

Renormalization Group Methods for Nonlinear Parabolic Equations

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ABSTRACT. Renormalization group (RG) methods are described for determining the key exponents related to the decay of solutions to nonlinear parabolic differential equations. Higher order (in the small coefficient of the nonlinearity) methods are developed. Exact solutions and theorems in some special cases confirm the RG results.

Keywords. Renormalization group, nonlinear parabolic equations, decay of solutions, asymptotic analysis, asymptotic self-similarity

I. INTRODUCTION.

Renormalization group methods (RG) have provided a powerful tool for calculation of key exponents that are otherwise extremely difficult to evaluate (see texts by Goldenfeld [1] and Creswick, Farach and Poole [2]). These methods were originally developed for quantum field theory and statistical mechanics, particularly critical phenomena, and have evolved into a broad philosophy rather than a single technique, as each new application often involves different methods.

Since differential equations are central to much of applied mathematics, it is important to examine whether RG can be used successfully to calculate important features of solutions.

There are several aspects of differential equations in which self-similarity is exhibited at an asymptotic fixed point. These include (i) decay of solutions for large time and space, (ii) finite time blow-up of solutions; and (iii) finite time extinction of solutions. In particular a key question involves the exponent that characterizes decay, blow-up or extinction. For systems of equations describing interface problems an interesting issue is (iv) the large time evolution of the interface, and, for example, the exponent that characterizes the length of the interface as a function of time.

Decay problems using renormalization group techniques were studied by Goldenfeld, Martin, Oono and Liu [3], Bricmont, Kupiainen and Lin [4], and Caginalp [5] (see other references therein). In particular, Goldenfeld et. al. used RG to calculate the decay exponent for the porous medium equation having a small nonlinear term, and showed that it differed from the classical heat equation.

An important set of goals has been to (a) make these methods more systematic within the context of applied mathematical methods, (b) define large classes of differential equations for which these methods lead to simple rules for asymptotic decay of solutions, (c) understand these classes of equations in terms of "universality classes" whereby different equations have similar behavior, (d) determine whether the results extend to larger values of the small parameter, ε , (e) verify the exponents numerically, or analytically with different methodology, (f) prove the dimensionality results of RG and conjectures made in [5].

In this paper we consider the heat equation with a small nonlinearity, with the restriction that the dimensionality (as discussed below) is consistent with the u_{xx} term of the equation. The role of

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dimensionality in determining decay of solutions has only recently been recognized [5]. We demonstrate that the RG transformation can be carried out to higher order in the small coefficient of the nonlinearity. Examples of exact solutions confirm the exponents obtained from the RG analysis.

II. RENORMALIZATION GROUP CALCULATIONS AND ASYMPTOTICS.

Let ε be a small, positive, dimensionless number and consider the diffusion equation with the nonlinearity of the form

$$u_t = \frac{1}{2}u_{xx} + \varepsilon F[x, u, u_x, u_{xx}] \quad (2.1a)$$

where t' is the original time variable, D is the diffusion coefficient and $t := 2Dt'$. The nonlinear term, F , is a linear sum of terms in the form (with the integers m, n, p, q constrained to be dimensionally compatible with the u_{xx} term):

$$x^m u^n u_x^p u_{xx}^q, \quad (n + p + q = 1, p + 2q - m = 2). \quad (2.2)$$

We consider the initial condition

$$u(x, 0; l) := g(x, l) = \frac{Q_0}{(2\pi l^2)^{1/2}} \exp\left(\frac{-x^2}{2l^2}\right) \quad (2.1b)$$

in which l is a small parameter, in order to study the decay from a sharply peaked Gaussian with amplitude Q_0 . Our procedure is to extract, for each order in ε , the leading order behavior in l^{-1} , so that only positive contributions to the decay are significant in the $O(\varepsilon^2)$ and higher. A key step in this process is to obtain a transformation that rescales variables. While RG methods usually involve an identity in this transformation, we utilize the basic ideas by using an identity up to a particular order in ε .

We first obtain a basic solution for the equation (2.1a) with the initial condition (2.1b). Using the Green's Function

$$G(x, t) := \frac{1}{(2\pi t)^{1/2}} \exp\left(\frac{-x^2}{2t}\right) \quad (2.3)$$

and taking the nonlinearity F as a source term one can express the solution of (2.1a, b) as

$$u(x, t) = \int_{-\infty}^{\infty} G(x-y, t)g(y)dy + \varepsilon \int_0^t ds \int_{-\infty}^{\infty} G(x-y, t-s) \cdot F[y, u(y, s), u_y(y, s), u_{yy}(y, s)]dy. \quad (2.4)$$

We solve (2.4) using an asymptotic expansion for small ε and write the formal sum,

$$u(x, t; \varepsilon, l) = u_0(x, t; l) + \varepsilon u_1(x, t; l) + \varepsilon^2 u_2(x, t; l) + \varepsilon^3 u_3(x, t; l) + \dots \quad (2.5)$$

so that l is not yet treated as a small number in comparison with ε here.

In this work, we consider the case $q = 0$ only, so that the nonlinearity will be completely specified by $p \geq 1$, as $n = 1 - p$ and $m = p - 2$, and the nonlinear term is given by $x^{p-2}u^{1-p}u_x^p$.

Proposition 1. *Consider the equation (2.1a) with the initial condition (2.1b). One has to leading order in ε and to leading order in l within $O(\varepsilon^r)$ the solution*

$$u(x, t; \varepsilon, l) = \frac{Q_0}{\sqrt{2\pi}} t^{-1/2} e^{-x^2/(2t)} \sum_{j=0}^r \frac{1}{j!} [\varepsilon A \log(t/l^2)]^j \quad (2.6)$$

for $p \geq 1$ and $q := 0$, where $A := A(p) := (-1)^p(1 \cdot 3 \cdots |2p - 3|)$, where only non-negative terms contribute for $O(\varepsilon^2)$ and beyond.

The details of the formal asymptotics will be published elsewhere.

III. THE RENORMALIZATION GROUP TRANSFORMATIONS.

In this section we calculate the anomalous exponent explicitly for the asymptotic relation (2.6) and obtain the similarity solution for large time and space. The result can be stated as follows (using true dimensions).

Proposition 2. *Suppose u can be expressed as :*

$$u(x, t'; \varepsilon, l) = \frac{T_0}{2\pi^{1/2}} \left(\frac{t'}{Q_1^2/D} \right)^{-1/2} e^{-x^2/(4Dt')} \sum_{j=0}^r \frac{1}{j!} [\varepsilon A \log(2Dt'/l^2)]^j \quad (3.1)$$

where A is defined as in Prop. 1. Then, to leading order in ε , u can be expressed as

$$u(x, t'; \varepsilon, l) = \left(\frac{t'}{Q_1^2/D} \right)^{-\frac{1}{2} + \varepsilon A} u_r^* \left(\frac{x}{(Dt'/Q_1^2)^{1/2}}, \frac{Q_1^2}{D} \right) \quad (3.1a)$$

so that the anomalous exponent is given by only εA . The fixed point function u_r^* has the following form

$$u_r^*(\xi, \tau) = \frac{T_0}{2\pi^{1/2}} \exp \left(-\frac{\xi^2}{4D\tau} \right) \sum_{j=0}^r \frac{1}{j!} \left[\varepsilon A \log \left(\frac{2D}{l^2} \tau \right) \right]^j. \quad (3.1b)$$

Verification: The verification of (3.1a, b) was given by Caginalp [5] for $r = 1$. We verify the claim for $r = 2$ in this section; the verification for the case $r \geq 3$ can be done by following the same procedure.

Stage 1: One needs to obtain an identity [up to $O(\varepsilon^2)$] of the form

$$u(b^\phi x, bt') = Z_2(b)u(x, t') \quad (3.2)$$

which is valid for a particular choice of Z_2 and ϕ and all $b > 1$. Note that the exponential term in (3.1) forces $\phi = \frac{1}{2}$ for each r . Rewriting (3.1) up to $O(\varepsilon^2)$ one has

$$\begin{aligned} u(b^{1/2}x, bt') &= \frac{T_0}{2\pi^{1/2}} \left(\frac{t'}{Q_1^2/D} \right)^{-1/2} e^{-\frac{x^2}{4Dt'}} \\ &\cdot \left\{ 1 + \varepsilon A \log(2Dt'/l^2) + \frac{1}{2!} [\varepsilon A \log(2Dt'/l^2)]^2 \right\} \\ &\cdot b^{-1/2} \left\{ 1 + \varepsilon A \log(b) + \frac{1}{2!} [\varepsilon A \log(b)]^2 \right\} \end{aligned} \quad (3.3)$$

so that (3.2) is satisfied with $\phi = \frac{1}{2}$ and

$$Z_2(b) := b^{-1/2} \left\{ 1 + \varepsilon A \log(b) + \frac{1}{2!} [\varepsilon A \log(b)]^2 \right\}. \quad (3.4)$$

Stage 2: By iteration one obtains (suppressing ε and l and ignoring $O(\varepsilon^3)$ terms)

$$u(b^{k/2}x, b^k t') = Z_2(b)^k u(x, t') \quad (3.5)$$

and a fixed point of this iteration will exist only if

$$u_2^*(x, t') := \lim_{k \rightarrow \infty} Z_2(b)^{-k} u(b^{k/2}x, b^k t') \quad (3.6)$$

is well defined. Now assuming the existence of a fixed point in this formal derivation we rewrite (3.6) for large but finite k as

$$u(b^{k/2}x, b^k t') \cong Z_2(b)^k u_2^*(x, t'). \quad (3.7)$$

Note that $b > 1$ was necessary for considering large time and space, and in fact for the assumption of approximate self-similarity that underlies the existence of the fixed point u_2^* . Letting

$$\bar{x} := b^{k/2}x \quad \text{and} \quad \bar{t} := b^k t' \quad (3.8)$$

one has (for large k)

$$u(\bar{x}, \bar{t}) \cong [Z_2(b)]^k u_2^*(\bar{x}b^{-k/2}, \bar{t}b^{-k}). \quad (3.9)$$

This means that for any large \bar{t} one can determine the u profile by setting $b^k := \bar{t}/(Q_1^2/D)$, so that the second argument remains unchanged as one examines different values of \bar{t} . Letting $t_1 := D\bar{t}/Q_1^2$, we can then write (3.9) as

$$u(\bar{x}, \bar{t}) \cong [Z_2(t_1^{1/k})]^k u_2^*(\bar{x}t_1^{-1/2}, Q_1^2/D). \quad (3.10)$$

Stage 3: The scaling exponent will be determined by the limit

$$\lim_{k \rightarrow \infty} [Z_2(t_1^{1/k})]^k = \lim_{k \rightarrow \infty} t_1^{-1/2} \left\{ 1 + \frac{\varepsilon A \log(t_1)}{k} + \frac{1}{2!} \left[\frac{\varepsilon A \log(t_1)}{k} \right]^2 \right\}^k \quad (3.11)$$

if it exists. To calculate this we first note

$$y := \frac{\varepsilon A \log(t_1)}{k} \quad (3.12)$$

$$1 + \frac{\varepsilon A \log(t_1)}{k} + \frac{1}{2!} \left[\frac{\varepsilon A \log(t_1)}{k} \right]^2 = \left[1 + \frac{1}{(1+i)} y \right] \left[1 + \frac{1}{(1-i)} y \right]. \quad (3.13)$$

Combining this with (3.12) and (3.11) and utilizing the asymptotic expansion $e^\delta \cong 1 + \delta$ for small δ one has

$$\begin{aligned} \left\{ 1 + \frac{\varepsilon A \log(t_1)}{k} + \frac{1}{2!} \left[\frac{\varepsilon A \log(t_1)}{k} \right]^2 \right\}^k &\cong \exp \left\{ \frac{\varepsilon A}{(1+i)} \log(t_1) \right\} \\ &\cdot \exp \left\{ \frac{\varepsilon A}{(1-i)} \log(t_1) \right\} = t_1^{\varepsilon A}. \end{aligned} \quad (3.14)$$

We then have, from (3.11),

$$\lim_{k \rightarrow \infty} [Z_2(t_1^{1/k})]^k = t_1^{-\frac{1}{2} + \varepsilon A}. \quad (3.15)$$

Using (3.15) in (3.10) and dropping superbar since (3.10) is valid for arbitrary large \bar{t} we have

$$u(x, t') = (Dt'/Q_1^2)^{-\frac{1}{2} + \varepsilon A} u_2^*(x (Dt'/Q_1^2)^{-1/2}, Q_1^2/D) \quad (3.16)$$

so that the anomalous exponent or "dimension" is $\alpha = -\varepsilon A$.

Stage 5: Using the same idea as in (3.2) we approximate u_2^* (up to $O(\varepsilon^2)$) and obtain (3.1b).

IV. EXACT RESULTS.

We consider, for small ε , equation (2.1a) and perform the change of variables

$$u(x, t) := e^{\varphi(\xi, \tau)}, \quad \tau := \log(t + t_0) \quad \text{and} \quad \xi := x(t + t_0)^{-1/2} \quad (4.1 - 4.3)$$

in order to rewrite the differential equation as

$$\varphi_\tau = \frac{1}{2} [\varphi_{\xi\xi} + \xi\varphi_\xi + \varphi_\xi^2] + \varepsilon F[\xi, 1, \varphi_\xi, \varphi_{\xi\xi} + \varphi_\xi^2]. \quad (4.4)$$

We will find exact solutions of two equations in the form (2.1a) in this section.

Example 1: Consider the equation

$$u_t = \frac{1}{2} u_{xx} + \varepsilon x^{-1} u_x. \quad (4.5)$$

Using (4.1) – (4.3) this equation can be rewritten as

$$\varphi_\tau = \frac{1}{2} [\varphi_{\xi\xi} + \varphi_\xi^2 + \xi\varphi_\xi] + \varepsilon \xi^{-1} \varphi_\xi. \quad (4.6)$$

One is looking for a non-negative exact solution to (4.6) in the form

$$\varphi(\xi, \tau) = \sigma \xi^2 + \alpha \tau, \quad (4.7)$$

where $\sigma, \alpha \in R$, so that

$$u(x, t - t_0) = t^{-\alpha} \exp\left(\frac{\sigma x^2}{t}\right). \quad (4.8)$$

This yields the following solution to (4.5):

$$u(x, t - t_0) = t^{-\frac{1}{2}-\varepsilon} e^{-\frac{x^2}{2t}}. \quad (4.9)$$

Example 2: For the equation

$$u_t = \frac{1}{2} u_{xx} + \varepsilon u^{-1} u_x^2 \quad (4.10)$$

a similar procedure yields the transformed equation

$$\varphi_\tau = \frac{1}{2} [\varphi_{\xi\xi} + \varphi_\xi^2 + \xi\varphi_\xi] + \varepsilon \varphi_\xi^2. \quad (4.11)$$

The substitutions above yield the non-negative exact solution,

$$u(x, t - t_0) = t^{\frac{-1}{2(1-2\varepsilon)}} \exp\left(\frac{-x^2}{(1-2\varepsilon)2t}\right). \quad (4.12)$$

Note: Letting $u(x, t) = [w(x, t)]^{\frac{1}{(1-2\varepsilon)}}$, where $\varepsilon \neq \frac{1}{2}$, one can transform (4.10) into linear (diffusion equation) form, *i.e.* $w_t = \frac{1}{2} w_{xx}$. Hence, the exact solution given in (4.12) could be obtained by using the fundamental solution to linear equation, namely $\Gamma(x, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$.

These two exact solutions agree with the RG calculations of [5] and Section 3 above.

Series-Integral solutions. We describe briefly an additional method for calculating these decay exponents. In particular we solve (4.4) using an asymptotic expansion for small ε and write the formal sum

$$\varphi(\xi, \tau; \varepsilon) = \phi_0(\xi, \tau) + \varepsilon\phi_1(\xi, \tau) + \varepsilon^2\phi_2(\xi, \tau) + \dots \quad (4.13)$$

Substituting (4.13) into (4.4) and retaining only $O(1)$ terms leads to the equation that corresponds to the linear part of (2.1a)

$$\phi_{0\tau} = \frac{1}{2} [\phi_{0\xi\xi} + \xi\phi_{0\xi} + \phi_{0\xi}^2] \quad (4.14)$$

so that a solution, corresponding to the fundamental solution, to (4.14) is

$$\phi_0 = -\frac{1}{2}\xi^2 - \frac{1}{2}\tau. \quad (4.15)$$

We proceed by using ϕ_0 in the equation (4.4) to generate the next term of (4.13), namely ϕ_1 . Similarly, we substitute (4.13) into (4.4) and retain up to $O(\varepsilon)$ terms to obtain the equation for ϕ_1 . Utilizing then (4.15) yields the equation

$$\phi_{1\tau} - \mathbf{L}\phi_1 = \Omega_0(\xi) \quad (4.16)$$

where

$$\mathbf{L}\phi = \frac{1}{2} [\phi_{\xi\xi} - \xi\phi_{\xi}] \quad (4.17)$$

$$\Omega_0(\xi) := F[\xi, 1, \phi_{0\xi}, \phi_{0\xi\xi} + \phi_{0\xi}^2] = F[\xi, 1, -\xi, \xi^2 - 1]. \quad (4.18)$$

We then write the general solution ϕ_1 as

$$\phi_1(\xi, \tau) := \varphi_1(\xi) - \alpha_1\tau + \widehat{\varphi}_1(\xi, \tau) \quad (4.19)$$

where

$$\varphi_1(\xi) = -2 \int_0^\xi dv e^{v^2/2} \int_0^v d\eta e^{-\eta^2/2} [\alpha_1 + \Omega_0(\eta)] \quad (4.20)$$

$$\alpha_1 = -\frac{\int_0^\infty \Omega_0(\eta) e^{-\eta^2/2} d\eta}{\int_0^\infty e^{-\eta^2/2} d\eta} \quad \text{and} \quad (4.21)$$

$$\widehat{\varphi}_{1\tau} = \mathbf{L}\widehat{\varphi}_1. \quad (4.22)$$

Notice that $\widehat{\varphi}_1 \equiv 0$ if initially $\widehat{\varphi}_1(., 0) = 0$ in (4.19).

These differential equations can then be analyzed to obtain the leading order term. Similarly, the process can be repeated to obtain successive terms in the series for ϕ . This yields some additional rigorous results that confirm the RG calculations of [5] including the following.

Theorem 1: Suppose that $F(x, u, p, q)$ is independent of q , that F/p^2 is smooth such that it and its first derivative are uniformly bounded. Then there exists a unique positive number $\alpha(\varepsilon)$, such that the equation (2.1) has a solution of the form

$$u(x, t, \varepsilon) = t^{-\alpha(\varepsilon)} e^{\phi(xt^{-1/2}, \varepsilon)}, \quad (4.23)$$

where α and ϕ have the limiting properties

$$\lim_{\varepsilon \rightarrow 0} \alpha(\varepsilon) = \frac{1}{2}, \quad \lim_{\varepsilon \rightarrow 0} \phi(\xi, \varepsilon) = -\frac{1}{4}\xi^2. \quad (4.24)$$

The exact results and theorem confirm that the decay exponents obtained in [5] for leading order in ε are exact.

V. CONCLUSION.

A number of important properties of solutions to parabolic differential equations can be described as asymptotically self-similar. These include decay, finite-time blow-up and extinction. Unlike the self-similarity of fractals, this property involves relationships that evolve, in the limit, toward self-similarity. We use an asymptotic analysis in several parameters in conjunction with renormalization group transformation that differs from standard RG theory in that the transformation is valid only to a particular order in the small parameter, ε , that is the coefficient of the nonlinearity. The asymptotic analysis and the RG transformations are for small ε and also small l (i.e., sharp Gaussian initial conditions). Thus the analysis is intended to pick out the most singular term in terms of l^{-1} . This means that terms that are higher order in ε and negative do not contribute.

The decay computed using the RG and asymptotic methods are technically for large time within an intermediate range, where the analysis is meaningful. The exact results and theorem confirm the conjecture that the exponent calculated is in fact the decay exponent at infinity.

REFERENCES.

- [1] N. Goldenfeld, Lectures on Phase Transitions and the Renormalization Group, Addison-Wesley, Reading, MA.(1992)
- [2] R. J. Creswick, H. A. Farach and C. P. Poole, Introduction to Renormalization Group Methods in Physics, Wiley, New York (1992).
- [3] N. Goldenfeld, O. Martin, Y. Oono and F. Liu, Anomalous dimensions and the renormalization group in a nonlinear diffusion process, Phys. Rev. Let. 64(1990)1361-1364.
- [4] J. Briçmont, A. Kupiainen and G. Lin, "Renormalization group and asymptotics of solutions of nonlinear parabolic equations" Comm. Pure, Appl. Math. 47 (1994) 893-921.
- [5] G. Caginalp, A renormalization group calculation of anomalous exponents for nonlinear diffusion, Phys.Rev. E 53 (1996) 66-73; Renormalization and scaling methods for nonlinear parabolic systems, Nonlinearity 10, (1997) 1217-1229.