_{3,1}^{(2)}$ is bounded below by

$$L_{3,1}^{(2)} > \int_1^{t/l^2} (1+z)^{-p} z^{p-1} \{\frac{1}{2!}[\log(z)]^2\} dz + C_{2,1}^{(3)} \int_1^{t/l^2} (1+z)^{-p} z^{p-1} dz$$
$$= \frac{1}{3!}\left[\log\left(\frac{t}{l^2}\right)\right]^3 + C_{2,1}^{(3)} \left[\log\left(\frac{t}{l^2}\right)\right] + C(p), \quad \text{(B6)}$$

where $C\,(p)$ is a constant depending on p. □

Proof of Lemma 2.5 Following the previous proof, for the case $t \leq l^2$ one can show

$$L_{3,2} < \int_0^{t/l^2} \lambda_1 (1+z)^{-p} z^{p-2} \{C_{2,1}^{(2)} z^{p+1} + C_{2,2}^{(1)} z^p\} dz$$
$$= \frac{\lambda_1 C_{2,1}^{(2)}}{2p} \left(\frac{t}{l^2}\right)^{2p} + \frac{\lambda_1 C_{2,2}^{(1)}}{2p-1} \left(\frac{t}{l^2}\right)^{2p-1} \quad \text{(B7)}$$

and

$$L_{3,2} \geq \int_0^{t/l^2} \lambda_1(1+z)^{-p} z^{p-2}\{C_{2,1}^{(1)} z^{p+1}\}dz$$
$$= \frac{\lambda_1 C_{2,1}^{(1)}}{2^{p+1}p}\left(\frac{t}{l^2}\right)^{2p} \geq 0. \tag{B8}$$

Similarly, for the case $t > l^2$ one first splits the integral (2.2.2) into two parts as follows:

$$L_{3,2} := \int_0^{l^2} \lambda_1(s+l^2)^{-p}s^{p-2}\{L_{2,1}+L_{2,2}\}ds + \int_{l^2}^{t} \lambda_1(s+l^2)^{-p}s^{p-2}\{\ L_{2,1}+L_{2,2}\}ds$$
$$:= L_{3,2}^{(1)} + L_{3,2}^{(2)}. \tag{B9}$$

A similar procedure above yields the following bounds:

$$\frac{\lambda_1 C_{2,1}^{(1)}}{2^{p+1}p} \leq L_{3,2}^{(1)} < \frac{\lambda_1 C_{2,1}^{(2)}}{2p} + \frac{\lambda_1 C_{2,2}^{(1)}}{2p-1}, \tag{B10}$$

$$L_{3,2}^{(2)} < \lambda_1 \int_1^{t/l^2} z^{-2}\{\frac{1}{2!}[\log(1+z)]^2\}dz + (C_{2,1}^{(4)}+C_{2,2}^{(2)})\lambda_1 \int_1^{t/l^2} z^{-2}dz < const \tag{B11}$$

and

$$L_{3,2}^{(2)} > \int_1^{t/l^2} \lambda_1(1+z)^{-p}z^{p-2}\{\frac{1}{2!}[\log(z)]^2\}dz + C_{2,1}^{(3)}\int_1^{t/l^2} \lambda_1(1+z)^{-p}z^{p-2}dz \geq C(p), \tag{B12}$$

where $C\,(p)$ is a constant depending on p. □

Proof of Lemma 2.6. We first write (2.3.3) as

$$L_k := \int_0^{t/l^2} \lambda_n(1+z)^{-p+(\frac{n-1}{2})} z^{p-(\frac{3n+1}{2})}[\log(1+z)]^n dz. \tag{B13}$$

We use (2.1.15a) with the inequality $(1+z)^{-p+(\frac{n-1}{2})} \leq 1$ (since $-p+\frac{n}{2}-\frac{1}{2} < -p+n-\frac{1}{2} < 0$ for $p \geq 2$, $n \geq 2$, $p \geq n$ and $n, p \in Z^+$) so that for $t \leq l^2$ we have the following bounds for L_k

$$0 \leq L_k < \int_0^{t/l^2} \lambda_n z^{p-(\frac{3n+1}{2})} z^n dz = \frac{\lambda_n}{p-\frac{n}{2}+\frac{1}{2}}\left(\frac{t}{l^2}\right)^{p-\frac{n}{2}+\frac{1}{2}}. \tag{B14}$$

Note that $p-(\frac{3n+1}{2})+n = \frac{p}{2}+\frac{p}{2}-\frac{n}{2}-\frac{1}{2} > 0$ for $p \geq 2$, $n \geq 2$, $p \geq n$ and n, $p \in Z^+$. Thus, for the large t/l^2 we first split the integral into two parts as

$$L_k = \int_0^1 \lambda_n(1+z)^{-p+(\frac{n-1}{2})} z^{p-(\frac{3n+1}{2})}[\log(1+z)]^n dz$$
$$+ \int_1^{t/l^2} \lambda_n(1+z)^{-p+(\frac{n-1}{2})} z^{p-(\frac{3n+1}{2})}[\log(1+z)]^n dz \tag{B15}$$
$$=: L_k^{(1)} + L_k^{(2)}$$

so that utilizing (B14) we have

$$0 \leq L_k^{(1)} < \frac{\lambda_n}{p-\frac{n}{2}+\frac{1}{2}}. \tag{B16}$$

For the second part of the integral, namely $L_k^{(2)}$, we use the inequality $(1+z)^{-p+(\frac{n-1}{2})} < z^{-p+(\frac{n-1}{2})}$ in order to obtain the following upper bound

$$0 < L_k^{(2)} = \int_1^{t/l^2} z^{-n-1}[\log(1+z)]^n dz$$
$$< \int_1^{\infty} z^{-n-1}[\log(1+z)]^n dz < const. \quad \text{(B17)}$$

□

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E-mail address: humst4@pitt.edu
E-mail address: caginalp@pitt.edu